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RANDOM SEQUENCES WITH NORMAL COVARIANCES

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The main goal of the paper is to introduce the notion of random sequences with normal covariances, describe the class of normal covariances and give their spectral decomposition. The motivation is to describe a large class of random sequences, containing weakly stationary case, that could be useful as mathematical models for real random sequences.

1. LOCALLY STATIONARY COVARIANCES

We shall start with the notion of a locally stationary covariance following Silverman who introduced this notion in his paper [1]. He considered the case of random processes defined on the whole real line only. A random process $\{Z(t)\}$, $t \in \mathbb{R}_1$, is locally stationary if its covariance function $R(\cdot, \cdot)$ can be expressed as a product

$$R(s, t) = R_1\left(\frac{s+t}{2}\right) R_2(s-t)\,, \tag{1}$$

where $R_1(\cdot) \geqq 0$ and $R_2(\cdot)$ is a stationary covariance. In this case we shall say that the covariance $R(\cdot, \cdot)$ is locally stationary too. We assume the expected value $\mathsf{E}\{Z(t)\}$ is vanishing for every real t. At the first sight one sees that the product (1) is not suitable for the definition of local stationarity for random sequences because the set $\mathbb{Z}$ of integers is not closed with respect to division by two. For this reason we suggest the following

Definition 1. Let $\{Z(n)\}$, $n \in \mathbb{Z}$, be a (complex) random sequence with finite second moments and vanishing expected value. We shall say that $\{Z(n)\}$, $n \in \mathbb{Z}$, is locally stationary, or its covariance function is locally stationary, when its covariance $R(\cdot, \cdot)$ can be written as

$$R(n, m) = R_1(n + m)\, R_2(n - m) \tag{2}$$

where $R_2(\cdot)$ is a stationary covariance.

In the case of a random process the product (1) immediately implies that $R_1(\cdot) \geqq 0$ in every case for $R(s, s) = R_1(s) = \mathsf{E}\{|Z(s)|^2\} \geqq 0$. On the other hand in the case of a random sequence we do not demand nonnegativity of $R_1(\cdot)$. Definition 1 yields only that $R_1(2n) \geqq 0$ for every $n \in \mathbb{Z}$; in general, $R_1(n + m)$ can be negative for $n + m$ odd. In case when $R_1(\cdot)$ is a covariance too the product (2) is a covariance automatically and we obtain a large class of locally stationary covariances. Let us describe this class detailly. As long as $R_1(\cdot)$ is a covariance then $R_1(\cdot)$ is a real function because $R_1(n + m) = \overline{R_1(m + n)} = R_1(m + n)$.

Definition 2. Let $\{Z(n)\}$, $n \in \mathbb{Z}$, be a (complex) random sequence with finite second moments and vanishing mean value. We shall say that $\{Z(n)\}$, $n \in \mathbb{Z}$ is symmetric if its covariance function $R(\cdot, \cdot)$ has the form

$$R(n, m) = R_1(n + m) .$$

A covariance function of this type will be called symmetric too.

Theorem 1. Every symmetric covariance function $R_1(\cdot)$ can be expressed in the form

$$R_1(n + m) = \int_{-\infty}^{\infty} \lambda^{n+m} \, \mathrm{d}F_1(\lambda)$$

where F_1 is a left continuous nondecreasing function with finite variation equal to $R_1(0)$.

Proof. The proof is quite similar to the proof of Theorem XII. 8.1 in [2]. Let x be a random variable belonging to the linear set of all finite linear combinations over the sequence $\{Z(n)\}$, $n \in \mathbb{Z}$, i.e.

$$x = \sum_{i=-N}^{N} \alpha_i \, Z(i)$$

where $\alpha_{-N}, \alpha_{-N+1}, \ldots, \alpha_{-1}, \alpha_0, \alpha_1, \ldots, \alpha_N$ are complex numbers. Let us consider the imbedding of $\{\alpha_{-N}, \alpha_{-N+1}, \ldots, \alpha_{-1}, \alpha_0, \alpha_1, \ldots, \alpha_N\}$ into a two-side sequence $\{\alpha_i\}_{i=-\infty}^{\infty}$ of complex numbers with finite number of nonzero elements. The set of these sequences with finite number of non-zero elements forms a linear set with respect to addition and multiplication by scalars coordinatewisely. Let us define on this set a bilinear form

$$\langle \alpha, \beta \rangle = \sum_{i=-L}^{L} \sum_{j=-L}^{L} \alpha_i \bar{\beta}_j \, R_1(i + j)$$

where $\alpha = \{\alpha_i\}_{i=-\infty}^{\infty}$, $\beta = \{\beta_j\}_{j=-\infty}^{\infty}$, $L = \mathrm{Max}\,(N, M)$. Denote $\|\alpha\|^2 = \sum_{i=-N}^{N} \sum_{j=-N}^{N} \alpha_i \bar{\alpha}_j \, . \, R(i + j)$ then $\|\cdot\|$ is a seminorm. We shall say that α is equivalent to $\beta (\alpha \sim \beta)$ if $\|\alpha - \beta\| = 0$. Instead of the original sequences we shall work with the classes of equivalence which form a unitary space with the norm $\|\cdot\|$. Let H be a completion of this unitary space with respect to the norm $\|\cdot\|$. Then H is a Hilbert space and let us define a shift operator T by the relation

$$(T\alpha)_i = (\alpha)_{i-1} , \quad i \in \mathbb{Z} .$$

To be well defined the operator T must map the null class $\{0\}$ into the null class again. If $\|\alpha\| = 0$, i.e.

$$\sum_{i=-N-1}^{N-1} \sum_{j=-N-1}^{N-1} \alpha_i \bar{\alpha}_j R(i+j) = 0 ,$$

then

$$\|T\alpha\|^2 = \sum_{i=-N-1}^{N-1} \sum_{j=-N-1}^{N-1} \alpha_{i-1} \bar{\alpha}_{j-1} R(i+j) =$$

$$= \sum_{i=-N-1}^{N-1} \sum_{j=-N-1}^{N-1} \alpha_{i-2} \bar{\alpha}_j R(i+j) = \langle T^2\alpha, \alpha \rangle = 0$$

because $|\langle T^2\alpha, \alpha\rangle| \leqq \|T^2\alpha\| \|\alpha\| = 0$. In a similar way, one can easily prove that the operator T is symmetric in H, i.e. the equality

$$\langle T\alpha, \beta \rangle = \langle \alpha, T\beta \rangle$$

holds for every pair α, β of elements in H which correspond to sequences with finite number of nonzero elements. It means the definition domain $D(T)$ of T is formed by all equivalence classes due to finite sequences, which form a linear set everywhere dense in H. As for every α where T is defined $T\bar{\alpha} = \overline{T\alpha}$ the operator T is of a real type with respect to complex adjoint property. This fact implies the existence of a maximal self-adjoint enlargement T_1 of T in H because the both deficiency-subspaces due to T have the same dimension. The self-adjoint operator T_1 has a spectral decomposition

$$T_1 = \int_{-\infty}^{\infty} \lambda \, dE_\lambda$$

where $\{E_\lambda\}_{\lambda=-\infty}^{\infty}$ is a resolution of the identity in H. Thanks to the fact that $T_1 = T$ on $D(T)$ and $D(T) = D(T^n)$ for every $n \geqq 0$ we can state

$$R_1(p) = \int_{-\infty}^{\infty} \lambda^p \, d\langle E_\lambda \, \alpha(0), \alpha(0) \rangle$$

where $(\alpha(0))_0 = 1$, $(\alpha(0))_i = 0$ otherwise. As we consider two-side sequences there exists the inverse operator T^{-1} to operator T defined by the relation

$$(T^{-1}\alpha)_i = (\alpha)_{i+1} .$$

The operator T^{-1} is defined on $D(T)$ and similarly we can prove that T^{-1} is symmetric and maps the null class $\{0\}$ into the same class of equivalence. T^{-1} is an operator of a real type as well and hence there exists a self-adjoint operator T_1^{-1} satisfying $T^{-1} \subset T_1^{-1}$. As $T_1^{-1} = (T_1)^{-1}$ on $D(T)$ then

$$T_1^{-1} x = \left(\int_{-\infty}^{\infty} \lambda \, dE_\lambda \right)^{-1} x = \int_{-\infty}^{\infty} \lambda^{-1} \, dE_\lambda x$$

for every $x \in D(T)$.

We obtained that for every $p \in \mathbb{Z}$

$$R_1(p) = \int_{-\infty}^{\infty} \lambda^p \, d\langle E_\lambda \, \alpha(0), \alpha(0) \rangle ,$$

i.e.

$$R_1(n+m) = \int_{-\infty}^{\infty} \lambda^{n+m} \, dF_1(\lambda) . \qquad \square$$

On the basis of Theorem 1 and the Karhunen theorem, (see [3]), we can derive a spectral decomposition of a symmetric sequence in the form of a stochastic integral understood in the quadratic mean sense

$$Z(n) = \int_{-\infty}^{\infty} \lambda^n \,\mathrm{d}\xi(\lambda)$$

where $\{\xi(\lambda)\}$, $\lambda \in \mathbb{R}_1$, is a martingale in the quadratic mean sense satisfying

$$Z(0) = \operatorname*{l.i.m.}_{\lambda \to \infty} \xi(\lambda)$$

$$0 = \operatorname*{l.i.m.}_{\lambda \to -\infty} \xi(\lambda)\,.$$

Theorem 2. Let $\{Z_1(n)\}$, $n \in \mathbb{Z}$ be a symmetric sequence, let $\{Z_2(n)\}$, $n \in \mathbb{Z}$ be a stationary sequence and let these sequences be mutually stochastically independent. Then their product $\{Z(n)\}$, $n \in \mathbb{Z}$,

$$Z(n) = Z_1(n)\,Z_2(n)\,, \quad n \in \mathbb{Z}$$

is a locally stationary sequence.

Proof.

$$\mathsf{E}\{Z(n)\,\overline{Z(m)}\} = \mathsf{E}\{Z_1(n)\,Z_2(n)\,\overline{Z_1(m)}\,\overline{Z_2(m)}\} = \mathsf{E}\{Z_1(n)\,\overline{Z_1(m)}\} \times$$
$$\times\ \mathsf{E}\{Z_2(n)\,\overline{Z_2(m)}\} = R_1(n+m)\,R_2(n-m)\,. \qquad \square$$

Theorem 3. Let $\{Z(n)\}$, $n \in \mathbb{Z}$ be locally stationary. Let $n_1 < n_2 < n_3 < \ldots < n_N$ be integers. Then the covariance matrix $R = \{\mathsf{E}\{Z(n_i)\,\overline{Z(n_j)}\}_{i=j=1}^{N}\}$ is the Hadamard product of a Toeplitz matrix and a Hankel matrix.

Proof. Local stationarity gives $\mathsf{E}\{Z(n_i)\,\overline{Z(n_j)}\} = R_1(n_i + n_j)\,R_2(n_i - n_j)$, for simplicity let us put $Z(n_i) = y(i)$, $i = 1, 2, \ldots, N$, then

$$\mathsf{E}\{y(i)\,\overline{y(j)}\} = S_1(i+j)\,S_2(i-j)\,.$$

Now, we have two matrices $S_1 = \{S_1(i+j)\}_{i=j=1}^{N}$, $S_2 = \{S_2(i-j)\}_{i=j=1}^{N}$ where S_1 is a Hankel matrix and S_2 is a Toeplitz matrix. Their Hadamard product is precisely the covariance matrix of the random variables $Z(n_1), Z(n_2), \ldots, Z(n_N)$. $\square$

Remark. It is evident that the Toeplitz matrix in Theorem 3 must be nonnegative definite because Definition 2 demands $R_2(\cdot)$ to be a covariance function. In general, the Hankel matrix need not be nonnegative definite although their Hadamard product must be a nonnegative definite matrix. On the other side, by use of Theorem 3 we obtain a known result that the Hadamard product of two non-negative definite matrices must be a nonnegative definite matrix also. It is evident as well that every finite subsequence of a symmetric sequence has its covariance matrix of the Hankel type. The contruction of a random sequence with a Hankel covariance matrix is based on property expressing a necessary and sufficient condition for nonnegative definiteness of a Hankel bilinear form. Let $\{X_0, X_1, \ldots, X_{N-1}\}$ be a sequence of random variables whose covariance function is a Hankel matrix. i.e. $\mathsf{E}\{X_i \overline{X}_j\} = r(i+j)$.

Then the matrix $R_1 = \{r(i + j)\}_{i=j=0}^{N-1}$ is positively definite if and only if

$$r(i + j) = \sum_{l=1}^{N} \varrho_l \theta_l^{i+j}, \quad i, j = 0, 1, \ldots, N - 1, \quad \varrho_l > 0, \quad \theta_l \in \mathbb{R}_1$$

(for details see [4]). This fact gives a hint for the construction of such a random sequence. Let $\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_N$ be i.i.d. random variables with $\mathsf{E}\{\varepsilon_i\} = 0$, $\mathsf{E}\{|\varepsilon_i|^2\} = 1$, $i = 1, 2, \ldots, N$. Let us define

$$X_i = \sum_{l=1}^{N} \varrho_l^{1/2} \theta_l^i \varepsilon_l, \quad i = 0, 1, \ldots, N - 1.$$

Then $\mathsf{E}\{X_i \overline{X}_j\} = \sum_{l=1}^{N} \varrho_l \theta_l^{i+j} = r(i + j)$, it means the covariance matrix of the sequence $\{X_0, X_1, \ldots, X_{N-1}\}$ is of the Hankel type. Surely, we can continue and instead of a finite sequence $\{\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_N\}$ we can consider an infinite sequence $\{\varepsilon_i\}_{i=-\infty}^{\infty}$. If the series

$$X_i = \sum_{l=-\infty}^{+\infty} \varrho_l^{1/2} \theta_l^i \varepsilon_l$$

is convergent in the quadratic mean sense for every $i \in \{-N + 1, -N + 2, \ldots \ldots, -1, 0, 1, \ldots, N - 1\}$, i.e. if the seties

$$\sum_{l=-\infty}^{+\infty} \varrho_l \theta_l^{2i}$$

is convergent for every $i \in \{-N + 1, -N + 2, \ldots, -1, 0, 1, \ldots, N - 1\}$, then the sequence $\{X_i\}_{i=-N+1}^{N-1}$ is symmetric.

We see that symmetric covariances are a special case of locally stationary covariances. If a locally stationary covariance is the product of a symmetric covariance and a stationary covariance by use of Theorem 1 we can express that locally stationary covariance $R(\cdot, \cdot)$ in the form

$$R(n, m) = \int_{-\infty}^{\infty} \int_{-\pi}^{\pi} \lambda^{m+n} \, \mathrm{e}^{\mathrm{i}\mu(n-m)} \, \mathrm{d}\mathrm{d}F_1(\lambda) \, F_2(\mu).$$

Every random sequence $\{Z(n)\}$, $n \in \mathbb{Z}$ with a covariance of this type can be by means of the Karhunen theorem expressed in the form of a stochastic integral understood in the quadratic mean sense

$$Z(n) = \int_{-\infty}^{\infty} \int_{-\pi}^{\pi} \lambda^n \, \mathrm{e}^{\mathrm{i}\mu n} \, \mathrm{d}\mathrm{d}\xi(\lambda, \mu).$$

The process $\xi(\cdot, \cdot)$ is defined at the whole plane and satisfies $\mathsf{E}\{\xi(\lambda_1, \mu_1) \overline{\xi(\lambda_2, \mu_2)}\} = F(\min(\lambda_1, \lambda_2), \min(\mu_1, \mu_2))$, where $F(\cdot, \cdot) = F_1(\cdot) F_2(\cdot)$. This property means that $\xi(\cdot, \cdot)$ is a plane martingale satisfying

$$P_{(\lambda,\mu)} \, \xi(u. v) = \xi(\lambda, \mu)$$

for every $\lambda \leqq u$, $v \geqq \mu$, $P_{(\lambda,\mu)}$ is the projector onto the subspace $H_{(\lambda,\mu)}$ generated by all random variables $\sum_{i=1}^{n} \alpha_i \, \xi(\lambda_i, \mu_i)$ with $\lambda_i \leqq \lambda$, $\mu_i \leqq \mu$ for every $i = 1, 2, \ldots, n$.

We see immediately also that

$$Z(0) = \operatorname*{l.i.m.}_{\lambda \to \infty} \xi(\lambda, \pi + \varepsilon), \quad \varepsilon > 0$$

if we have put $\xi(-\infty, -\pi) = 0$. In this way we obtain the system of the subspaces $\{H_{(\lambda,\mu)}\}$, $(\lambda, \mu) \in \mathbb{R}_2$ with $H_{(-\infty,-\infty)} = \{0\}$, $H_{(\infty,\infty)}$ is the Hilbert space generated by the all random variables $\xi(\lambda, \mu)$, $\lambda \in (-\infty, \infty)$, $\mu \in (-\infty, \infty)$. Every subspace $H_{(\lambda,\mu)}$ defines the corresponding projector $P_{(\lambda,\mu)}$ and one can easily prove that the system $\{P_{(\lambda,\mu)}\}$, $(\lambda, \mu) \in \mathbb{R}_2$ forms a resolution of the identity in the space $H_{(\infty,\infty)}$. The system $\{P_{(.,.)}\}$ defines in the space $H_{(\infty,\infty)}$ a normal operator S

$$S = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{z + \bar{z}}{2} e^{\frac{z-\bar{z}}{2}} \, dP_{(\lambda,\mu)}, \quad z = \lambda + i\mu$$

with definition domain $\mathscr{D}(S) = \{x \in H_{(\infty,\infty)}: \iint_{-\infty}^{\infty} \lambda^2 \, d\langle P_{(\lambda,\mu)} x, x\rangle < \infty\}$. As $P_{(\lambda,\mu)} Z(0) = \xi(\lambda, \mu)$ we obtain that $S\,Z(0) = Z(1)$ and, in general, for every $n \in \mathbb{Z}$ $S^n Z(0) = Z(n)$. Now we shall use a general formula

$$\iint_{\Omega'} f(w) \, dP'_{(v_1,v_2)} = \iint_{\Omega} f(\varphi(z)) \, dP_{(\lambda,\mu)}$$

where $\varphi(z): \Omega \to \Omega'$ is a measurable mapping and $P'(\Delta') = \iint_{\varphi^{-1}(\Delta')} dd P_{(\mu,\mu)}$. In our case if we put $\varphi(z) = \frac{1}{2}(z + \bar{z}) e^{(z+\bar{z})/2}$ and $f(w) = w$ then the operator S in $H_{(\infty,\infty)}$ can be described as

$$S = \iint_{-\infty}^{\infty} w \, dd P'_w .$$

Let $H(z(\cdot))$ be a subspace of $H_{(\infty,\infty)}$ generated by all random variables $Z(n)$, $n \in \mathbb{Z}$. Let T be equal to S in $H(z(\cdot))$, i.e. T is an operator in $H(z(\cdot))$ with $\mathscr{D}(T) = \mathscr{D}(S) \cap \cap H(z(\cdot))$ and $Tx = Sx$ for every $x \in \mathscr{D}(T)$. There is no problem to prove that T is in the space $H(z(\cdot))$ a normal operator too.

We can summarize and state that every locally stationary covariance being the product of a symmetric covariance and a stationary covariance creates a normal operator and hence this covariance can be written in the following form

$$R(n, m) = \iint_{-\infty}^{\infty} w^n \bar{w}^m \, dd G(u, v), \quad w = u + iv . \tag{3}$$

Definition 3. A covariance function that can be expressed in the form (3) will be called a normal covariance. The notion of a normal covariance for random processes was introduced by the author in [5].

Similarly, as it was done in the case of a locally stationary covariance that is the product of a symmetric covariance and a stationary covariance, every normal covariance generates a normal operator in the Hilbert space of values due to the underlying random sequence. This normal operator is the shift-operator in the linear set of all linear combinations over the values of $\{Z(n)\}$, $n \in \mathbb{Z}$; in other words, $T^n Z(0) = Z(n)$ holds for every $n \in \mathbb{Z}$. By means of the Karhunen theorem every random sequence having a normal covariance can be expressed in the form of a stochastic integral understood in the quadratic mean

$$Z(n) = \iint_{-\infty}^{\infty} z^n \, \mathrm{dd}\xi(\lambda, \mu) , \quad z = \lambda + \mathrm{i}\mu$$

where $\xi(\cdot, \cdot)$ is a plane martingale. If a complex function $f(\cdot, \cdot)$ defined in the plane satisfies for every $n \in \mathbb{Z}$

$$\iint_{-\infty}^{\infty} |z|^{2n} \, |f(\lambda, \mu)|^2 \, \mathrm{dd}G(\lambda, \mu) < \infty ,$$

then the random sequence $\{Y(n) = \iint_{-\infty}^{\infty} z^n f(\lambda, \mu) \, \mathrm{dd}\xi(\lambda, \mu)\}$, $n \in \mathbb{Z}$ is again a random sequence having a normal covariance because

$$\mathsf{E}\{Y(n) \, \overline{Y(m)}\} = \iint_{-\infty}^{\infty} z^n \, \bar{z}^m |f(\lambda, \mu)|^2 \, \mathrm{dd}G(\lambda, \mu) .$$

2. NORMAL COVARIANCES AND RKHS

As familiarly known, for details see [6], every covariance function formes a kernel for the reproducing kernel Hilbert space (RKHS). Here we shall investigate the RKHS due to a normal covariance $R(\cdot, \cdot)$. Every covariance determines its RKHS unambiguously and vice versa. By definition the RKHS is the space of all complex sequences $\{m(n)\}$, $n \in \mathbb{Z}$, determined by

$$m(k) = \mathsf{E}\{\xi \overline{Z(k)}\} , \quad k \in \mathbb{Z} ,$$

where $\xi \in H(z(\cdot))$, $H(z(\cdot))$ is the Hilbert space generated by all finite linear combinations over the values of $\{Z(n)\}_{n=-\infty}^{\infty}$. We see immediately that $R(n, \cdot) \in$ RKHS for every $n \in \mathbb{Z}$ because

$$R(n, k) = \mathsf{E}\{Z(n) \, \overline{Z(k)}\} .$$

The property characterizing the RKHS is its "reproducing property"

$$m(n) = \langle m(\cdot); R(n, \cdot)\rangle , \quad n \in \mathbb{Z} ,$$

if we define a scalar product in the RKHS by the relation

$$\langle m_1(\cdot); m_2(\cdot)\rangle = \mathsf{E}\{\xi_1 \bar{\xi}_2\}$$

where $m_i(n) = \mathsf{E}\{\xi_i \, \overline{Z(n)}\}$, $i = 1, 2$. There exists an isometric and isomorphic mapping I between the RKHS and $H(z(\cdot))$ given by $m(\cdot) \longleftrightarrow \xi$. We have seen that every normal covariance is closely connected with a normal operator. This operator is defined in $H(z(\cdot))$ on an everywhere dense linear subset and is given by

$$T Z(n) = Z(n + 1) , \quad n \in \mathbb{Z} .$$

By means of I we can transform T into the RKHS by the relation

$$T m(k) = \mathsf{E}\{T\xi \, \overline{Z(k)}\}$$

where $\xi \in L(z(\cdot))$, i.e. $L(z(\cdot)) = \{\xi \colon \xi = \sum_{i=1}^{N} \alpha_i \, Z(n_i), n_i \in \mathbb{Z}, \alpha_i \text{ complex}\}$. Especially, we have

$$TR(n, k) = \mathsf{E}\{Z(n + 1) \, \overline{Z(k)}\} = R(n + 1, k) .$$

It means, the operator T in the RKHS is defined on the linear subset $L = \{m(\cdot) \in \in \text{RHKS}: m(\cdot) = \sum_{i=1}^{N} \alpha_i R(n_i, \cdot)\}$ which is also everywhere dense in the RKHS. The operator T in the RKHS has a normal enlargement because the operator $\mathcal{T}$ in $H(z(\cdot))$ is normal and there exists the mapping I. We know the covariance $R(\cdot, \cdot)$ of $\{Z(n)\}_{n=-\infty}^{\infty}$ can be written as

$$R(n, m) = \iint_{-\infty}^{\infty} z^n \bar{z}^m \, \mathrm{dd}G(\lambda, \mu)$$

hence $R(n, k) = \iint_{-\infty}^{\infty} z^n \bar{z}^k \, \mathrm{dd}G(\lambda, \mu)$. Thus for every $\xi \in L(z(\cdot))$, $\xi = \sum_{i=1}^{N} \alpha_i Z(n_i)$ for the corresponding $\xi = I\, m_\xi(\cdot)$

$$m_\xi(k) = \iint_{-\infty}^{\infty} \sum_{i=1}^{N} \alpha_i z^{n_i} \bar{z}^k \, \mathrm{dd}G(\lambda, \mu)\,.$$

This fact implies that every element $m(\cdot) \in$ RKHS can be expressed as

$$m(k) = \iint_{-\infty}^{\infty} f(z)\, \bar{z}^k \, \mathrm{dd}G(\lambda, \mu)\,,$$

where $f(z)$ belongs to the closure $\mathrm{cl}\,\{\sum_{i=1}^{N} \alpha_i z^{n_i}\}$ of all linear combinations $\sum_{i=1}^{N} \alpha_i z^{n_i}$, α_i complex, $n_i \in \mathbb{Z}$ with respect to the following norm $(\iint |f(z)|^2 \, \mathrm{dd}G(\lambda, \mu))^{1/2}$. In case when the system $\{z^n\}$, $n \in \mathbb{Z}$, is complete in the space $\mathscr{L}_2(\mathbb{R}_2, G(\cdot, \cdot))$ then the RKHS is isometric and isomorphic to the whole space $\mathscr{L}_2(\mathbb{R}_2, G(\cdot, \cdot))$. In every case, the RKHS is isometric and isomorphic to the subspace spanned by all functions of the type $\sum_{1}^{N} \alpha_i z^{n_i}$, α_i complex, $n_i \in \mathbb{Z}$. At the first sight the operator T in the RKHS can be easily described as

$$T m(k) = \iint_{-\infty}^{\infty} \sum_{i=1}^{N} \alpha_i \, z^{n_i+1} \, \bar{z}^k \, \mathrm{dd}G(\lambda, \mu)$$

if $\xi \in L(z(\cdot))$. In general,

$$T m(k) = \iint_{-\infty}^{\infty} f(z)\, z\bar{z}^k \, \mathrm{dd}G(\lambda, \mu)$$

if

$$m(k) = \iint_{-\infty}^{\infty} f(z)\, \bar{z}^k \, \mathrm{dd}G(\lambda, \mu)\,.$$

Theorem 4. The normal operator T in the space $H(z(\cdot))$ is bounded if and only if the corresponding operator T in the RKHS is closed with respect to multiplication by z, i.e. if $f(\cdot) \in \mathrm{cl}\,\{\sum_{i=1}^{N} \alpha_i z^{n_i}\}$ then $f(\cdot)\,.\,z \in \mathrm{cl}\,\{\sum_{i=1}^{N} \alpha_i z^{n_i}\}$ too.

Proof. Let T be bounded in $H(z(\cdot))$. Then T can be in the unique way enlarged onto the whole space $H(z(\cdot))$ and $TT^* = T^*T$. By means of the mapping I the operator T in the RKHS can be defined everywhere in the RKHS as a normal operator too. Denoting $TT^* = A$ we have a positive symmetric operator in the RKHS and

$\|T m(\cdot)\| = \|A^{1/2} m(\cdot)\|$ where $A^{1/2}$ is the square root of A. As

$$\|T m(\cdot)\|^2 = \iint_{-\infty}^{\infty} |f(z)|^2 \, |z|^2 \, \mathrm{d}\mathrm{d}G(\lambda, \mu)$$

we see that together with $f(\cdot) \in \mathrm{cl}\{\sum_{i=1}^{N} \alpha_i z^{n_i}\}$ $z f(\cdot) \in \mathrm{cl}\{\sum_1^N \alpha_i z^{n_i}\}$ also.

On the other hand, let with every function $f(\cdot) \in \mathrm{cl}\{\sum_1^N \alpha_i z^{n_i}\}$ $z f(\cdot)$ belong to $\mathrm{cl}\{\sum_{i=1}^{N} \alpha_i z^{n_i}\}$, i.e. if

$$\iint_{-\infty}^{+\infty} |f(z)|^2 \, \mathrm{d}\mathrm{d}G(\lambda, \mu) < \infty$$

then

$$\iint_{-\infty}^{\infty} |f(z)|^2 \, |z|^2 \, \mathrm{d}\mathrm{d}G(\lambda, \mu) < \infty .$$

It means that the operator T in the RKHS is defined everywhere and T is normal. The adjoint operator T^* is determined by multiplication by $\bar{z}$, i.e.

$$T^* \, m(k) = \iint_{-\infty}^{\infty} f(z) \, \bar{z}^{k+1} \, \mathrm{d}\mathrm{d}G(\lambda, \mu) , \quad k \in \mathbb{Z} .$$

The operator $T^*T = TT^*$ is then symmetric everywhere in the RKHS defined hence T^*T must be bounded. At the same moment T in the RKHS must be bounded too as

$$\|T m(\cdot)\| = \|A^{1/2} m(\cdot)\|$$

for every $m(\cdot) \in$ RKHS. The mapping I between the RKHS and $H(z(\cdot))$ gives immediately the boundedness of the operator T in $H(z(\cdot))$. □

3. CHARACTERIZATION OF NORMAL COVARIANCES

The main goal of this part is to describe the class of normal covariances. We have seen that this class if sufficiently large because every stationary covariance is normal and every symmetric covariance too. Their product is a normal covariance also.

Theorem 5. Let $R(\cdot, \cdot)$ be a covariance defined on $\mathbb{Z} \times \mathbb{Z}$. Then $R(\cdot, \cdot)$ is normal if and only if $R(\cdot, \cdot)$ satisfies the following "reproducing property" in its RKHS

$$R(n + 1, m) = \langle R(n, k); R(m, k + 1)\rangle .$$

Proof. First, let us suppose $R(\cdot, \cdot)$ is normal. Then $R(n, m) = \iint_{-\infty}^{\infty} z^n (\bar{z})^m \, . \, \mathrm{d}\mathrm{d}F(\lambda, \mu)$, $z = \lambda + \mathrm{i}\mu$. By use the transformation $\lambda + \mathrm{i}\mu \leftrightarrow |z| \, \mathrm{e}^{\mathrm{i}\varrho}$ we can express $R(\cdot, \cdot)$ in the form

$$R(n, m) = \int_0^{\infty} \int_{-\pi}^{\pi} |z|^{n+m} \, \mathrm{e}^{\mathrm{i}\varrho(n-m)} \, \mathrm{d}\mathrm{d}G(|z|, \varrho)$$

where the function $G(\cdot, \cdot)$ is induced by the mentioned transformation from the function $F(\cdot, \cdot)$. For every $n \in \mathbb{Z}$ $R(n, \cdot) \in$ RKHS and

$$R(n + 1, m) = \int_0^{\infty} \int_{-\pi}^{\pi} |z|^{1+n+m} \, \mathrm{e}^{\mathrm{i}\varrho(n+1-m)} \, \mathrm{d}\mathrm{d}G(|z|, \varrho) =$$
$$= \int_0^{\infty} \int_{-\pi}^{\pi} |z|^{n+m} \, \mathrm{e}^{\mathrm{i}\varrho(n-m)} \, |z| \, \mathrm{e}^{\mathrm{i}\varrho} \, \mathrm{d}\mathrm{d}G(|z|, \varrho) .$$

Similarly,

$$R(m, j + 1) = \int_0^\infty \int_{-\pi}^{\pi} |z|^{m+j+1} e^{i\varrho(m-j-1)} \, \mathrm{dd}G(|z|, \varrho) =$$
$$= \int_0^\infty \int_{-\pi}^{\pi} |z|^{m+j} e^{i\varrho(m-j)} |z| \, e^{-i\varrho} \, \mathrm{dd}G(|z|, \varrho) \, ,$$
$$R(n, j) = \int_0^\infty \int_{-\pi}^{\pi} |z|^{n+j} e^{i\varrho(n-j)} \, \mathrm{dd}G(|z|, \varrho) \, .$$

Now, by use of the "reproducing property" in the RKHS we have that

$$\langle R(n, j); R(m, j + 1)\rangle = \int_0^\infty \int_{-\pi}^{\pi} |z|^n e^{i\varrho n} \, . \, \overline{|z|^m e^{i\varrho m} \, . \, |z| \, e^{-i\varrho}} \, \mathrm{dd}G(|z|, \varrho) =$$
$$= \int_0^\infty \int_{-\pi}^{\pi} |z|^{n+m+1} \, . \, e^{i\varrho(n+1-m)} \, \mathrm{dd}G(|z|, \varrho) = R(n + 1, m) \, .$$

This form of a normal covariance shows that every normal covariance can be written by means of a function $S(\cdot, \cdot)$ defined on $\mathbb{Z} \times \mathbb{Z}$ as $R(n, m) = S(n + m, n - m)$, where

$$S(p, q) = \int_0^\infty \int_{-\pi}^{\pi} |z|^p e^{i\varrho q} \, \mathrm{dd}G(|z|, \varrho) = \int_0^\infty \int_{-\pi}^{\pi} e^{\omega p} e^{i\varrho q} \, \mathrm{dd}H(\omega, \varrho) \, .$$

Now, let us suppose that a given covariance function $R(\cdot, \cdot)$ satisfies

$$R(n + 1, m) = \langle R(n, j); R(m, j + 1)\rangle \, .$$

Let us define a shift operator T in the RKHS generated by the covariance $R(\cdot, \cdot)$

$$T R(n, \cdot) = R(n + 1, \cdot) \, .$$

We see that the operator T can be defined on the linear set L of all linear combinations of the form $\sum_{i=1}^{N} \alpha_i R(n_i, \cdot) \in \text{RKHS}$ which form an everywhere dense subset in the RKHS

$$T \sum_{i=1}^{N} \alpha_i R(n_i, \cdot) = \sum_{i=1}^{N} \alpha_i R(n_i + 1, \cdot) \, ;$$

in other words, the definition domain $\mathscr{D}(T)$ of T is equal to L. For us the adjoint operator T^* to T will be very important. Let us prove that $\mathscr{D}(T^*) \supset L$. As T^* is linear it is sufficient to prove that $R(m, \cdot) \in \mathscr{D}(T^*)$ for every $m \in \mathbb{Z}$. By the definition of T^* $R(m, \cdot) \in \mathscr{D}(T^*)$ if and only if

$$\langle T m(\cdot); R(m, \cdot)\rangle = \langle m(\cdot); T^* R(m, \cdot)\rangle$$

holds for every $m(\cdot) \in \mathscr{D}(T)$ and $T^* R(n, \cdot)$ must be defined unambiguously. If $m(\cdot) = \sum_{i=1}^{N} \alpha_i R(n_i, \cdot)$, then

$$\langle T m(\cdot); R(m, \cdot)\rangle = \sum_{i=1}^{N} \alpha_i \langle R(n_i + 1, \cdot); R(m, \cdot)\rangle =$$
$$= \sum_{i=1}^{N} \alpha_i \langle R(n_i, \cdot); R(m, (\cdot) + 1)\rangle = \langle m(\cdot); R(m, (\cdot) + 1)\rangle \, .$$

We have used the assumption of Theorem 5. This fact implies that the operator T^* is a shift-operator in the argument in the RKHS, i.e.

$$T^* R(m, \cdot) = R(m, (\cdot) + 1) \, .$$

and $\mathscr{D}(T^*) \supset L$. Thus the definition domain of T^* is everywhere dense in the RKHS and hence there exists a closed enlargement $\bar{T}$ of T. First, we shall prove that $TT^* m(\cdot) = T^*T m(\cdot)$ for every $m(\cdot) \in L$.

The set L is invariant with respect to T because if $m(\cdot) = \sum_{i=1}^{N} \alpha_i R(n_i, (\cdot))$ then $T m(\cdot) = \sum_{i=1}^{N} \alpha_i R(n_i + 1, (\cdot)) \in L$ as well. We must prove that $T^* m(\cdot)$ belongs to L also. T^* is linear, $T^* m(\cdot) = T^* \sum_{i=1}^{N} \alpha_i R(n_i, (\cdot)) = \sum_{i=1}^{N} \alpha_i R(n_i, (\cdot) + 1)$ and as we see one can apply the operator T at this moment

$$TT^* m(\cdot) = \sum_{i=1}^{N} \alpha_i R(n_i + 1, (\cdot) + 1).$$

On the other hand,

$$T^*T m(\cdot) = T^* \sum_{i=1}^{N} \alpha_i R(n_i + 1, (\cdot)) = \sum_{i=1}^{N} \alpha_i R(n_i + 1, (\cdot) + 1).$$

We have proved that $TT^* = T^*T$ on the subset L. Further, we obtain that $\|T m(\cdot)\| = \|T^* m(\cdot)\|$ for every $m(\cdot) \in L$ because for every pair $m_1(\cdot), m_2(\cdot) \in L$ the following equality

$$\langle T^*T m_1(\cdot); m_2(\cdot)\rangle = \langle T m_1(\cdot); T m_2(\cdot)\rangle =$$
$$= \langle m_1(\cdot); T^*T m_2(\cdot)\rangle = \langle m_1(\cdot); TT^* m_2(\cdot)\rangle = \langle T^* m_1(\cdot), T^* m_2(\cdot)\rangle$$

holds. The operator T^* is closed, i.e. when $m_n(\cdot) \to m(\cdot)$ and $T^* m_n(\cdot) \to M(\cdot)$ in the RKHS then $m(\cdot) \in \mathscr{D}(T^*)$ and $T^* m(\cdot) = M(\cdot)$. Thus

$$\|T^* m_n(\cdot) - T^* m_p(\cdot)\| = \|T^*(m_n(\cdot) - m_p(\cdot))\| =$$
$$= \|T(m_n(\cdot) - m_p(\cdot))\| \to 0 \quad \text{as} \quad n, p \to \infty$$

in every case when $m_n(\cdot) \in L$. This fact implies a possibility how to construct a closure of T. Let us define a new operator $\bar{T}$ in the RKHS, which will be an enlargement of T, $T \subset \bar{T}$, by the following procedure:

$$\mathscr{D}(\bar{T}) = \{m(\cdot) \in \text{RKHS}: \exists\{m_n(\cdot)\} \subset L, m_n(\cdot) \to m(\cdot), T m_n(\cdot) \to t(\cdot) \in \text{RKHS}\}.$$

Then we put $\bar{T} m(\cdot) = t(\cdot)$. It is evident that $L \subset \mathscr{D}(\bar{T})$, $T \subset \bar{T}$ and let us prove that $\bar{T}$ is defined in the unique way. Suppose $m_n(\cdot) \to m(\cdot)$, $p_n(\cdot) \to m(\cdot)$, $T m_n(\cdot) \to t(\cdot)$, $T p_n(\cdot) \to s(\cdot)$. Then

$$\langle m(\cdot); t(\cdot) - s(\cdot)\rangle = \lim_{n\to\infty} \langle m(\cdot); T(m_n(\cdot) - p_n(\cdot))\rangle =$$
$$= \lim_{n\to\infty} \langle T^* m(\cdot); m_n(\cdot) - p_n(\cdot)\rangle = \langle T^* m; 0\rangle = 0$$

for every $m(\cdot) \in \mathscr{D}(T^*)$. As $\overline{\mathscr{D}(T^*)} = \text{RKHS}$, $t(\cdot) = s(\cdot)$ must be and $\bar{T}$ is determined unambiguously. Further, we shall prove that $(\bar{T})^* = T^*$ and $\mathscr{D}(\bar{T}) = \mathscr{D}(T^*)$. As $T \subset \bar{T}$ it implies immediately $(\bar{T})^* \subset T^*$. Let $m(\cdot) \in \mathscr{D}(T^*)$, $p(\cdot) \in \mathscr{D}(\bar{T})$. Then

$$\langle \bar{T} p(\cdot); m(\cdot)\rangle = \lim_{n\to\infty} \langle T p_n(\cdot); m(\cdot)\rangle = \lim_{n\to\infty} \langle p_n(\cdot), T^* m(\cdot)\rangle = \langle p(\cdot), T^* m(\cdot)\rangle.$$

We see $m(\cdot) \in \mathscr{D}(\bar{T})^*$ and $T^* m(\cdot) = (\bar{T})^* m(\cdot)$, hence $T^* \subset (\bar{T})^*$ and the equality $T^* = (\bar{T})^*$ is proved. Now let $m(\cdot) \in \mathscr{D}(\bar{T})$. It means there exists a sequence $\{m_n(\cdot)\} \subset L$ such that $m_n(\cdot) \to m(\cdot)$, $T m_n(\cdot) \to \bar{T} m(\cdot)$, i.e. $\|T m_n(\cdot) - T m_p(\cdot)\| = \|T^* m_n(\cdot) - T^* m_p(\cdot)\| \to 0$ as $n, p \to \infty$. Hence $T^* m_n(\cdot) \to t^*(\cdot)$ and as T^* is closed and $L \subset \mathscr{D}(T^*)$, $m(\cdot) \in \mathscr{D}(T^*)$, $T^* m(\cdot) = t^*(\cdot)$. We have proved $\mathscr{D}(\bar{T}) \subset \mathscr{D}(T^*)$. Similarly, if $m(\cdot) \in \mathscr{D}(T^*)$ there exists a sequence $\{m_n(\cdot)\} \subset L$, $m_n(\cdot) \to m(\cdot)$ and $T^* m_n(\cdot) \to t^*(\cdot)$. At the same moment $\|T m_n(\cdot) - T m_p(\cdot)\| = \|T^* m_n(\cdot) - T^* m_p(\cdot)\| \to 0$ as $n, p \to \infty$, hence $\{T m_n(\cdot)\}$ is a Cauchy sequence in a complete space that implies the existence of a limit $t(\cdot) \in \text{RKHS}$, $t(\cdot) = \lim_{n \to \infty} T m_n(\cdot)$ and thanks to the closeness of $\bar{T}$ we can state that $\mathscr{D}(T^*) \subset \mathscr{D}(\bar{T})$. We have proved that $\mathscr{D}(\bar{T}) = \mathscr{D}(T^*)$. We know that $\|T m(\cdot)\| = \|T^* m(\cdot)\|$ for every $m(\cdot) \in L$. This fact together with the closeness of $\bar{T}$ and T^* imply that for every $m(\cdot) \in \mathscr{D}(\bar{T})$ $\|\bar{T} m(\cdot)\| = \|T^* m(\cdot)\|$ Now, there is no problem to prove that $\bar{T}$ is a normal operator in the RKHS, i.e. $\bar{T}$ must be defined on an everywhere dense linear subset, $\bar{T}$ must be closed satisfying $\bar{T}(\bar{T})^* = (\bar{T})^* \bar{T}$. Suppose $m(\cdot) \in \mathscr{D}(\bar{T}T^*)$. It means $T^* m(\cdot) \in \mathscr{D}(\bar{T})$ and in every case $\mathscr{D}(\bar{T}T^*) \subset \mathscr{D}(T^*) = \mathscr{D}(\bar{T})$. We can consider $\bar{T} m(\cdot)$ and to prove $\bar{T} m(\cdot) \in \mathscr{D}(T^*)$ we must investigate $\langle \bar{T} p(\cdot); \bar{T} m(\cdot)\rangle$ for every $p(\cdot) \in \mathscr{D}(\bar{T})$. Thus $\langle \bar{T} p(\cdot); \bar{T} m(\cdot)\rangle = \langle T^* p(\cdot); T^* m(\cdot)\rangle = \langle p(\cdot); \bar{T}T^* m(\cdot)\rangle$ and as immediately follows $\langle \bar{T} p(\cdot); \bar{T} m(\cdot)\rangle$ is a continuous linear functional on $\mathscr{D}(\bar{T})$ and it implies $\bar{T} m(\cdot) \in \mathscr{D}(T^*)$. In the opposite direction, let $m(\cdot) \in \mathscr{D}(T^*\bar{T})$ that means $m(\cdot) \in \mathscr{D}(\bar{T})$ with $\bar{T} m(\cdot) \in \mathscr{D}(T^*)$. We have $\mathscr{D}(T^*\bar{T}) \subset \mathscr{D}(\bar{T}) = \mathscr{D}(T^*)$ and we can consider $T^* m(\cdot)$. As $T^{**} = \bar{T}$ ($\bar{T}$ is closed) $T^* m(\cdot)$ belongs to $\mathscr{D}(\bar{T})$ if and only if $T^* m(\cdot)$ belongs to $\mathscr{D}(T^{**})$, i.e. $\langle T^* p(\cdot); T^* m(\cdot)\rangle$ must be a continuous linear functional on $\mathscr{D}(T^*)$. At the first sight $\langle T^* p(\cdot); T^* m(\cdot)\rangle = \langle p(\cdot); \bar{T}T^* m(\cdot)\rangle$ and hence $T^* m(\cdot) \in \mathscr{D}(\bar{T}) = \mathscr{D}(T^{**})$. We have proved that $\mathscr{D}(\bar{T}T^*) = \mathscr{D}(T^*\bar{T})$. We proved sooner that $TT^* = T^*T$ on the linear subset L; because L is everywhere dense in $\mathscr{D}(\bar{T}T^*)$ too the equality $\bar{T}T^* = T^*\bar{T}$ must hold on the whole definition domain $\mathscr{D}(\bar{T}T^*)$. We proved that $\bar{T}$ is a normal operator in the RKHS. As follows from the general theory of unbounded operators in a Hilbert space every normal operator possesses a spectral resolution, in our case

$$\bar{T} = \iint_{-\infty}^{\infty} z \, \mathrm{d}P_z$$

where $\{P_z\}$, $z \in \mathbb{C}$, is a complex resolution of the identity in the RKHS. The adjoint operator T^* is then expressed in the form

$$T^* = \iint_{-\infty}^{\infty} \bar{z} \, \mathrm{d}P_z$$

and their common definition domain $\mathscr{D}(\bar{T}) = \mathscr{D}(T^*)$ is the linear subset in the RKHS of all $m(\cdot) \in \text{RKHS}$ for which

$$\iint_{-\infty}^{\infty} |z| \, \mathrm{d}\langle P_z \, m(\cdot), m(\cdot)\rangle$$

exists. We defined the operator $\bar{T}$ as a shift-operator in L, i.e. $\bar{T} R(n, \cdot) = R(n + 1, \cdot)$. This gives $\bar{T}^n R(0, \cdot) = R(n, \cdot)$ and this property holds for every $n \in \mathbb{Z}$. Especially, we see that the operator $\bar{T}$ has an inverse operator T^{-1} on $L(\bar{T} = T$ on $L)$ and

$$\bar{T}^{-1} m(\cdot) = T^{-1} m(\cdot) = (\iint_{-\infty}^{\infty} z \, dP_z)^{-1} m(\cdot) = \iint_{-\infty}^{\infty} z^{-1} \, dP_z \, m(.)$$

holds for every $m(\cdot) \in L$. In this way we obtained that

$$R(n, (\cdot)) = \iint_{-\infty}^{\infty} z^n \, dP_z \, R(0, (\cdot))$$

holds for every $n \in \mathbb{Z}$. The property of the complex resolution of the identity

$$P_{z_1} P_{z_2} = P_{z_2} P_{z_1} = P_{\min(z_1, z_2)}$$

$(\min(z_1, z_2) = \min(\operatorname{Re} z_1, \operatorname{Re} z_2), \min(\operatorname{Im} z_1, \operatorname{Im} z_2))$ yields a possibility to write the covariance function $R(\cdot, \cdot)$ in the following form

$$R(n, m) = \iint_{-\infty}^{\infty} z^n (\bar{z})^m \, d\langle P_z R(0, \cdot); R(0, \cdot)\rangle =$$
$$= \iint_{-\infty}^{\infty} z^n (\bar{z})^m \, ddF(\lambda, \gamma), \quad z = \lambda + i\mu$$

where $F(\cdot, \cdot)$ is a two-dimensional distribution function with a finite variation equal to $R(0, 0)$. As the relation between a normal operator and its resolution of the identity is a one-to-one correspondence the function $F(\cdot, \cdot)$ is determined by the covariance function $R(\cdot, \cdot)$ unambiguously. □

Another characterization of normal covariances is possible by means of the notion of nonnegative definiteness.

Theorem 6. A covariance function $R(\cdot, \cdot)$ defined on $\mathbb{Z} \times \mathbb{Z}$ is normal if and only if $R(\cdot, \cdot)$ can be expressed in the form

$$R(n, m) = S(n + m, n - m)$$

where the function $S(\cdot, \cdot)$, which is defined on $\mathbb{Z} \times \mathbb{Z}$, is nonnegative definite in the following sense:

$$\sum_n \sum_m \sum_p \sum_q \alpha_{nm} \bar{\alpha}_{pq} S(n + p, m - q) \geqq 0$$

for every finite collection $\{\alpha_{nm}\}$ of complex numbers, $n \in \mathbb{Z}$, $m \in \mathbb{Z}$.

Proof. Let $R(\cdot, \cdot)$ be normal. Then as shown in the proof of Theorem 5 $R(\cdot, \cdot)$ can be expressed as

$$R(n, m) = \int_{-\infty}^{\infty} \int_{-\pi}^{\pi} e^{x(n+m)} \, e^{iy(n-m)} \, ddH(x, y)$$

and it implies the existence of a function $S(\cdot, \cdot)$ defined on $\mathbb{Z} \times \mathbb{Z}$ such that

$$R(n, m) = S(n + m, n - m) \, .$$

Further, let $\{\alpha_{nm}\}$ be any finite collection of complex numbers. Then

$$\sum_n \sum_m \sum_p \sum_q \alpha_{nm} \bar{\alpha}_{pq} S(n + p, m - q) =$$
$$= \sum_n \sum_m \sum_p \sum_q \alpha_{nm} \bar{\alpha}_{pq} \int_{-\infty}^{\infty} \int_{-\pi}^{\pi} e^{x(n+p)} \, e^{iy(m-q)} \, ddH(x, y) =$$
$$= \int_{-\infty}^{\infty} \int_{-\pi}^{\pi} \sum_n \sum_m \sum_p \sum_q \alpha_{nm} \bar{\alpha}_{pq} \, e^{xn} \, e^{iym} \, \overline{e^{xp} \, e^{iyq}} \, ddH(x, y) =$$
$$= \int_{-\infty}^{\infty} \int_{-\pi}^{\pi} \left|\sum_n \sum_m \alpha_{nm} \, e^{xn} \, e^{iym}\right|^2 ddH(x, y) \geqq 0 \, .$$

Let, on the contrary, there exist a function $S(\cdot, \cdot)$ on $\mathbb{Z} \times \mathbb{Z}$ nonnegative definite in the mentioned sense such that $R(n, m) = S(n + m, n - m)$. Let L be the linear set of all complex valued functions $f(\cdot, \cdot)$ defined on $\mathbb{Z} \times \mathbb{Z}$ such that $f(u, v) = 0$ except a finite subset in $\mathbb{Z} \times \mathbb{Z}$. If $f(\cdot, \cdot), g(\cdot, \cdot) \in L$ then we shall define

$$\langle f; g \rangle = \sum_n \sum_m \sum_p \sum_q f(n, m) \overline{g(p, q)} S(n + p, m - q).$$

Then $\langle f; g \rangle$ is an Hermite bilinear form and according to assumption $\langle f; f \rangle^2 \geqq 0$. Instead of the original functions we shall consider classes of equivalence

$$f \sim g \Leftrightarrow \|f - g\| = 0,$$

in other words, we shall consider the factor space L/N_0, where $N_0 = \{f \in L: \|f\| = 0\}$. Let H be a completion of L/N_0 with respect to the norm $\|\cdot\|$. Then H is a Hilbert space. In a simple way the bilinear form $\langle \cdot; \cdot \rangle$ can be translated from L into H. Now, let T be a shift operator defined on L by the relation

$$Tf(u, v) = f(u - 1, v - 1).$$

T is well defined because if $\|f\| = 0$ then $\|Tf\| = 0$ also. Let us put $Sf(u, v) = f(u - 1, v + 1)$ and let us prove that $\|Tf\|^2 = \langle Tf; Tf \rangle = \langle STf; f \rangle$. By definition

$$\begin{aligned} \|Tf\|^2 &= \sum_n \sum_m \sum_p \sum_q Tf(n, m) \overline{Tf(p, q)} S(n + p, m - q) = \\ &= \sum_n \sum_m \sum_p \sum_q f(n - 1, m - 1) \overline{f(p - 1, q - 1)} S(n + p, m - q) = \\ &= \sum_n \sum_m \sum_p \sum_q f(n, m) S(n + p + 2, m - q) \overline{f(p, q)} = \\ &= \sum_n \sum_m \sum_p \sum_q f(n - 2, m) \overline{f(p, q)} S(n + p, m - q) = \langle STf; f \rangle \end{aligned}$$

because $STf(u, v) = Sf(u - 1, v - 1) = f(u - 2, v)$, It means that $\|Tf\|^2 = \langle STf; f \rangle \leqq \|STf\| \|f\|$ and if $\|f\| = 0$ then $\|Tf\| = 0$ also. Thanks to this property the operator T can be translated into space H with the definition domain $\mathscr{D}(T) = L/N_0$. The operator T is defined in H on an everywhere dense linear subset. In a similar way one can prove that the operator S is also well defined and can be translated into the Hilbert space H. Let us show that $S \subset T^*$, where T^* is the adjoint operator to T in H. Let $f \in L/N_0$, $g \in L/N_0$. Then

$$\begin{aligned} \langle Tf; g \rangle &= \sum_n \sum_m \sum_p \sum_q f(n - 1, m - 1) \overline{g(p, q)} S(n + p, m - q) = \\ &= \sum_n \sum_m \sum_p \sum_q f(n, m) \overline{g(p, q)} S(n + 1 + p, m + 1 - q) = \\ &= \sum_n \sum_m \sum_p \sum_q f(n, m) \overline{g(p - 1, q + 1)} S(n + p, m - q) = \langle f; Sg \rangle. \end{aligned}$$

It means that $S = T^*$ on L/N_0 in H and hence $\mathscr{D}(T^*)$ is everywhere dense in H. At this moment the operator T has a closed enlargement $\overline{T}$ in H, $\overline{T}$ is unambiguously defined. There is no problem to prove (following the proof of the previous Theorem 5) that the operator $\bar{T}$ is normal in H, i.e. $\bar{T}$ is closed, $\bar{T}\bar{T}^* = \bar{T}^*\bar{T}$ and $\mathscr{D}(\bar{T})$ is an everywhere dense subset in H. For every normal operator there exists a complex resolution of the identity in H and such an operator can be expressed as

$$\bar{T} = \iint_{-\infty}^{\infty} z \, \mathrm{d}P_z\,, \quad z = \lambda + \mathrm{i}\mu\,.$$

Let $\delta(u, v)$ be the element in H with $\delta(0, 0) = 1$ and $\delta(u, v) = 0$ otherwise. Then $\delta(u, v) \in L/N_0$ and

$$\langle \bar{T}\,\delta(u, v); \delta(u, v)\rangle = \sum_n \sum_m \sum_p \sum_q \delta(n - 1, m - 1)\, \delta(p, q)\, S(n + p, m - q) =$$
$$= S(1, 1) = R(1, 0)\,.$$

Similarly, for every $s \in \mathbb{Z}$, $t \in \mathbb{Z}\langle \bar{T}^s\,\delta(u, v); \bar{T}^t(u, v)\rangle = S(s + t, s - t) = R(s, t)$. By means of properties of the complex resolution $\{P_z\}, z \in \mathbb{C}$ we immediately obtain that

$$R(s, t) = \iint_{-\infty}^{\infty} z^s(\bar{z})^t \, \mathrm{dd}F(\lambda, \mu)\,,$$

where $F(\lambda, \mu) = \langle P_z\, \delta(u, v); \delta(u, v)\rangle$. We have proved that the covariance $R(\cdot, \cdot)$ is normal. □

It is worth mentioning that Theorem 6 as special cases contains the Hamburger moment problem and the Herglotz lemma. The Hamburger moment problem is very closely connected with characterization of symmetric covariance functions, the Herglotz lemma describes weakly stationary covariances. In this sense Theorem 6 is a generalization of these both cases.

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