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APPROXIMATE SOLUTION OF STOCHASTIC PROGRAMMING PROBLEMS WITH RECOURSE*

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We present a method for solving approximately the linear stochastic programming problem with complete recourse. The problem is set in Banach space of (Riemann integrable) functions and we deduce conditions that guarantee stability of approximations of a sequence of finite-dimensional problems.

1. INTRODUCTION

In this paper we present an approximation scheme for linear stochastic programming problems (LSP) with recourse. The problem which we shall deal with may be formulated by introducing a Banach space $B(S)$ of bounded and (Riemann) integrable functions.

Let (S, Σ, m), $S \subset R^l$ be a probability space. Consider the sets A from the σ-algebra Σ with m-measure zero of their boundary. These sets constitute an algebra $\Sigma_0 \subset \Sigma$ (cf. [13]). Let m_0 be the restriction of the probability measure m to algebra Σ_0. The space $B(S) \triangleq B(S, \Sigma_0, m_0)$ denotes now the class of bounded Σ_0-measurable functions. This is a Banach space with sup-norm topology [13]. For example if $S = [0, 1]$ and m is the Lebesgue measure on $[0, 1]$ then $B(S)$ is simply the space of Riemann integrable functions. Let us formulate now the problem of interest: minimize

$$c^{\mathrm{T}}x + \int_S q^{\mathrm{T}}(s)\, y(s)\, m_0(\mathrm{d}s) \tag{1.1}$$

over $[x, y(\cdot)] \in R^r \times B(S, \Sigma_0, m_0; R^v)$ satisfying

$$x \in X \triangleq \{x \mid Dx \geqq d\}\,, \quad X \subset R^r\,, \tag{1.2}$$

and almost surely (a.s.)

$$y(s) \in S \triangleq \{y \mid Gy \geqq g\}\,, \quad y \subset R^v\,, \tag{1.3}$$

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and

$$(1.4) \qquad Ax + Wy(s) \geqq b(s)\,.$$

Here $c \in R^r$, $q(s) \in R^v$, $s \in S$, $k \times r$- and $k \times v$-matrices A and W are fixed and let the sets X and Y be convex bounded polyhedras.

Several authors have examined approximation and stability problems in stochastic programming with recourse (see e.g. [1], [3]–[10]). In general they use probability-theoretic approach (except [7], [8]). For instance in [6] partition of initial space of elementary events is fulfilled by a finite sub-σ-algebra, on elements of which the desired function is regarded to be constant.

In this paper we use a functional-analytical approach. Assuming the existence of a sequence of discrete measures that converges weakly to the probability measure m we deduce conditions that guarantee stability of our approximation. In [7] we used this approach to present discrete stability conditions in L^p-spaces. For discretization in this paper we restrict the class of integrable functions to the more convenient class of Σ_0-measurable functions $B(S)$.

In the next section we present some necessary auxiliary results and notions concerning the discrete convergence of mappings. In Section 3 we present conditions that guarantee stability of discrete approximation.

2. DISCRETIZATION OF THE PROBLEM

Let us introduce some notions from the discrete convergence of mappings [11], [12].

Let E and E_n ($n \in N = \{1, 2, 3, \ldots\}$) be Banach spaces with norms $\|\cdot\|$ and $\|\cdot\|_n$, respectively and let $\mathscr{P} = (p_n)$ be a system of *linear connection operators* p_n: $E \to E_n (n \in N)$ such that for every $y \in E$

$$(2.1) \qquad \|p_n y\|_n \to \|y\| \quad (n \in N)\,.$$

Definition 2.1. A sequence (y_n) $(n \in N)$ with $y_n \in E_n$ *$\mathscr{P}$-converges* (or *converges discretely*) to $y \in E$ if $\|y_n - p_n y\|_n \to 0$ $(n \in N)$. We denote this convergence by $y_n \to y$.

Remark 2.1. Denote by $|\cdot|$ the Euclidean norm of a vector $x \in R^r$ and by $|x_n - x| \to 0$ $(n \in N)$ the sense of convergence in Euclidean norm.

Let us define now *discrete convergence* in space $B(S)$.

Let b_n be an nv-dimensional space of vectors $y_n = (y_{1n}, \ldots, y_{nn})^{\mathrm{T}}$, $y_{in} \in R^v$, $i = 1, \ldots, n$, with max-norm

$$\|y_n\|_n = \max_{1 \leqq i \leqq n} |y_{in}|\,.$$

Let the support S of the measure m be bounded.

Let some l-dimensional quadrature process be given:

$$\lim_{n\to\infty} \sum_{i=1}^{n} h(s_{in})\, m_{in} = \int_S h(s)\, m(\mathrm{d}s) \tag{2.2}$$

for every continuous function $h(s)$. Here $m_{in} > 0$, $i = 1, \ldots, n$,

$$\sum_{i=1}^{n} m_{in} = m(S) \quad \text{and} \quad (s_{1n}, \ldots, s_{nn})^{\mathrm{T}} = s_n$$

are some fixed different points in S.

In this paper we have to restrict our initial probability measure m:

A 1) $m\{s \mid |s - s_0| = \text{const}\} = 0$ $\forall s_0 \in S$. For example the restriction A 1) is fulfilled if the probability measure m has a density.

Define for spaces $B(S)$ and $b_n (n \in N)$ the system of *connection operators* $\mathscr{P} = (p_n)$, $p_n \colon B(S) \to b_n$, in the following way:

$$p_n y = (y(s_{in}),\ \ i = 1, \ldots, n) \tag{2.3}$$

where $s_n = (s_{1n}, \ldots, s_{nn})^{\mathrm{T}}$ is a point taken from the quadrature formula (2.2).

Remark 2.2. The convergence $\|p_n y\|_n \to \|y\|$ $(n \in N)$ $\forall y \in B(S)$ is a direct consequence from the convergence of quadrature process (2.2) [13].

Lemma 2.1. For the convergence of quadrature process (2.2) it is necessary and sufficient that there exists a collection of sets $\{A_{in}\}_{i=1}^{n}$ $(n \in N)$, $m(A_{in}) > 0$, from algebra Σ_0 such that

1) $\bigcup_{i=1}^{n} A_{in} = S$; 2) $A_{in} \cap A_{jn} = \emptyset$, $i \neq j$;

3) $\operatorname{diam} A_{in} \to 0$ as $n \to \infty$; 4) $s_{in} \in A_{in}$;

5) $\max_{1 \leq i \leq n} |m_{in}\, m_0(A_{in})^{-1} - 1| \to 0$ as $n \to \infty$

(here $\operatorname{diam} A = \sup_{s,t \in A} |s - t|$).

Lemma 2.1. is proved and formulated in a somewhat different form in [13].

3. CONDITIONS FOR CONVERGENCE OF DISCRETE APPROXIMATION

Let the fixed recourse be relatively complete in the following sense:

A 2) for all $x \in X$ and a.a. $s \in S$ there exists an $y \in S$ such that

$$Ax + Wy \geqq b(s)\,.$$

A 3) $q(\cdot), b(\cdot) \in B(S)$.

Instead of the problem (1.1)–(1.4) consider the following problem (3.1n)–(3.2n) in the finite-dimensional space $R^r \times b_n$: minimize

$$c^T x + \sum_{i=1}^{n} q^T(s_{in})\, y_{in} m_{in} \tag{3.1n}$$

over $[x, y_n] \in R^r \times b_n$ where $x \in X$, $y_{in} \in S$, $i = 1, \ldots, n$, and

$$Ax + W y_{in} \geqq b(s_{in}), \quad i > 1, \ldots, n. \tag{3.2n}$$

Let us introduce the following notations:

$$Q = \{[x, y(s)] \mid x \in X,\ y(s) \in S,\ Ax + W y(s) \geqq b(s) \text{ for a.a. } s \in S\},$$

$$Q_n = \{[x, y_n] \mid x \in X,\ y_{in} \in S,\ Ax + W y_{in} \geqq b(s_{in}),\ i = 1, \ldots, n\}.$$

Proposition 3.1. Let 1) the function $b(s)$ be locally Lipschitzian with constant L; 2) the condition A 2) be fulfilled; 3) the sets $X = \{x \mid Dx \geqq d\}$ and $Y = \{y \mid Gy \geqq g\}$ be bounded polyhedra; 4) the support S of the measure m be bounded; 5) the quadratute process converge. Then the sequence of solutions of problems ((3.1 n)–(3.2n)) $(n \in N)$ is dicretely compact and its limit points are admissible for the problem (1.1)–(1.4).

Proof. Since we deal with a linear problem (3.1n)–(3.2n) in finite-dimensional space with convex bounded polyhedra X and Y then the minimum is attained on the boundary of the constraint set Q_n. It is easy to see that in all cases

$$|w_{j(n)}(\bar{y}_{in} - \bar{y}_{kn})| \leqq |b_{j(n)}(s_{in}) - b_{j(n)}(s_{kn})| \tag{3.3}$$

where $w_{j(n)}$ is the $j(n)$th row of the (constant) recourse matrix W, $b_{j(n)}$ is the $j(n)$th component of the vector-valued function $b(s)$ and $\bar{y}_n$ is the solution of (3.1n)–(3.2n).

Define now the following function $\bar{y}_n(s)$ in $B(S)$:

$$\bar{y}_n(s) = \bar{y}_{in} \quad \text{as} \quad s \in A_{in}.$$

Here the sets A_{in} are taken from the convergence criterion of the quadrature process (Lemma 2.1). Since $\bar{y}_{in} \in S$, $i = 1, \ldots, n$, the sequence $(\bar{y}_n(\cdot))\ n \in N)$ is bounded in $B(S)$ and

$$\sup_{s \in A_{in}} |w_{j(n)}(\bar{y}_n(s_{in}) - \bar{y}_n(s))| \leqq \sup_{s \in A_{in}} |b_{j(n)}(s_{in}) - b_{j(n)}(s)| \leqq L \sup_{s \in A_{in}} |s_{in} - s| \leqq$$
$$\leqq L \operatorname{diam} A_{in}.$$

Due to condition 5) diam $A_{in} \to 0$ as $n \to \infty$ and hence, the compactness criterion in space $B(S)$ (see [2] IV. 5.6) is fulfilled for the sequence $(\bar{y}_n(\cdot))$ $(n \in N)$. Due to the supplementarity property ([12] p. 648) the sequence $(\bar{y}_n)$ $(n \in N)$ of solutions of problems (3.1n)–(3.2n) is $\mathscr{P}$-compact (or discretely compact).

Prove the second part of the proposition. Let us introduce the set $G \subset R^r \times B(S) \times B(S)$,

$$G = \{[x, y(s), b(s)] \mid Ax + W y(s) - b(s) \geqq 0 \text{ for a.a. } s \in S\}$$

and define the function $b_n(s)$:

$$b_n(s) = b(s_{in}) \quad \text{as} \quad s \in A_{in}.$$

Then $[\bar{x}_n, \bar{y}_n(s), b_n(s)] \in G$ for $n = 1, 2, \ldots$. Since the sequence $([\bar{x}_n, \bar{y}_n(s), b_n(s)])$ $(n \in N' \subset N)$ converges to $[\bar{x}, \bar{y}(s), b(s)]$ in the sense of sup-norm topology, it converges for almost all $s \in S$ and therefore the set G is closed. Consequently, $[\bar{x}, \bar{y}(s)] \in Q$ i.e. the limit point of the sequence $([\bar{x}_n, \bar{y}_n])$ $(n \in N')$ is admissible. □

Relying on the Proposition 3.1 we can now formulate and prove the main result of this paper on discrete stability of the sequence of problems (3.1n)–(3.2n). To make our presentation short let us introduce some notations:

f^* – the optimal value of the problem (1.1)–(1.4),
f_n^* – the optimal value of the problem (3.1n)–(3.2n),
$z = [x, y(s)], \quad z_n = [x_n, y_n]$,
$f(z) = c^T x + \int q^T(s)\, y(s)\, m(ds)$,
$f_n(z_n) = c^T x + \sum_{i=1}^{n} q^T(s_{in})\, y_{in} m_{in}$.

Theorem 3.1. Let the following assumptions be satisfied:
a) quadrature process (2.2) converges;
2) conditions A 1) – A 3) are fulfilled;
3) the sets $X = \{x \mid Dx \geqq d\}$ and $Y = \{y \mid Gy \geqq g\}$ are bounded polyhedra;
4) the support S of the measure m is bounded;
5) the function $b(s)$ is locally Lipschitzian with constant L.
Then

$$f_n^* \to f^* \quad \text{as} \quad n \to \infty .$$

Proof. Since the limit points of the sequence of solutions $[\bar{x}_n, \bar{y}_n]$ of problems (3.1n)–(3.2n) $(n \in N)$ are admissible then

$$f^* \leqq \liminf_{n\to\infty} f_n(\bar{z}_n) .$$

Let us show that

$$\limsup_{n\to\infty} f_n(\bar{z}_n) \leqq f^* .$$

Since $[\bar{x}_n, p_n\bar{y}] \in Q_n$, $\bar{x}_n \in X$, $p_n\bar{y} \in Y$, then

$$\lim_{n\to\infty} f_n(\bar{z}_n) - f(\bar{z}) \leqq \lim_{n\to\infty} f_n(p_n\bar{z}) - f(\bar{z})$$

(here $p_n\bar{z} = [\bar{x}, p_n\bar{y}]$). From the convergence of quadrature process (2.2) we can conclude that for a small $\varepsilon > 0$ there exists an index n_1 such that for $n \geqq n_1$ we have

$$|f_n(p_n\bar{z}) - f(\bar{z})| = |\sum_{i=1}^{n} q^T(s_{in})\, \bar{y}(s_{in})\, m_{in} - \int q^T(s)\, \bar{y}(s)\, m(ds)| < \varepsilon .$$

Hence, for $n \geqq n_1$ $f_n^* \leqq f^* + \varepsilon$. □

Corollary 3.1. The limit point $[\bar{x}, \bar{y}(s)]$ of discretely converging subsequence $([\bar{x}_n, \bar{y}_n])$ $(n \in N')$ is optimal to the problem (1.1)–(1.4).

Example. Let us explain the idea of the discrete approximation method. Let $s \in [0, 1]$, $c = \frac{1}{2}$, $q = 1$, $A = W = 1$, $X = Y = R^1$, $m(ds) = ds$, $b(s) = s$. Then the problem (1.1)–(1.4) becomes:

$$\min \{\tfrac{1}{2}x + \int_0^1 y(s)\, ds\} = f^* = \tfrac{1}{2}\,.$$

$$\text{(3.3)} \qquad x + y(s) \geqq s$$

$$s \in [0, 1]$$

$$y \in B[0, 1]$$

Clearly $\bar{x} = 0$, $\bar{y}(s) = s$, $s \in [0, 1]$.

Let us discretize the problem (3.3): let $n \in N$ be fixed, $s_{in} = i/n$, $i = 1, 2, \ldots, n$. Consider now the following minimization problem:

$$\text{(3.3n)} \qquad \min \{\tfrac{1}{2}x + 1/n \sum_{i=1}^{n} y_{in}\} = f_n^* = + \tfrac{1}{2} + 1/(2n)\,.$$

$$x + y_{in} \geqq i/n\,, \quad i = 1, \ldots, n\,,$$

$$y_{in} \in [0, 1]\,, \qquad i = 1, \ldots, n\,.$$

Clearly $\bar{x}_n = 0$, $\bar{y}_{in} = i/n$, $i = 1, \ldots, n$.

Hence $f_n^* \to f^*$, $\bar{x}_n = \bar{x}$ and $\bar{y}_n \to \bar{y}$ discretely as $n \to \infty$.

Remark. If $y \in L^p[0, 1]$, $1 \leqq p \leqq \infty$, then we must use instead of p_n the piecewise integral connection operator p_n' in the form [7]:

$$(p_n' y)_{in} = n^{-1} \int_{i-1/n}^{i/n} y(s)\, ds\,, \quad i = 1, \ldots, n\,.$$

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