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THE L^2-OPTIMAL CONVOLVING FUNCTIONS IN RECONSTRUCTION CONVOLUTION ALGORITHMS

FRANTIŠEK MATÚŠ

Numerical solutions of Radon's integral equation based on a convolution reconstruction algorithm are investigated. We require the best approximation of an unknown image (due to the L^2-norm on a given area) by a reconstruction formula depending on a free convolving function. In special cases, existence such an L^2-optimal convolving function is verified and conditions for the uniqueness are found. Equations for estimation of the numerical counterpart of this function by a given interpolating function are developed. A noise contamination in the projections is discussed, too. The obtained results are compared by numerical reconstruction from simulated data.

1. INTRODUCTION

In many scientific applications it is necessary to determine a distribution – image – of a physical property of an object under investigation. Often it is not possible to find it directly, but we may get sets of its line or strip integrals (corresponding to particular angles of view) – projections of image. Then, an image reconstruction from given projections is our basic problem.

This situation may be mathematically described in the following way: An image is represented by an integrable function f of two real variables $x = (x_1, x_2)$ and projections by the Radon transform $\hat{f}$ of f:

$$\hat{f}(\theta, r) = \int f(r \cos \theta - u \sin \theta, r \sin \theta + u \cos \theta) \, du ,$$

i.e. by the function $\hat{f}$ on the set of all lines in the plane with values equal to line integrals of f along these lines.

An image reconstruction from projections is based on the development of techniques for solving variations of the above integral equation. We deal with the reconstruction method known as the convolution algorithm which has several free parameters inherent in any implementation of this method. The choice of these parameters (convolving and interpolating functions) must take into account the characteristics of the projections to ensure accuracy of the reconstructed image. This paper contains, we believe, the first serious attempt to find a method for construction these data dependent parameters (this open problem is formulated in [2], Chapter 2).

The convolution reconstruction algorithm was derived as an equivalent discrete approximation to the Radon inversion formula (see [1]):

$$f(x) = f(x_1, x_2) = \frac{1}{4\pi^2} \int_0^{\pi} \int_{-\infty}^{+\infty} |\sigma| \, e^{i\sigma(x_1 \cos\theta + x_2 \sin\theta)} \int_{-\infty}^{+\infty} \hat{f}(\theta, r) \, e^{-i\sigma r} \, dr \, d\sigma \, d\theta \, .$$

All ideas are demonstrated in the special case of parallel geometry of projections; we assume there are known the projections (L, K positive integers, K odd)

(2) $$\varrho_{lk} = \hat{f}(l\Delta, kd) \quad \text{where} \quad \Delta = \pi/L; \quad l = 0, 1, \ldots, L-1$$
$$\text{and} \quad d > 0; \quad k = -\tfrac{1}{2}(K-1), \ldots, -1, 0, 1, \ldots, \tfrac{1}{2}(K-1);$$

and for simplicity $\varrho_{lk} = 0$ for other integers l, k. Let ϱ denote the set of all these numbers.

Replacing the function $|\sigma|$ in (1) by an integrable function $\varphi(\sigma)$, the Radon inversion formula gives a numerically useful estimation of an image:

(3) $$f(x_1, x_2) \sim \Delta \sum_{lk} \varrho_{lk} v(d^{-1}(x_1 \cos l\Delta + x_2 \sin l\Delta - kd))$$

where:

(4) $$v(t) = \frac{1}{4\pi^2} \int_{-\infty}^{+\infty} \varphi\left(\frac{\sigma}{d}\right) e^{i\sigma t} \, d\sigma \, .$$

The functions v, φ correspond to the convolving or filtering functions, in sense of [2]. The above derived forms of the convolution algorithm and the convolving function are suitable for purposes of this paper and will be used without any exception.

For instance, if φ is given by

$$\varphi(\sigma) = |\sigma| \quad \sigma \text{ belongs to } [-\pi/d, \pi/d]$$
$$\varphi(\sigma) = 0 \quad \text{in other case},$$

then the convolving function is equal to

$$v(t) = \frac{1}{2d} \left[\frac{\sin \pi t}{\pi t} + \frac{\cos(\pi t) - 1}{(\pi t)^2} \right].$$

Numerical applications use the convolving function of the form

(5) $$v(t) = \sum_j v_j g(t - j)$$

where g is a chosen interpolating function; in the last example the values $v_j = v(j)$ are equal to the known coefficients due to Ramachandran and Laksminarayanan (see also [2]).

2. FORMULATION OF THE PROBLEM

Now we introduce the problem treated in the next paragraphs:

We suppose to know the set ϱ of real numbers (having above described structure with parameters K, L, d, Δ), which corresponds according (2) to the Radon transform $\hat{f}_0$ of an unknown fixed function f_0. The support of f_0 is assumed to be included in the known set U:

$$U = \{(x_1, x_2);\ |x_1 \cos l\Delta + x_2 \sin l\Delta| < \tfrac{1}{2}Kd,\ l = 0, 1, \ldots, L-1\}$$

(later we will suppose U being a measurable subset of this set only). We shall look for such a convolving function v^* for the reconstruction formula (3) to be the best estimation (in some sense defined later) of the function f_0. This function v^*, if exists, is called optimal and the main problem of this paper is to express (or to estimate) it by the known data.

To use the reconstruction formula (3) for such assumed function f_0 we need not know the convolving function v outside the interval $I = (\frac{1}{2} - K, K - \frac{1}{2})$, i.e. the convolving functions will be investigated in this interval only. All the preliminary considerations to define exactly basic operators of our problem are now available. Let $\boldsymbol{G}$ be a linear operator which maps every real function v square integrable on the interval I into the function $\boldsymbol{G}v$ of two variables, defined by:

$$\boldsymbol{G}v(x) = \Delta\chi_U(x) \sum_{lk} \varrho_{lk} v\left(\frac{1}{d}(x_1 \cos l\Delta + x_2 \sin l\Delta - kd)\right),$$

i.e., $\boldsymbol{G}$ is a mapping from $\mathscr{L}^2(I)$ into $\mathscr{L}^2(U)$ (χ_U is the characteristic function of U).

It is easy to verify that the adjoint operator $\boldsymbol{G}^*$ is a mapping from $\mathscr{L}^2(U)$ into $\mathscr{L}^2(I)$ given by

$$\boldsymbol{G}^* f(t) = \Delta d \sum_{lk} \varrho_{lk} \hat{f}(l\Delta, (t + k)\, d)\,.$$

Both these operators are bounded and if we denote by $\|\varrho\| = \sum_{lk} |\varrho_{lk}|$, then an upper bound for their norms may be expressed as

$$\|\boldsymbol{G}\| = \|\boldsymbol{G}^*\| \leqq \Delta d \sqrt{(K)}\, \|\varrho\|\,.$$

The composition of $\boldsymbol{G}$ and $\boldsymbol{G}^*$ is an operator on the Hilbert space $\mathscr{L}^2(I)$. It is self-adjoint, bounded and positive, we shall write $\boldsymbol{A} = \boldsymbol{G}^*\boldsymbol{G}$. For future references the following notation is developed:

$$\boldsymbol{G} = \sum_l \boldsymbol{G}_l\,,$$

where

$$\boldsymbol{G}_l v(x) = \Delta\chi_U(x) \sum_k \varrho_{lk} v\left(\frac{1}{d}(x_1 \cos l\Delta + x_2 \sin l\Delta - kd)\right),$$

and

$$\boldsymbol{A} = \boldsymbol{A}_1 + \boldsymbol{A}_2\,,$$

where

$$A_1 = \sum_l G_l^* G_l \quad \text{and} \quad A_2 = \sum_{l \neq l'} G_l^* G_{l'} .$$

For a fixed function $f_0 \in \mathscr{L}^2(U)$ we introduce the functional $\mathscr{I}$ on $\mathscr{L}^2(I)$

$$\mathscr{I}v = \|f_0 - Gv\|^2 ;$$

$\mathscr{I}$ may be rewritten also as

$$\mathscr{I}v = \|f_0\|^2 + (v, Av) - 2(G^*f_0, v) . \tag{6}$$

Now we can reformulate the basic problem exactly:

For known ϱ, U, unknown fixed function $f_0 \in \mathscr{L}^2(U)$ (described numerically by ϱ) we will look for a convolving function $v^* \in \mathscr{L}^2(I)$ satisfying

$$\inf \mathscr{I}v = \mathscr{I}v^* . \tag{7}$$

As the reconstruction Gv^* of f_0 is the best due to the norm on $\mathscr{L}^2(U)$, any v^* satisfying (7) will be called the L^2-optimal convolving function of the function f_0.

3. EXISTENCE OF THE SOLUTION

At the first sight $\inf \mathscr{I}v \geqq 0$ but existence or uniqueness of an L^2-optimal convolving function are non-trivial questions. The following two sections are devoted to these problems.

A connection between A and $\mathscr{I}$ is a consequence of general propositions (they can be found in [3] or [4]).

Theorem 1. A function v^* is the L^2-optimal convolving function of f_0 if and only if

$$Av^* = G^*f_0 . \tag{8}$$

We shall prove that the equation (8) has always a solution, i.e. $G^*f_0 \in \mathscr{R}(A)$ (the symbols $\mathscr{R}$, $\mathscr{N}$ denote the range and the kernel of the operator in brackets). Analysis of the operator A is not trivial, we shall now present auxiliary operators, which make the situation clear.

First, F_l maps $\mathscr{L}^2(I)$ into $\mathscr{L}^2(J)$

$$F_l\, v(t) = \Delta \chi_J(t) \sum_k \varrho_{lk} v(t - k) , \quad \text{where} \quad J = (-\tfrac{1}{2}K, \tfrac{1}{2}K)$$

and χ_J is the characteristic function of the interval J. The operator F_l was derived from G_l and it is easy to prove that there are positive constants C, c such that for every v of $\mathscr{L}^2(I)$

$$C\|F_l v\| \geqq \|G_l v\| \geqq c\|F_l v\|$$

holds owing to the form of the set U.

Let F be an operator from $\mathscr{L}^2(I)$ into $\mathscr{L}^2(J)^L$ (the last symbol should be in this paper always understood as the Cartesian product of the L exemplars of the space

$\mathscr{L}^2(J)$; the scalar product of $\mathscr{L}^2(J)^L$ is assumed to be the sum of the scalar products of all components) defined by

$$Fv = (F_0 v, F_1 v, \ldots, F_{L-1} v) .$$

The following lemma is an immediate consequence of the preceding considerations.

Lemma 1.

$$\mathscr{N}(A_1) = \bigcap_l \mathscr{N}(G_l) = \bigcap_l \mathscr{N}(F_l) = \mathscr{N}(F)$$

As A_1 is a self-adjoint, positive operator, $\mathscr{L}^2(I)$ can be written as the direct sum of $\mathscr{N}(A_1)$ and $\mathscr{N}(A_1)^\perp = \overline{\mathscr{R}(A_1)}$ (the bar above the set denotes the closure of this set, the symbol $\perp$ the orthogonal subspace).

Lemma 2. The restriction of A_1 on $\mathscr{N}(A_1)^\perp$ is a strongly positive operator.

Proof. It is sufficient to establish that the restriction of F on $\mathscr{N}(F)^\perp$ has a continuous inverse operator because

$$(A_1 v, v) = \sum_l \|G_l v\|^2 \geqq c^2 \sum_l \|F_l v\|^2 = c^2 \|Fv\|^2 ,$$

but the last proposition is technically rather complicated. To find the structure of F it is suitable to replace the space $\mathscr{L}^2(I)$ by $\mathscr{L}^2(I_0)^{2K-1}$ where $I_0 = (-\frac{1}{2}, \frac{1}{2})$. A natural isomorphism conserving the scalar products is denoted simply by the brackets $[\![\ ]\!]$:

$$v \text{ belongs to } \mathscr{L}^2(I) \text{ if and only if } [\![v]\!] \in \mathscr{L}^2(I_0)^{2K-1} ,$$

$[\![v]\!]$ is the vector of the functions v^k on I_0: $v^k(t) = v(t-k)$, $k = 1 - K, \ldots$ $\ldots, -1, 0, 1, \ldots, K-1$. The same situations are between $\mathscr{L}^2(J)$, $\mathscr{L}^2(J)^L$ and $\mathscr{L}^2(I_0)^K$, $\mathscr{L}^2(I_0)^{KL}$. In this notation the operator F_l may be given by

$$[\![F_l v]\!] = \mathbf{Q}_l [\![v]\!] ,$$

where $\mathbf{Q}_l$ is a matrix with K rows and $2K-1$ columns; its jth row contains beginning from the jth column the numbers $\Delta\varrho_{lk}$ $(k = \frac{1}{2}(1-K), \ldots, \frac{1}{2}(K-1))$, other elements of $\mathbf{Q}_l$ are equal to zero.

Hence, the operator F is expressible as follows

$$[\![Fv]\!] = \mathbf{Q}[\![v]\!]$$

where $\mathbf{Q}$ is a matrix with KL rows and $2K-1$ columns; $\mathbf{Q}$ contains the matrices $\mathbf{Q}_l$ as blocks in column $(l = 0, 1, \ldots, L-1)$. The kernel $\mathscr{N}(\mathbf{Q})$ of the matrix $\mathbf{Q}$ is a subspace of $(2K-1)$-dimensional Euclidean space; let $\mathbf{P}$ and $\mathbf{P}^\perp$ be the matrices of the orthogonal projectors on $\mathscr{N}(\mathbf{Q})$ and $\mathscr{N}(\mathbf{Q})^\perp$. Then a natural decomposition of every $[\![v]\!]$ into $\mathbf{P}[\![v]\!]$ and $\mathbf{P}^\perp[\![v]\!]$ gives the available decomposition of $\mathscr{L}^2(I_0)^{2K-1}$, i.e.

$$\mathscr{L}^2(I) = \mathscr{N}(F) \oplus \mathscr{N}(F)^\perp , \quad \text{where} \quad \mathscr{N}(F) = \{v; \mathbf{P}[\![v]\!] = [\![v]\!]\}$$
$$\text{and} \quad \mathscr{N}(F)^\perp = \{v; \mathbf{P}^\perp[\![v]\!] = [\![v]\!]\} .$$

Finally, let $\mathbf{Q}^-$ be the matrix satisfying $\mathbf{Q}^-\mathbf{Q} = \mathbf{P}^\perp$ and q be the norm of $\mathbf{Q}^-$. Then for $v \in \mathcal{N}(F)^\perp$

$$\|v\| = \|\mathbf{P}^\perp[\![v]\!]\| = \|\mathbf{Q}^-\mathbf{Q}[\![v]\!]\| \leqq q\|\mathbf{Q}[\![v]\!]\| = q\|Fv\|$$

takes place and the existence of the inverse bounded operator to the restriction of F on $\mathcal{N}(F)^\perp$ is proved. □

Theorem 2. For every $f_0 \in \mathcal{L}^2(U)$ there is an L^2-optimal convolving function v^*.

Proof. We decompose $\mathcal{L}^2(I)$ into the direct sum $\mathcal{L}^2(I) = \mathcal{N}(A_1) \oplus \mathcal{R}(A_1)$. The inclusion $\mathcal{N}(A_1) \subset \mathcal{N}(A_2)$ implies that $\mathcal{R}(A_1)$ is also an invariant subspace of the operator A. If we restrict all the operators on $\mathcal{R}(A_1) = \mathcal{S}$, we obtain

$$A|_{\mathcal{S}} = A_1|_{\mathcal{S}}\left(E|_{\mathcal{S}} + A_1|_{\mathcal{S}}^{-1} A_2|_{\mathcal{S}}\right).$$

The operator A_2 is compact (see (13)), then $A_1|_{\mathcal{S}}^{-1} A_2|_{\mathcal{S}}$ is also compact and hence the ranges of $(E|_{\mathcal{S}} + A_1|_{\mathcal{S}}^{-1} A_2|_{\mathcal{S}})$ and A are closed sets ([5] contains all general propositions, which was used in this section without references).

Finally

$$\mathcal{L}^2(I) = \mathcal{N}(G) \oplus \overline{\mathcal{R}(G^*)} = \mathcal{N}(A) \oplus \mathcal{R}(A)$$

and from

$$\mathcal{N}(A) = \mathcal{N}(G) \quad \text{we get} \quad \mathcal{R}(A) = \overline{\mathcal{R}(G^*)}.$$

Therefore, if $f_0 \in \mathcal{L}^2(U)$, then $G^*f_0 \in \mathcal{R}(A)$ and the proposition follows straightforward from Theorem 1.

4. UNIQUENESS OF THE SOLUTION

In this section we give a necessary and sufficient condition for uniqueness of the L^2-optimal convolving function but under the strong restriction only: we shall consider the operator G on the dense linear manifold $\mathcal{D}(I)$ in $\mathcal{L}^2(I)$ which consists of all $(L-1)$-times continuously differentiable functions. Then the uniqueness of an L^2-optimal convolving function is equivalent to the proposition $\mathcal{N}(G) \cap \mathcal{D}(I) = \{0\}$ and therefore we shall first investigate this equality.

Let $\mathbf{T}_n$ be a matrix with $n+1$ rows and $L-1$ columns ($L > 1$, for $L = 1$ is the situation simple and is omitted here), $n = 0, 1, \ldots, L-2$. The jth column of $\mathbf{T}_n$ ($j \leqq n$) contains zero elements only, the $(j+1)$th column ($j \geqq n$) contains the values

$$\sum_{lk} \varrho_{lk} \frac{j!}{(j-n)!} \cos^m l\Delta \,.\, \sin^{n-m} l\Delta \,.\, (-k)^{j-n}\,, \quad \text{for} \quad m = 0, 1, \ldots, n\,.$$

Let $\mathbf{T}$ be the matrix consisting of the matrices $\mathbf{T}_n$ in the column ($n = 0, 1, \ldots, L-2$).

Lemma 3. A nonzero polynomial of the degree less then $L-1$ belongs to $\mathcal{N}(G)$ if and only if the matrix $\mathbf{T}$ is singular.

Proof. The polynomial $\sum_{j=0}^{L-2} v_i t^i = v(t) \neq 0$ belongs to $\mathcal{N}(\boldsymbol{G})$ only if

$$\frac{\partial^n \boldsymbol{G} v}{\partial x_1^m, \partial x_2^{n-m}}(0) = 0 \quad \text{for} \quad n = 0, 1, \ldots, L-2, \quad m = 0, 1, \ldots, n,$$

i.e. if

$$\sum_{j=n}^{L-2} v_i \sum_{lk} \varrho_{lk} \frac{j!}{(j-n)!} \cos^m l\Delta \,.\, \sin^{n-m} l\Delta \,.\, (-k)^{j-n} = 0,$$

what is equivalent to the singularity of the matrix $\mathbf{T}$. □

Corollary. If all values ϱ_{lk} are nonnegative and at least one is positive then Lemma 3 is satisfied.

Theorem 3. $\mathcal{N}(\boldsymbol{G}) \cap \mathcal{D}(I) = \{0\}$ if and only if both matrices $\mathbf{Q}$, $\mathbf{T}$ are nonsingular.

Proof. Necessity of these conditions is evident, we shall only concern with sufficiency. Let $v \in \mathcal{N}(\boldsymbol{G}) \cap \mathcal{D}(I)$ and let the condition of the uniqueness be true. The following symbols will be used to represent a special type of the differentiable operators

$$\boldsymbol{D}_j = \prod_{l=0\; l\neq j}^{L-1} \left(-\sin l\Delta \frac{\partial}{\partial x_1} + \cos l\Delta \frac{\partial}{\partial x_2}\right).$$

Then

$$\boldsymbol{D}_j \boldsymbol{G} v(x) = \Delta \chi_U(x) \sum_k \varrho_{jk} v^{(L-1)} \left(\frac{1}{d}(x_1 \cos j\Delta + x_2 \sin j\Delta - kd)\right) \prod_{l=0\; l\neq j}^{L-1} \frac{\sin(l-j)\Delta}{d}$$

and hence

$$\boldsymbol{F}_j v^{(L-1)} = 0.$$

The first condition gives $v^{(L-1)} = 0$ and the second condition according to Lemma 3 arrives at the desired result $v = 0$. □

5. NOISE IN PROJECTIONS

The following situation will be now investigated: Let ϱ_{lk} be the sum of a number $\tilde{\varrho}_{lk}$ and a random variable having the normal probability distribution $N(0, \sigma^2)$. Let these random variables be stochastically mutually independent. The indices ϱ, $\tilde{\varrho}$ are used to represent a dependence of operators or functionals on the sets of ϱ_{lk} and $\tilde{\varrho}_{lk}$. Then $\mathcal{I}_\varrho v$ according to (6) is a random variable and the expectation od this variable $\mathsf{E}\{\mathcal{I}_\varrho v\}$ defines a new functional on $\mathcal{L}^2(I)$; we denote it by $\mathsf{E}\mathcal{I}_\varrho$. We can generalize: v^* is called the L^2-optimal convolving function for the function $f_0 \in \mathcal{L}^2(U)$ under a noise in projections if

(9) $$\inf \mathsf{E}\mathcal{I}_\varrho v = \mathsf{E}\mathcal{I}_\varrho v^*.$$

Let $\boldsymbol{B}$ be the operator on $\mathscr{L}^2(I)$ equal to

$$\boldsymbol{B}\, v(t) = \Delta^2 d L \sum_k \chi^*(t + k)\, v(t)$$

where $\chi^*(t)$ is the Radon transform of χ_U at the line determined by $(l\Delta, td)$ (independent of l). The simple computation gives

$$\mathsf{E}\mathscr{I}_\varrho v = \|f_0\|^2 + (v, (\boldsymbol{A}_{\bar\varrho} + \sigma^2 \boldsymbol{B})\, v) - 2(\boldsymbol{G}^*_{\bar\varrho} f_0, v)\,.$$

We observe, $\boldsymbol{B}$ is a self-adjoint, strongly positive operator

$$(\boldsymbol{B}v, v) \geqq L\,.\,K\,.\,\Delta^2\,.\,d^2 \operatorname{tg} \tfrac{1}{2}\Delta\,.\,\|v\|^2$$

and then $\boldsymbol{A}_{\bar\varrho} + \sigma^2 \boldsymbol{B}$ $(\sigma^2 > 0)$ has these properties, too. Hence Theorem 1 implies

Theorem 4. For every $f_0 \in \mathscr{L}^2(U)$ under a noise in projections there is the unique L^2-optimal convolving function v^*.

Despite of seemingly more complicated notion of the projections, the definitive results of this section was reached much easier then before, and have all desirable properties.

6. NUMERICAL ASPECTS

For numerical purposes we shall minimize the functional $\mathscr{I}$ on a finite-dimensional subspace $\mathscr{H}(I)$ of the space $\mathscr{L}^2(I)$. The area U will be assumed here to be measurable. Let g be a given interpolating function, we define $\mathscr{H}(I)$ as the set of all linear combinations of the functions g_i

$$g_i(t) = g(t - i)\,; \quad i = 1 - K, \ldots, 0, \ldots, K - 1$$

restricted on the interval I.

If it is required the L^2-optimal convolving function to be even, then an appropriate choice of the subspace $\mathscr{H}(I)$ is: $\mathscr{H}(I)$ must be the set of all linear combinations of the functions

$$g_i + g_{-i}\,; \quad i = 0, 1, \ldots, K - 1\,.$$

This approach is in this paper omitted.

Let us suppose $v \in \mathscr{H}(I)$, v being in the form (5): $v = \sum_{j=1-K}^{K-1} v_j g_j$ (v_j – real numbers). Minimization of $\mathscr{I}$ on $\mathscr{H}(I)$ is then equivalent to minimization of the following function of $2K - 1$ variables

$$\mathscr{I}(v_j) = \|f_0 - \sum_j v_j \boldsymbol{G} g_j\|^2 \tag{10}$$

and v is a minimum of (10) if and only if

$$\sum_{j=1-K}^{K-1} v_j(\boldsymbol{A} g_j, g_i) = (\boldsymbol{G}^* f_0, g_i)\,; \quad i = 1 - K, \ldots, 0, \ldots, K - 1$$

takes place. As f_0 is not known, it will be useful to replace the function $\boldsymbol{G}^* f_0$ by the function: $\sum_j \boldsymbol{G}^* f_0(j)\, g_j$; the right-hand sides of the last equations are now known

$$b_i = \Delta \,.\, d \,.\, \sum_{jlk} \varrho_{lk} \varrho_{lj+k} \,.\, (g_j, g_i) \tag{11}$$

We introduce the forms of the operators $\boldsymbol{A}_1, \boldsymbol{A}_2$

$$\boldsymbol{A}_1\, v(t) = \Delta^2 \,.\, d \sum_{lkk'} \varrho_{lk} \varrho_{lk'} \chi^*(t + k)\, v(t + k - k') \tag{12}$$

$$\boldsymbol{A}_2\, v(t) = \Delta^2 \,.\, d^2 \sum_{l \neq l'} \sum_{kk'} \varrho_{lk} \varrho_{l'k'} \sec |l - l'|\, \Delta \,.\, \int \chi_U((t + k)\, d, ud) \,. \tag{13}$$

$$.\, v((t + k) \cos (l' - l)\, \Delta + u \sin (l' - l)\, \Delta - k')\, \mathrm{d}u \,.$$

Then

$$(\boldsymbol{A}_1 g_j, g_i) = \Delta^2 d \sum_{lkk'} \varrho_{lk} \varrho_{lk'} \int \chi^*(t + k)\, g_{j+k'-k}(t)\, g_i(t)\, \mathrm{d}t \tag{14}$$

and

$$(\boldsymbol{A}_2 g_j, g_i) = \Delta^2 d^2 \sum_{l \neq l'} \sum_{kk'} \varrho_{lk} \varrho_{l'k'} \sec |l - l'|\, \Delta \,.\, \iint \chi_U(td, ud) \,. \tag{15}$$

$$g_{j+k'}(t \,.\, \cos (l' - l)\, \Delta + u \,.\, \sin (l' - l)\, \Delta) \,.\, g_{i+k}(t)\, \mathrm{d}t\, \mathrm{d}u \,.$$

Finally, the developed system of linear equations for finding an estimation of the L^2-optimal convolving function on $\mathscr{H}(I)$ has the form

$$\sum_{j=1-K}^{K-1} v_j[(\boldsymbol{A}_1 g_j, g_i) + (\boldsymbol{A}_2 g_j, g_i)] = b_i \,; \quad i = 1 - K, \ldots, K - 1 \tag{16}$$

where all terms, described by (11), (14), (15), are a priori known. If g_i belongs to $\mathscr{D}(I)$ $(i = 1 - K, \ldots, K - 1)$, then Theorem 3 gives conditions for strong positiveness of $\boldsymbol{A}$ and under these conditions equations (16) have a unique solution; in other cases, existence and uniqueness of a solution of (16) are difficult questions.

For projections under noise (according to the last paragraph) we get analogously the equations

$$\sum_{j=1-K}^{K-1} v_j[(\boldsymbol{A} g_j, g_i) + \sigma^2 \,.\, (\boldsymbol{B} g_j, g_i)] = b_i \,; \quad i = 1 - K, \ldots, K - 1 \tag{17}$$

where

$$(\boldsymbol{B} g_j, g_i) = \Delta^2 d L \sum_k \int \chi^*(t + k)\, g_j(t)\, g_i(t)\, \mathrm{d}t \,. \tag{18}$$

These equations (17) have under conditions of Theorem 4 a unique solution.

7. NUMERICAL EXAMPLE

We consider the following example to illustrate our results.

Let

$$V_1 = \{x; x_1^2 + x_2^2 \leqq 8^2\}$$
$$V_2 = \{x; (x_1 - 1)^2 + (x_2 - 1)^2 \leqq 4^2\}$$

and

$$f_0 = \chi_{V_1} - 0.4\chi_{V_2} .$$

Let the parameters of parallel geometry of projections be chosen as follows

$$K = 15 \quad L = 20 \quad 1/d = 0.93 ,$$

and the set U be given as at the beginning. The set ϱ was computed analytically from the function f_0 (see (2)) and a noise was added, if was considered.

The convolving function in the form (5) with the simpliest interpolating function (nearest point interpolation)

$$g(t) = \begin{cases} 1 & \text{if } t \in [-\frac{1}{2}, \frac{1}{2}) , \\ 0 & \text{otherwise} . \end{cases}$$

is used. Numerical counterpart of the L^2-optimal convolving function was computed as a solution of equations (17) (the integral terms in (14), (15) and (18) was estimated for simplicity). In cases $\sigma^2 = 0.0$, $\sigma^2 = 0.2$ (the mean values $\tilde{\varrho}_{lk}$ was replaced by the prepared values of projections ϱ_{lk} simply) we got the results represented in Figure 1.

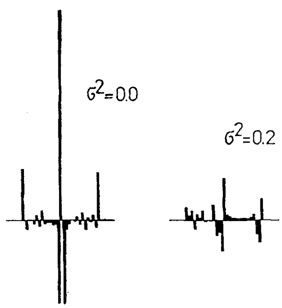

Fig. 1.

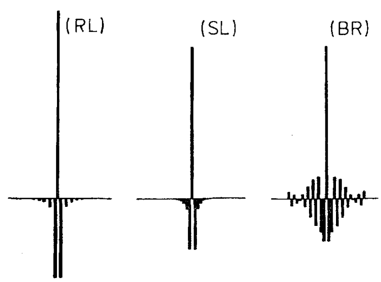

Fig. 2.

These convolving functions are compared with three others usually used (see [2]):

1) v_j are earlier given coefficients due to Ramachandran and Laksminarayanan (RL),

2) $v_j = \dfrac{1}{\pi^2 d} \dfrac{2}{1 - 4j^2}$ coefficients of Shepp-Logan (SL),

3) $v_j^C = \dfrac{C^2}{\pi^2 d}\left(\dfrac{\pi}{2Cj} \sin(\pi Cj) - \dfrac{1}{C^2 j^2} \sin^2\left(\dfrac{\pi}{2} Cj\right)\right)$ coefficients of Bracewell-Riddle ($C \in (0, 1)$, we choose $C = 0.9$) (BR);

the corresponding convolving functions are in Figure 2.

Four indicators of the reconstruction quality are suggested:

1) $a_1 = (\sum_{x \in \mathscr{U}} ((f_0 - Gv)(x))^2)^{1/2}$ square difference,

2) $a_2 = \max_{x \in \mathscr{U}} |(f_0 - Gv)(x)|$ maximal absolute difference,

3) $a_3 = \operatorname{aveg}_{x \in \mathscr{U}} |(f_0 - Gv)(x)|$ average of absolute differences

4) a_4^δ – percentage of the points x of $\mathscr{U}$ for which the number $|(f_0 - Gv)(x)|$ exceeds a given boundary δ.

The set $\mathscr{U}$ is intended to be

$\mathscr{U} = \{x; x_1^2 + x_2^2 \leqq 8.01^2, x_1, x_2 \text{ integers}\}$;

the points of $\mathscr{U}$ have the both coordinates integer numbers and for these points the reconstruction is performed.

The table below reports on the values of the above mentioned indicators in all foregoing cases of our example and prove that methods of this paper may lead to interesting convolving functions quite different from known.

Noise	$\sigma^2 = 0{\cdot}0$				$\sigma^2 = 0{\cdot}2$			
Convolving functions	L^2-optimal	RL	SL	BR	L^2-optimal	RL	SL	BR
a_1	0·92	1·38	1·59	1·60	1·30	1·49	1·64	1·66
a_2	0·258	0·273	0·333	0·348	0·384	0·335	0·358	0·377
a_3	0·043	0·068	0·075	0·079	0·063	0·078	0·082	0·083
$a_4^{0.1}$	8·1	27·9	31·0	30·5	17·8	27·9	28·9	29·9

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Ing. František Matúš, Ústav teorie informace a automatizace ČSAV (Institute of Information Theory and Automation — Czechoslovak Academy of Sciences), Pod vodárenskou věží 4, 182 08 Praha 8. Czechoslovakia.