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UNIFICATION OF THE ABSTRACT DUALITY SCHEME

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The paper shows that the abstract duality scheme introduced in Tran Quoc Chien [6] is powerful enough that every duality model can be involved in it.

0. INTRODUCTION

Undoubtedly, the duality theory is the most delicate subject in optimization theory. Many efforts have been devoted to it (see the references in [5]) because of the beauty of symmetry, the economic significance and the numerical applications due to the duality concept. However, the more authors study this subject the more versions of duality are suggested, and one can hardly orient himself in the ocean of literature. In Tran Quoc Chien [1]–[7] the author has attempted to unify some duality models in one unique scheme, namely the abstract duality scheme.

The hypothesis that the abstract duality scheme is general enough to involve all duality models is treated in the present paper and the answer is, fortunately, positive.

1. PRELIMINARIES

In this section we recall (see [6]) some notions and notations which will be used in the paper.

Throughout this work we suppose that Y is a real ordered linear space and the positive cone Y_+ of Y has a nonempty core (cor $Y_+ \neq \emptyset$).

For elements $a, b \in Y$ we denote

$$a \geqq b \quad \text{iff} \quad a - b \in Y_+$$
$$a > b \quad \text{iff} \quad a - b \in Y_{++} = Y_+ \setminus \{0\}$$
$$a \gg b \quad \text{iff} \quad a - b \in \operatorname{cor} Y_+$$
$$a \ngtr b \quad \text{iff} \quad a \nless b .$$

For subsets A and B in Y we define

$$A \succ B \quad \text{iff} \quad \forall a \in A \ \forall b \in B \colon a \succ b$$

where $\succ$ may be any relation of $\geqq$, $>$, $\gg$ and $\gneqq$.

A nonempty set $\Omega \subset Y$ is called a *Y_+-quasiinterval* if

$$\Omega = (\Omega - Y_+) \cap (\Omega + Y_+) .$$

$\Omega \subset Y$ is called a *Y_+-interval* if there exist $\alpha, \beta \in Y \cup \{-\infty, \infty\}$ such that $\Omega = \{y \in Y \mid \alpha R_1 y \ \& \ y R_2 \beta\}$, where R_1 and R_2 may be any relation of $\leqq$, $<$ and $\ll$. α resp. β is called the lower bound resp. upper bound of Ω and indicated by $\inf \Omega$ resp. $\sup \Omega$. Clearly, a Y_+-interval is a Y_+-quasiinterval.

Given a subset A and a Y_{++}-quasiinterval Ω in Y, an element $a \in \Omega$ is called a *supremal* of A with respect to Ω if $a \in \operatorname{lin}(A - Y_+)$ and $(a + Y_{++}) \cap \Omega \cap \operatorname{lin}(A - Y_+) = \emptyset$. A point $a \in A$ is called a *maximum* of A if $A \not\gneqq a$. The set of all supremals with respect to Ω resp. all maxima of A are denoted by $\operatorname{Sup}_\Omega A$ resp. $\operatorname{Max} A$. Analogously, an *infimal* with respect to Ω, a *minimum*, $\operatorname{Min} A$ and $\operatorname{Inf}_\Omega A$ are defined. If $\Omega = Y$ then the letter Ω is omitted.

2. ABSTRACT DUALITY SCHEME

In the sequel suppose that Ω is a Y_+-quasiinterval in Y and $\mathscr{P}$ and $\mathscr{D}$ are arbitrary fixed sets. Further, suppose that $P \colon \Omega \to \mathscr{P}$ and $D \colon \Omega \to \mathscr{D}$ are multivalued maps fulfilling the following conditions:

(i) primal availability:

$$y_1 < y_2 \Rightarrow P(y_1) \supset P(y_2) .$$

(ii) dual availability:

$$y_1 < y_2 \Rightarrow D(y_1) \subset D(y_2) .$$

Put

$$\mathscr{P}_0 = \bigcup_{y \in \Omega} P(y) , \quad \mathscr{D}_0 = \bigcup_{y \in \Omega} D(y)$$

and

$$\mu(p) = \{y \in \Omega \mid p \in P(y)\} \quad p \in \mathscr{P}_0$$
$$\nu(d) = \{y \in \Omega \mid d \in D(y)\} \quad d \in \mathscr{D}_0 .$$

Problems

(AP) $$\operatorname{Max-Sup}_\Omega \mu(\mathscr{P}_0)$$

resp.

(AD) $$\operatorname{Min-Inf}_\Omega \nu(\mathscr{D}_0)$$

are called the *abstract primal* resp. *abstract dual*.

A point $p^* \in \mathscr{P}_0$ is called a *Max-optimal* resp. a *Sup_Ω-optimal solution* of problem (AP) if

$$\mu(p^*) \cap \operatorname{Max} \mu(\mathscr{P}_0) \neq \emptyset , \quad \text{resp.} \quad \mu(p^*) \cap \operatorname{Sup}_\Omega \mu(\mathscr{P}_0) \neq \emptyset .$$

A point $d^* \in \mathscr{D}_0$ is called a *Min-optimal* resp. an *Inf$_\Omega$-optimal solution* of (AD) if

$$\nu(d^*) \cap \operatorname{Min} \nu(\mathscr{D}_0) \neq \emptyset\,, \quad \text{resp.} \quad \nu(d^*) \cap \operatorname{Inf}_\Omega \nu(\mathscr{D}_0) \neq \emptyset\,.$$

2.1. Definition. Problems (AP) and (AD) are said to satisfy the *weak duality* resp. *Sup-Inf strong duality* if

$$\mu(\mathscr{P}_0) \mathrel{\overline{\gtrless}} \nu(\mathscr{D}_0)\,, \quad \text{resp.} \quad \operatorname{cor}(\Omega) \cap \operatorname{Sup}_\Omega \mu(\mathscr{P}_0) = \operatorname{cor}(\Omega) \cap \operatorname{Inf}_\Omega \nu(\mathscr{D}_0)\,.$$

2.2. Theorem (weak duality). The weak duality holds if and only if the condition (iii) weak duality condition:

$$D(y) \neq \emptyset \Rightarrow P(y') = \emptyset \quad \forall y' > y$$

holds.

Proof. Let (iii) hold. Then for $y' \in \mu(\mathscr{P}_0)$ and $y'' \in \nu(\mathscr{D}_0)$ we have $P(y') \neq \emptyset$ and $D(y'') \neq \emptyset$ and by (iii) $y' \mathrel{\overline{\gtrless}} y''$. Now suppose (iii) does not hold. There exist y', y'' such that $y' > y''$ and $D(y'') \neq \emptyset$ and $P(y') \neq \emptyset$. Hence $y' \in \mu(\mathscr{P}_0)$ and $y'' \in \nu(\mathscr{D}_0)$ and the weak duality is not satisfied. □

2.3. Theorem (Sup-Inf strong duality). Suppose that $Y_{++} = \operatorname{cor} Y_+$ and (iii) is valid. Then, if the following condition
(iv) Sup-Inf strong duality condition:

$$\forall y' \in \operatorname{cor} \Omega (P(y) = \emptyset \ \forall y \in \Omega \ (y > y') \Rightarrow D(y) \neq \emptyset \ \forall y \in \Omega \ (y > y'))$$

holds, then the Sup-Inf strong duality holds.

If, in addition, Ω is a Y_+-interval, then condition (iv) is necessary for Sup-Inf strong duality to be satisfied.

Proof. The first part of the theorem is Theorem 2.5 in [6]. So, let Ω be a Y_+-interval. If

$$\operatorname{cor}(\Omega) \cap \operatorname{Sup}_\Omega \mu(\mathscr{P}_0) = \operatorname{cor}(\Omega) \cap \operatorname{Inf}_\Omega \nu(\mathscr{D}_0) = \emptyset$$

there is nothing to prove. So suppose there exists

$$y_0 \in \operatorname{cor}(\Omega) \cap \operatorname{Sup}_\Omega \mu(\mathscr{P}_0) = \operatorname{cor}(\Omega) \cap \operatorname{Inf}_\Omega \nu(\mathscr{D}_0)\,.$$

Let $y' \in \operatorname{cor}(\Omega)$ be such that

$$P(y) = \emptyset \quad \forall y \in \Omega \ (y > y')\,.$$

Since Ω is a Y_+-interval one can choose $\bar{y} \in \Omega$ with $\bar{y} < y'$ and $\bar{y} < y_0$. It is easy to verify that there exists a point y_1 of the segment $[\bar{y}, y']$ which belongs to $\operatorname{cor}(\Omega) \cap \operatorname{Sup}_\Omega \mu(\mathscr{P}_0)$. By the Sup-Inf strong duality $y_1 \in \operatorname{cor}(\Omega) \cap \operatorname{Inf}_\Omega \nu(\mathscr{D}_0)$ that means, by definition, $D(y) \neq \emptyset \ \forall y \in \Omega \ (y > y_1)$. Hence we have

$$D(y) \neq \emptyset \quad \forall y \in \Omega \quad (y > y')\,. \qquad \square$$

3. UNIFICATION THEOREM

We shall introduce a symmetric duality model in the most general manner as follows: Fix a Y_+-quasiinterval $\Omega \subset Y$. Let X be a nonempty set and $F: X \to Y$ be a multivalued map. The problem

(P) $$\text{Max-Sup}_\Omega F(X)$$

is called primal.

Suppose that problem (P) has a dual problem of the following form

(D) $$\text{Min-Inf}_\Omega G(Z),$$

where Z is a nonempty set and $G: Z \to Y$ is a multivalued map.

3.1. Lemma (cf. [6] Lemma 1.5).

(i) $\text{Max}\, A = \text{Max}\,(A - Y_+)$, $\text{Min}\, A = \text{Min}\,(A + Y_+)$

(ii) $\text{Sup}_\Omega A = \text{Sup}_\Omega(\text{lin}\, A) = \text{Sup}_\Omega(A - \text{lin}\, Y_+)$

(iii) $\text{Inf}_\Omega A = \text{Inf}_\Omega(\text{lin}\, A) = \text{Inf}_\Omega(A + \text{lin}\, Y_+)$.

3.2. Lemma. Let $A \subset \Omega$ then

(i) $\Omega \cap (A - Y_+) - Y_+ = A - Y_+$,

(ii) $\Omega \cap (A + Y_+) + Y_+ = A + Y_+$,

(iii) $\text{Max}\, A = \text{Max}\,(\Omega \cap (A - Y_+))$,

(iv) $\text{Sup}_\Omega A = \text{Sup}_\Omega(\Omega \cap (A - Y_+))$,

(v) $\text{Min}\, A = \text{Min}\,(\Omega \cap (A + Y_+))$,

(vi) $\text{Inf}_\Omega A = \text{Inf}_\Omega(\Omega \cap (A + Y_+))$.

Proof. (i) $A \subset \Omega \cap (A - Y_+) \Rightarrow A - Y_+ \subset \Omega \cap (A - Y_+) - Y_+$. Conversely, $\Omega \cap (A - Y_+) \subset A - Y_+ \Rightarrow \Omega \cap (A - Y_+) - Y_+ \subset A - Y_+ - Y_+ = A - Y_+$.

The proof of equality (ii) is similar. Combining (i), (ii) and Lemma 3.1 (i) we obtain (iii) and (v).

Further, since $A \subset A - Y_+ \subset A - \text{lin}\, Y_+$ we have, considering Lemma 3.1 (ii), $\text{Sup}_\Omega A = \text{Sup}_\Omega(A - Y_+)$. Now using (i) we obtain (iv). (vi) can be proved analogously. □

3.3. Theorem (unification theorem). Suppose that $F(X) \subset \Omega$ and $G(Z) \subset \Omega$. There exists a pair of abstract primal (AP) and abstract dual (AD) that is equivalent to the dual pair (P) and (D) in the sense:

(i) $\text{Max}\, \mu(\mathcal{P}_0) = \text{Max}\, F(X)$
$\text{Sup}_\Omega\, \mu(\mathcal{P}_0) = \text{Sup}_\Omega F(X)$
$\text{Min}\, \nu(\mathcal{D}_0) = \text{Min}\, G(Z)$
$\text{Inf}_\Omega\, \nu(\mathcal{D}_0) = \text{Inf}_\Omega G(Z)$

(ii) The weak duality resp. Sup-Inf strong duality are valid simultaneously for the pairs (AP), (AD) and (P), (D).

Proof. Put

$$\mathcal{P} = X, \quad \mathcal{D} = Z$$

$$P(y) = \{x \in X \mid F(x) \geqq y\} \quad y \in \Omega$$
$$D(y) = \{z \in Z \mid G(z) \leqq y\} \quad y \in \Omega .$$

Obviously, the maps $P(y)$ resp. $D(y)$ satisfy the primal availability resp. dual availability.

Following Section 2 set

$$\mathscr{P}_0 = \bigcup_{y \in \Omega} P(y), \quad \mathscr{D}_0 = \bigcup_{y \in \Omega} D(y),$$
$$\mu(p) = \{y \in \Omega \mid p \in P(y)\} \quad p \in \mathscr{P}_0$$

and

$$\nu(d) = \{y \in \Omega \mid d \in D(y)\} \quad d \in \mathscr{D}_0 .$$

We obtain then the following abstract primal

(AP) $\qquad \text{Max-Sup}_\Omega\, \mu(\mathscr{P}_0)$

and abstract dual

(AD) $\qquad \text{Min-Inf}_\Omega\, \nu(\mathscr{D}_0) .$

It is easy to verify the following equalities

$$\mathscr{P}_0 = X , \quad \mathscr{D}_0 = Z$$
$$\mu(\mathscr{P}_0) = \Omega \cap (F(x) - Y_+)$$
$$\nu(\mathscr{D}_0) = \Omega \cap (G(Z) + Y_+) .$$

Now using Lemma 3.2 we see that the pair of just constructed abstract problems (AP) and (AD) satisfies statements (i) and (ii). □

Conclusion. Although simple the Unification Theorem is of a great significance. It unifies all duality models in a unique theory on the basis of the abstract duality scheme. By this scheme one can establish dual problems for a large class of optimization problems. It is successfully applied in vector optimization as we have seen in [1]–[7] where other methods can hardly be used.

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