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ON A NONLINEAR DIFFERENTIAL GAME OF EVASION WITH CONSTRAINTS

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A strategy of evasion for a class of nonlinear differential games with linear constraints is constructed.

1. INTRODUCTION

A game of evasion is a mathematical idealization of a conflict situation of the following type: Let two moving, controllable objects O_1 and O_2 be given. These objects can be represented e.g. by planes, or by some another technical devices. The aim of the object O_2 is to avoid a given subset M of the phase space and the aim of the object O_1 is to force the object O_2 to fall into the set M. The evasion game consists of finding a strategy of choosing some controls for the object O_2, to ensure that this object remains all the time outside the set M. A differential game of evasion is a game of evasion which is described by a system of differential equations depending on some control parameters. If the motion of the objects of the game is constrained to a subset of the state space, then the game is called an evasion game with constraints.

A definition of an evasion and pursuing strategy, given by P. Isaacs, has some disadvantages. These are supposed to be optimal in some sense and thus definition is too restrictive. In order to improve possibilities of calculations of strategies, L. S. Pontryagin (see [17], [18]) proposed another definition of strategy. Today there are many papers concerning linear and nonlinear differential games based on Pontryagin's definition of strategy and it is hardly possible to give a list of all these papers. We only refer the reader to the papers [11–20] and to the book [7], where many such papers are quoted.

The aim of this paper is to show that the methods for solving nonlinear differential games of evasion (in Pontryagin's sense) developed recently, can also be successfully applied to nonlinear differential evasion games with constraints. We have been motivated by the paper [12], where a certain type of so called quasilinear differential game of evasion with linear constraints is solved. Naturally, there are many open problems concerning nonlinear evasion games with constraints. For example, it is possible to consider the game described by a control system in more general form than that in our paper with some type of linear or nonlinear constraints. Another class of problems concerns evasion games with constraints played by more that two players (see e.g. [12]). Moreover, from the practical point of view, it is natural to give also some integral constraints on control functions e.g. $\int_0^\infty \|u(t)\|^2 \, dt < \infty$, $\int_0^\infty \|v(t)\|^2 \, dt < \infty$, where u and v are control functions of the pursuer and evader, resp.; see e.g. [1]).

Before we give the precise formulation of our problem, let us briefly recall the mathematical description of an evasion process. Let the motion of the controllable objects O_1 and O_2 be described by the nonlinear differential equations

$$\dot{x} = f(x, u), \tag{1}$$

$$\dot{y} = g(y, v), \tag{2}$$

respectively, where $x, y \in \mathbb{R}^m$, $u \in \mathbb{R}^p$ is a control parameter of the pursuer and $v \in R^q$ is a control parameter being in possession of the evader. If the control u is given by a function $u = u(t)$, then after substituting it into equation (1) we obtain the equation $\dot{x}(t) = f(x(t), u(t))$ and we can solve this equation with the initial condition $x(0) = x_0$. Analogously, if $v = v(t)$, the we can solve the equation $\dot{y}(t) = g(y(t), v(t))$ with the initial condition $y(0) = y_0$. Let $x = (x_1, x_2)$, $y = (y_1, y_2)$, where x_1, y_1 are geometric coordinates and x_2, y_2 are velocity coordinates of the objects O_1 and O_2 respectively. Let us define the set $M = \{(x_1, x_2, y_1, y_2) \in \mathbb{R}^m \times \mathbb{R}^m : x_1 = y_1\}$. If $x_1(t) = y_1(t)$ for some $t \in (0, \infty)$, i.e. $(x(t), y(t)) \in M$, then the objects O_1 and O_2 have the same geometric coordinates. This can be interpreted as capture of the object O_2 by the pursuer O_1 at time t. If the control function $v = v(t)$ can be chosen in such a way that $(x(t), y(t)) \notin M$ for all $t \in (0, \infty)$ and for arbitrary control function $v = v(t)$ belonging to a given class of function, then this means that the evader O_2 cannot be captured by the pursuer in finite time. Now, let us write equations (1) and (2) as one equation

$$\dot{z} = F(z, u, v), \tag{3}$$

where $z = (x, y) \in \mathbb{R}^{2m}$, $F(z, u, v) = (f(x, u), g(y, v))$. Then $M = \{(z_1, z_2, z_3, z_4) \in \mathbb{R}^{2m} : z_1 = z_3\}$ and the condition $z(t) \notin M$ means that the objects O_1 and O_2 have their geometric coordinates different at time t.

Now let us assume that the evasion game is described by the equation

$$\dot{z} = Cz + f(u, v), \tag{4}$$

where $z \in \mathbb{R}^n$, C is a constant matrix, $f(u, v)$ is a continuous function on the set $U \times V \subset \mathbb{R}^p \times \mathbb{R}^q$, where U and V are nonempty, compact sets. Let $M \subset \mathbb{R}^n$ be a linear subspace of R^n. The evasion game described by an equation of the form (4), which is linear in z and nonlinear in (u, v), and by a linear subspace M is called a quasilinear differential game of evasion. The following theorem concerning such a quasilinear differential game of evasion has been proved by R. V. Gamkrelidze and G. L. Kharatishvili (see [4]).

Theorem 1. Let $U \subset \mathbb{R}^p$, $V \subset \mathbb{R}^n$ be nonempty, compact sets, $f(u. v)$ be a continuous function on $U \times V$ with values in R^n and let a linear subspace M of $\mathbb{R}^n$ be given. Assume that the following conditions are satisfied:

(1) $\dim M \leqq n - 2$

(2) There exists a 2-dimensional subspace W of L, where L is the orthogonal complement to the subspace M in $\mathbb{R}^n$, and an integer k such that:

(a) Each set

$$\pi C^i f(U, V), \quad i = 1, 2, \ldots, k - 2$$

is a point

(b) The set

$$R_0 = \bigcap_{u \in U} \operatorname{co} \pi C^{k-1} f(u, V)$$

contains an interior point (relatively to W), where $\pi \colon \mathbb{R}^n \to W$ is the orthogonal projection onto L and co A denotes the convex hull of the set A.

Then there exists an evasion strategy (in the sense of Definition 2 from Section 2) for the game described by equation (4) and the subspace M and the following holds: If $\varrho(z(t), M)$ is the Euclidean distance of the point $z(t)$ from the set M, then

$$(5) \qquad \varrho(z(t), M) \geqq K(1 + \|z(t)\|)^{-k} \quad \text{for all} \quad t \in [0, \infty),$$

where $\|\cdot\|$ is the Euclidean norm on $\mathbb{R}^n$ and the constant $K > 0$ depends on the equation, but not on strategies.

Example. Let the dynamics of objects O_1 and O_2, respectively be described by the equations

$$(5) \qquad \begin{aligned} \ddot{x} + a\dot{x} &= \varrho\tilde{u} \\ \ddot{y} + b\dot{y} &= \sigma\tilde{v}, \end{aligned}$$

where $x, y, \tilde{u}, \tilde{v} \in \mathbb{R}^{2n}$, $n \geqq 1$, $\|\tilde{u}\| \leqq 1$, $\|\tilde{v}\| \leqq 1$, a, b, ϱ, σ are positive numbers and $\sigma > \varrho$. If $z = (z_1, z_2, z_3) = (x - y, \dot{x}, \dot{y})$, then the system (5) can be written in the form

$$(6) \qquad \begin{aligned} \dot{z}_1 &= z_2 - z_3 \\ \dot{z}_2 &= -az_2 + \varrho\tilde{u} \\ \dot{z}_3 &= -bz_3 + \sigma\tilde{v} \end{aligned}$$

and we can write this system in the form (4), where

$$C = \begin{bmatrix} 0 & I & -I \\ 0 & -aI & 0 \\ 0 & 0 & -bI \end{bmatrix},$$

$u = (0, \varrho\tilde{u}, 0)^*$, $v = (0, 0, \sigma\tilde{v})^*$, $f(u, v) = (0, \varrho\tilde{u}, \sigma\tilde{v})^*$ (w^* is the transpose of w), I is the $n \times n$, unit matrix $U = \{(0, \varrho\tilde{u}, 0)\colon \|\tilde{u}\| \leqq 1\}$, $V = \{(0, 0, \sigma\tilde{v})\colon \|\tilde{v}\| \leqq 1\}$, $M = \{z = (z_1, z_2, z_3) \in \mathbb{R}^{3n}\colon z_1 = 0\}$. The subspace $L = (z_1, z_2, z_3) \in \mathbb{R}^{3n}\colon z_2 = 0$, $z_3 = 0\}$ is the orthogonal complement of M in $\mathbb{R}^{3n}$ and the orthogonal projection $\pi\colon \mathbb{R}^{3n} \to W$ is represented by the matrix

$$\pi = \begin{bmatrix} I & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

Obviously, $\pi f(u, v) = \{0\}$ for all $(u, v) \in U \times V$, i.e. $\pi f(U, V) = \{0\}$ and one can easily calculate that $\pi C f(u, v) = (\varrho\tilde{u} - \sigma\tilde{v}, 0, 0)^*$ The set $\pi C f(u, V)$ has the form $\pi C f(u, V) = \{(\varrho u, -\sigma v, 0, 0)^*\colon v \in V\}$ and one can show that this set is the ball in L with center at the origin and radius $\sigma - \varrho$. Therefore the assumption $\sigma > \varrho$ implies that the assumption (2)–(b) of Theorem 1 is also satisfied. By this theorem there exists an evasion strategy for the game described by the system (6) and the subspace M. Moreover, the inequality (5) with $k = 2$ is satisfied.

2. THE EVASION THEOREM

Consider a system of ordinary differential equations

$$\dot{z} = P(z, u, v)\,, \tag{7}$$

where $z \in \mathbb{R}^n$ is the state variable, u, v are control parameters (u – the parameter for the pursuer, v – the parameter for the evader), $u \in U$, $v \in V$, $U \subset \mathbb{R}^k$, $V \subset \mathbb{R}^l$ are nonempty, compact sets. We suppose that the following conditions are satisfied:

(A) $P(z, u, v)$ is continuous on $\mathbb{R}^n \times U \times V$

(B) There exist constants $a > 0$, $b > 0$ such that

$$|(z, P(z, u, v))| \leqq a\|z\|^2 + b \quad \text{for all} \quad (z, u, v) \in \mathbb{R}^n \times U \times V$$

((x, y) is the scalar product of x and y)

(C) For any $R > 0$ there exists a constant $C_R > 0$ such that

$$\|P(z, u, v) - P(\bar{z}, u, v)\| \leqq C_R\|z - \bar{z}\|$$

for all $(u, v) \in U \times V$ and $z, \bar{z} \in B_R = \{w \in \mathbb{R}^n\colon \|w\| \leqq R\}$.

The conditions (A)–(C) ensure that for any measurable functions $u(t)$, $v(t)$ defined on the interval $[0, \infty)$ and such that $u(t) \in U$, $v(t) \in V$ for all $t \geqq 0$ and any $z_0 \in \mathbb{R}^n$ there exists a unique solution of the initial value problem

$$\dot{z} = P(z, u(t), v(t))\,, \quad z(0) = z_0 \tag{8}$$

defined on $[0, \infty)$ (see [2]).

We denote by $\mathcal{M}(I, Y)$ the set of all measurable functions defined on the interval $I = [0, \infty)$ with values in the set $Y \subset \mathbb{R}^p$.

Definition 1. A mapping v: $\mathcal{M}(I, U) \times \mathbb{R}^n \to \mathcal{M}(I, V)$ (the values of v we denote by $v(u, z)$, or $v^u(z, \cdot)$) is said to be a strategy, if it possesses the property: If $u_1, u_2 \in \mathcal{M}(I, U)$, $z_0 \in \mathbb{R}^n$, $T > 0$, then the equality $u_1(t) = u_2(t)$ almost everywhere on $[0, T]$ implies the equality $v^{u_1}(z_0, t) = v^{u_2}(z_0, t)$ almost everywhere on $[0, T]$.

Definition 2. Let M be a linear subspace of $\mathbb{R}^n$. A strategy v: $\mathcal{M}(I, U) \times \mathbb{R}^n \to \mathcal{M}(I, V)$ is called an evasion strategy, if for any measurable function $u \in \mathcal{M}(I, U)$ and any $z_0 \in \mathbb{R}^n$, $z_0 \notin M$, the solution $z(t)$ of the initial value problem (8) with $v(t) = v^u(z_0, t)$ does not intersect the subspace M for any $t \geqq 0$. The problem to find an evasion strategy is called an evasion problem, or the differential game of evasion.

We consider an evasion problem with constraints (to be specified later), where the system (7) has the following special form:

$$\begin{aligned} &\dot{x} = A(x, y) + B(x, y, u, v)\,, \\ &\dot{y} = \beta y + g(x, y, u, w, v) + f(u, w)\,, \end{aligned} \tag{9}$$

where $x \in \mathbb{R}^n$, $y \in \mathbb{R}^m$, $\beta > 0$, $u \in U$, $w \in W$, $v \in V$, $U \subset \mathbb{R}^k$, $W \subset \mathbb{R}^l$, $V \subset \mathbb{R}^r$ are compact sets, u is a parameter for the pursuer, w, v are parameters for the evader, A: $\mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$, B: $\mathbb{R}^n \times \mathbb{R}^m \times U \times W \times V \to \mathbb{R}^n$, g: $\mathbb{R}^n \times \mathbb{R}^m \times U \times W \times V \to \mathbb{R}^n$, f: $U \times W \to \mathbb{R}^m$. In our case the subspace M is a subspace of the state space $R^n \times \mathbb{R}^m$. Let the following constraints be given:

$$(q_k, y) \geqq 0\,, \quad k = 1, 2, \ldots, s\,, \tag{10}$$

where $q_1, q_2, \ldots, q_s$ are given constant vectors in $\mathbb{R}^m$.

We consider the system (9) with the constraints (10) and we wish to find an evasion strategy $(w^u(x_0, y_0, \cdot), v^u(x_0, y_0, \cdot))$ (it has two component w^u and v^u) such that for any $u \in \mathcal{M}(I, U)$ and any $(x_0, y_0) \in \mathbb{R}^n \times \mathbb{R}^m$ satisfying the inequalities $(q_k, y_0) \geqq 0$, $k = 1, 2, \ldots, s$, the solution $z(t) = (x(t), y(t))$ of the system (9) with $u = u(t)$, $v = v^u(x_0, y_0, t)$, $w = w^u(x_0, y_0, t)$, which fulfils the initial condition $z(0) = (x_0, y_0)$, satisfies the inequalities $(q_k, y(t)) \geqq 0$ for $k = 1, 2, \ldots, s$ and for all $t \in [0, \infty)$. We call this game the differential game of evasion with constraints given by the system (9), the subspace M and the constraints (10).

Now, let us rewrite the system (9) into the form

$$\dot{z} = P_0(z) + F(z, u, w, v)\,, \tag{11}$$

where $z = (x, y)$, $P_0(z) = (A(x, y), \beta y)$, $F(z, u, w, v) = (B(z, u, w, v), g(z, u, w, v) + f(u, w))$. We assume that

(a) A is C^r-differentiable (i.e. P_0 is C^r-differentiable)

(b) The mapping $P(z, u, v) = P_0(z) + F(z, u, v)$ satisfies the conditions (A)–(C).

We define recurrently: $C_0(z) = id$ (the identity), $C_1(z) = D\,P_0(z)$ (the Jacobian of P_0 at z), $C_i(z) = D\,C_{i-1}(z)\,P_0(z)$ for $i > 1$.

Let us formulate the so called evasion conditions:

(E 1) For any $\bar{z} = (\bar{x}, \bar{y}) \notin M$ there exists a linear subspace $L = L(\bar{z})$ in $\mathbb{R}^{n+m}$ orthogonal to M, $\dim L \geqq 2$, an open neighbourhood $U_{\bar{z}}$ of the point $\bar{z}$ and a natural number $p = p(\bar{z}) \leqq r$ such that the following conditions are fulfiled:

(i) $\{\pi\, C_j(z)\, F(z, u, w, v)\colon (u, w, v) \in U \times W \times V\} = \{0\}$ for all $z \in U_{\bar{z}}$ and and all $j = 0, 1, \ldots, p-2$, where $\pi = \pi(\bar{z})$ is the orthogonal projection of R^{n+m} onto the subspace $L(\bar{z})$

(ii) $\bigcap_{u \in U} \bigcap_{w \in W} \operatorname{co}\{\pi\, C_{p-1}(\bar{z})\, F(\bar{z}, u, w, v)\colon v \in V\}$ contains an interior point with respect to $L(\bar{z})$), where co B is the convex hull of B.

(E 2) There exists a point ω in the set $\bigcap_{u \in U} f(u, W)$ such that $(\omega, q_k) \geqq 0$ and $\|g(z, u, w, v)\| \leqq (\omega, q_k)/\|q_k\|$ for $k = 1, 2, \ldots, s$ and for all $(z, u, w, v) \in \mathbb{R}^{n+m} \times U \times W \times V$, where $f(u, W) = \{f(u, w)\colon w \in W\}$.

Theorem 2. Suppose that for the differential game of evasion given by the system (9), a linear subspace M of $\mathbb{R}^{n+m}$ and the constraints (10), the conditions (a), (b) and the evasion conditions (E 1), (E 2) are fulfilled, where $\operatorname{codim} M = n + m - \dim M \geqq 2$. Then there exist closed sets W, W_1 in $\mathbb{R}^{n+m}$ such that $M \subset \operatorname{int} W_1 \subset \operatorname{int} W$, a positive function $\gamma\colon [0, \infty) \times [0, \infty) \to \mathbb{R}$, a positive function $T\colon [0, \infty) \to \mathbb{R}$ with values in the interval $(0, 1)$ and an evasion strategy $(w^u(z, \cdot), v^u(z, \cdot))$ such that for any measurable function $u \in \mathscr{M}(I, U)$ each corresponding solution $z(t) = (x(t), y(t))$ of the system (11) satisfies:

(1) If $z_0 = (x_0, y_0) \notin W$, then $\varrho(z(t), M) \geqq \gamma(\varrho(z_0, M), \|z_0\|)$ for all $t \geqq 0$ and $z(T(\|z_0\|)) \notin W$, where $\varrho(z, M)$ is the distance of the point z from the subspace M.
(2) If $z(t_1) \notin W$ for some $t_1 \in [0, \infty)$, then $z(t) \notin W_1$ for all $t \geqq t_1$.
(3) If $z(t_1) \in W$, then there exists a $t_2 \in [t_1, t_1 + T(\|z(t_1)\|)]$ such that $z(t_2) \notin W$.
(4) If $z_0 = (x_0, y_0) \notin M$ is the initial value for the solution $z(t) = (x(t), y(t))$ and $(q_k, y_0) \geqq 0$, $k = 1, 2, \ldots, s$, then $(q_k, y(t) \geqq 0$ for all $k = 1, 2, \ldots, s$ and all $t \in [0, \infty)$.

Remark. General results for nonlinear evasion games without constraints of that kind as we have formulated in the previous theorem were proven by B. Kaśkosz [5] (see also [6], [7]). We are using her method in the proof of the theorem. This method is somewhat related to the one developed by R. V. Gamkrelidze and G. L. Kharatishvili (see [4]). Both methods are applicable also to evasion games described by some systems of integro-differential equations of Volterra type (see [9], [10]) and similar results to those contained in the theorem may be proven also for the evasion game with constraints described by these integro-differential equations. Some results concerning the evasion game with constraints of the form (10) described by the system of quasilinear differential equations

$$\begin{aligned} \dot{x}_i &= A_i x_i + B_i y_i + f_i(u, v)\,, \quad i = 1, 2, \ldots m\,, \\ \dot{y} &= \beta y + f_{m+1}(u, v)\,, \end{aligned}$$

$x_i \in \mathbb{R}^{n_i}$, $y \in \mathbb{R}^n$, A_i, B_i are constant matrices, u, v are control parameters, are obtained in the paper [12].

Before proving the above theorem, we define so called extended evasion game and formulate some indispensable lemmas.

Let us denote by (EG) the evasion game described by the system (11), the subspace M and the constraints (10).

Definition 3. By the extended evasion game corresponding to the evasion game (EG) we mean the evasion game described by the system of differential equations

$$\dot{z} = P_0(z) + \sum_{i=1}^{n+m+1} \mu_i F(z, u, w, v_i), \tag{12}$$

where $z = (x, y)$, $u \in U$, $w \in W$, $v_i \in V$, $i = 1, 2, \ldots, n + m + 1$, $\mu = (\mu_1, \mu_2, \ldots$ $\ldots, \mu_{n+m+1}) \in \{\mu \in \mathbb{R}^{n+m+1}: \sum_{i=1}^{n+m+1} \mu_i = 1,\ \mu_j \geqq 0,\ j = 1, 2, \ldots, n + m + 1\}$ ($u \in U$ is the control parameter for the pursuer and $\tilde{v} = (\mu_1, \mu_2, \ldots, \mu_{n+m+1}, w, v_1, v_2, \ldots$ $\ldots v_{n+m+1}) \in \Delta \times W \times V \times V \times \ldots \times V$ is the control parameter for the evader), by the subspace M and by the constraints (10). We denote this game by $(\mathrm{EG})_{\mathrm{ex}}$. An evasion strategy for the game $(\mathrm{EG})_{\mathrm{ex}}$ is denoted by $\tilde{v}^u(z, t) = \tilde{v}^u(t) = (\mu_1^u(t), \ldots$ $\ldots, \mu_{n+m+1}^u(t), w^u(t), v_1^u(t), \ldots, v_{n+m+1}^u(t))$.

Lemma 1 (see [5, Theorem 2.1]). Let $\tilde{v}^z(z, \cdot)$ be an evasion strategy for the game $(\mathrm{EG})_{\mathrm{ex}}$ and T be a given positive number. Then for any $\varepsilon > 0$ there exists an evasion strategy $v^u(z, \cdot)$ for the game (EG) such that for each measurable function $u \in$ $\in \mathcal{M}(I. U)$ and any $z_0 \in \mathbb{R}^{n+m}$ the corresponding solutions $z_1(t)$, $z_2(t)$ of the systems (10) and (11), respectively, satisfy the inequality

$$\|z_1(t) - z_2(t)\| < \varepsilon \quad \text{for all} \quad t \in [0, T]\,. \tag{13}$$

Lemma 2 (see [3]). Let $f: \mathbb{R} \times \mathbb{R}^r \to \mathbb{R}^s$, $(t, u_1, \ldots, u_r) \to f(t, u_1, \ldots, u_r)$ be continuous in t on the interval $[0, T]$. Let $Q(t)$ be a compact set in $\mathbb{R}^r$, which is upper semicontinuous in t. Let $R(t) = \{f(t, u): u \in Q(t)\}$ and $y: [0, T] \to \mathbb{R}^s$ be a mesurable function such that $y(t) \in R(t)$ for all $t \in [0, T]$. Then there exist measurable functions $u_1, u_2, \ldots, u_r$ defined on $[0, T]$ such that $f(t, u_1(t), u_2(t), \ldots, u_r(t)) = y(t)$ for all $t \in [0, T]$.

Lemma 3 (see [17]). Let Q be a cube in $\mathbb{R}^n$ with center at the origin and sides parallel to the axes and let p be a natural number. Then there exists a positive constant Θ such that for any curve $w_p(t)$ in $\mathbb{R}^n$ whose components are polynomials of degrees not greater than p there exists a point $w_0 \in A$ such that

$$\|w_p(t) - w_0 t^p\| \geqq \Theta t^p \quad \text{for all} \quad t \in [0, \infty)\,.$$

Proof of Theorem 2. There exists a vector $\omega \in \bigcap_{u \in U} f(u, W)$ satisfying the second evasion condition (E 2). Let u be any feasurable function defined on the interval

$J = [0, 1]$ with values in the set U. Consider the equation

$$f(u(t), w) = \omega\,, \tag{14}$$

where by the solution of this equation we mean a measurable function $w = w(t)$ defined on the interval J with values in the set W, which fulfils this equation almost everywhere on J. The existence of such solution of (14) follows from Lemma 2. Denote this solution by $w^u_\omega(t)$.

Let $u(t)$, $v(t)$ be any measurable functions defined on J with values in U and V, respectively. Let $z(t) = (x(t), y(t))$ be a solution of the system (11) corresponding to the functions $u(t)$, $w^u_\omega(t)$ and $v(t)$, which satisfies the initial condition $z(0) = (x_0, y_0)$, where $(y_0, q_k) \geqq 0$, $k = 1, 2, \ldots, s$. The function $y(t)$ has the form:

$$y(t) = e^{\beta t} y_0 + \int_0^t e^{\beta(t-s)} g(x(s), y(s), u(s), w^u_\omega(s), v(s))\, ds + \int_0^t e^{\beta(t-s)} f(u(s), w^u_\omega(s))\, ds\,.$$

Since $f(u(t), w^u_\omega(t)) = \omega$ almost everywhere on J, we have

$$\begin{aligned}(y(t), q_k) &= e^{\beta t}(y_0, q_k) + \int_0^t e^{\beta(t-s)}(\omega, q_k)\, ds + \\ &+ \int_0^t e^{\beta(t-s)}(g(x(s), y(s), u(s), w^u_\omega(s), v(s)), q_k)\, ds \geqq \\ &\geqq e^{\beta t}(y_0, q_k) + (\omega, q_k)\int_0^t e^{\beta(t-s)}\, ds - \\ &- \int_0^t e^{\beta(t-s)} \|g(x(s), y(s), u(s), w^u_\omega(s), v(s))\|\ \|q_k\|\, ds \geqq e^{\beta t}(y_0, q_k) + \\ &+ (\omega, q_k)\int_0^t e^{\beta(t-s)}\, ds - \int_0^t e^{\beta(t-s)}(\omega, q_k)\, ds = e^{\beta t}(y_0, q_k) \geqq 0\end{aligned}$$

for all $t \in [0, \infty)$.

Since the constraints conditions (10) are satisfied for arbitrary measurable function $v \in \mathscr{M}(J, V)$, we have the possibility to construct the additional components of the evasion strategy by Kaśkosz's method.

Let us choose any point $\bar{z} \notin M$ and let $U_{\bar{z}}$ be its open neighbourhood, for which the evasion condition (E 1)-(i) is satisfied. Then there is an open neighbourhood $\tilde{V}_{\bar{z}}$ of the point $\bar{z}$, $\tilde{V}_{\bar{z}} \subset U_{\bar{z}}$ and a number $\tilde{T}_{\bar{z}} \in (0, 1)$ such that for any initial value $z_0 \in \tilde{V}_{\bar{z}}$ each corresponding solution of the system (11) and also of the system (12) lies in $U_{\bar{z}}$ for all $t \in [0, \tilde{T}_{\bar{z}}]$. One can show (see [5]) that if $z(t)$ is a solution of the system (11) satisfying the initial condition $z(0) = z_0$, then

$$\begin{aligned}z(t) &= z_0 + P_0(z_0)\, t + \ldots + C_{p-1}(z_0)\, P_0(z_0)\, \frac{t^p}{p!} + \\ &+ \int_0^t \Big[F(z(s), u(s), w(s), v(s)) + C_1(z(s))\, F(z(s), u(s), w(s), v(s))\,(t-s) +\end{aligned}$$

$$+ \ldots + C_{p-2}(z(s))\, F(z(s), u(s), w(s), v(s)) \frac{(t-s)^{p-2}}{(p-2)!}\Bigg] \mathrm{d}s +$$

$$+ \int_0^t C_{p-1}(z(s))\, F(s), u(s), w(s), v(s)) \frac{(t-s)^{p-1}}{(p-1)!} \mathrm{d}s +$$

$$+ \int_0^t C_p(z(s)) \left(P_0(z(s)) + F(z(s), u(s), w(s), v(s)\right) \frac{(t-s)^p}{p!} \mathrm{d}s .$$

Using the evasion condition (E 1)-(i) we obtain

$$(15) \qquad \pi\, z(t) = w_p(z_0, t) + R(t^{p+1}) +$$

$$+ \int_0^t \pi\, C_{p-1}(z(c))\, F(z(s), u(s), w(s), v(s)) \frac{(t-s)^{p-1}}{(p-1)!} \mathrm{d}s ,$$

where $R(t^{p+1}) = \int_0^t \pi\, C_p(z(s)) \left(P_0(z(s)) + F(z(s), u(s), w(s), v(s)\right)(t-s)^p/p!\, \mathrm{d}s$,

$$(16) \qquad \|R(t^{p+1})\| \leqq N_{\bar{z}} t^{p+1} \quad \text{for} \quad t \in [0, \tilde{T}_{\bar{z}}] ,$$

$N_{\bar{z}}$ is a positive constant, $w_p(z_0, t) = \pi z_0 + \pi P_0(z_0)\, t + \ldots + \pi C_{p-1}(z_0)\, P_0(z_0)\,.$ $.\, t^p/p!$ and $\pi = \pi(\bar{z})$ is the projection of $\mathbb{R}^{n+m}$ onto the subspace $L = L(\bar{z})$.

Now we shall describe a construction of a local evasion strategy near the point $\bar{z} \in M$. The evasion condition (E 2)-(ii) implies that there is a cube Q_z with center at the point $\bar{z}$ and sides parallel to the axes (we suppose orthogonal coordinate system on L) and such that

$$(17) \qquad Q_{\bar{z}} \subset \bigcap_{u \in U} \bigcap_{w \in W} \operatorname{co} \{\pi\, C_{p-1}(\bar{z})\, F(\bar{z}, u, w, v) \colon v \in V\} =$$

$$= \bigcap_{u \in U} \bigcap_{w \in W} \{\pi\, C_{p-1}(\bar{z}) \sum_{i=1}^{n+m+1} \mu_i\, F(\bar{z}, u, w, v_i) \colon \mu \in \Delta,\ v_i \in V,\ i = 1, 2, \ldots, n+m+1\}$$

Let us put $Q = (1/p!) Q_{\bar{z}}$ and choose open neighbourhoods $V_{\bar{z}}, \tilde{V}_{\bar{z}}^0$ of z and a number $\tilde{T}_{\bar{z}}^0 \in (0, \tilde{T}_{\bar{z}})$ such that if $z_0 \in V_{\bar{z}}$, then any solution of (11) and (12), respectively, with the initial value z_0 remains in $\tilde{V}_{\bar{z}}^0$ for all $t \in [0, \tilde{T}_{\bar{z}}^0]$ and (18)

$$(18) \qquad \|\pi\, C_{p-1}(z)\, F(z, u, w, v) - \pi\, C_{p-1}(\bar{z})\, F(\bar{z}, u, w, v)\| \leqq \tfrac{1}{2} \Theta_{\bar{z}}\, p!$$

for all $z \in \tilde{V}_{\bar{z}}^0 \subset \tilde{V}_z$ (we may suppose that $\tilde{V}_{\bar{z}}^0 \subset \tilde{V}_{\bar{z}}$) and for all $(u, w, v) \in U \times W \times V$, where $\Theta = \Theta_{\bar{z}}$ is the positive number, for which the inequality from Lemma 3 is valid (here the point $w_0 = w_{z_0}$ corresponds to the cube $Q = (1/p!) Q_{\bar{z}}$ and to the curve $w_p(z_0, t)$). From (17) we obtain that if $(1/p!) w_{z_0} \in Q_{\bar{z}}$, then for any $u \in U$ and $w \in W$ there exists $v(u, w) = (\mu_1(u, w), \ldots, \mu_{n+m+1}(u, w), v_1(u, w), \ldots, v_{n+m+1}(u, w)) \in$ $\in \Delta \times V \times \ldots \times V$ such that

$$(19) \qquad \pi\, C_{p-1}(\bar{z}) \sum_{i=1}^{n+m+1} \mu_i(u, w)\, F(\bar{z}, u, w, v_i(u, w)) = w_{z_0} p!\,.$$

Let us take a basis ξ in $\mathbb{R}^{(n+m+1)(l+1)}$ ($l = \dim V$) and choose from the set of all solutions of the equation (19) the lexicographic maximum $v_0(u)$ with respect to the basis ξ. Then by Lemma 2 the function $v_0(u(t), w(t))$ is a measurable function for any measurable functions $u(t)$ and $w(t)$. Define the local evasion strategy for the game $(EG)_{ex}$ as the mapping $v^u_{\bar{z}}(z_0, t) = (\mu_1(u(t), w^u_\omega(t)), \ldots, \mu_{n+m+1}(u(t), w^u_\omega(t)), w^u_\omega(t), v_1(u(t), w^u_\omega(t)), \ldots, v_{n+m+1}(u(t), w^u_\omega(t)))$. Then for the solution $\tilde{z}(t)$ of the system (12) with the initial value z_0, corresponding to the value of the strategy we have

$$\|\pi\, \tilde{z}(t)\| = \Big\|w_p(z_0, t) + \int_0^t \pi\, C_{p-1}(\tilde{z}(t)) \sum_{i=1}^{n+m+1} \mu_i(u(s), w^u_\omega(s))\,.$$
$$.\, F(z(s), u(s), w^u_\omega(s), v_i(u(s), w^u_\omega(s)) \frac{(t-s)^{p-1}}{(p-1)!}\, ds + R(t^{p+1}) =$$
$$= \Big\|w_p(z_0, t) + R(t^{p+1}) + \int_0^t \pi\, C_{p-1}(\bar{z}) \sum_{i=1}^{n+m+1} \mu_i(u(s), w^u_\omega(s))\,.$$
$$.\, F(\bar{z}, u(s), w^u_\omega(s), v_i(u(s), w^u_\omega(s)) \frac{(t-s)^{p-1}}{(p-1)!}\, ds +$$
$$+ \int_0^t \sum_{i=1}^{n+m+1} \mu_i(u(s), w^u_\omega(s)) \left[\pi\, C_{p-1}(\tilde{z}(s))\, F(\tilde{z}(s), u(s), w^u_\omega(s)\,,\right.$$
$$\left. v_i(u(s), w^u_\omega(s)) - \pi\, C_{p-1}(\bar{z})\, F(\bar{z}, u(s), w^u_\omega(s), v_i(u(s), w^u_\omega(s))\right]\,.$$
$$.\, \frac{(t-s)^{p-1}}{(p-1)!}\Big\|\, ds \geqq \|w_p(z_0, t) + w_{z_0} t^p\| - \frac{\Theta_{\bar{z}}}{2}\, p! \int_0^t \frac{(t-s)^{p-1}}{(p-1)!}\, ds -$$
$$- N_{\bar{z}} t^{p+1} \geqq \Theta_{\bar{z}} t^p - \frac{\Theta_{\bar{z}}}{2}\, t^p - N_{\bar{z}} t^{p+1} = \left(\frac{\Theta_{\bar{z}}}{2} - N_{\bar{z}} t\right) t^p\,.$$

Obviously there exists a $T_{\bar{z}} \in (0, \tilde{T}^0_{\bar{z}}]$ such that $\Theta_{\bar{z}}/2 - N_{\bar{z}} T_{\bar{z}} =^{\mathrm{def}} \tilde{K}_{\bar{z}} > 0$ and from the above estimation we have

(20) $$\|\pi\, \tilde{z}(t)\| \geqq \tilde{K}_{\bar{z}} t^p \quad \text{for all} \quad t \in [0, T_{\bar{z}}]\,.$$

Choose a positive constant $C_{\bar{z}}$ such that $C_{\bar{z}} > (T_z)^{-1}$ and

(21) $$\|\pi(P_0(z) + P(z, u, w, v))\| < \tfrac{1}{2} C_{\bar{z}} \quad \text{for all} \quad z \in V_{\bar{z}} \quad \text{and all} \quad (u, w, v) \in U \times W \times V\,.$$

If $z_0 \in V_{\bar{z}}$ and $t \in [0, T_{\bar{z}}]$, then the solution $z(t)$ of the equation (11) with the initial value z_0 satisfies the inequality

(22) $$\varrho(z(t), M) \geqq \varrho(z_0, M) - t(C_{\bar{z}}/2) \quad \text{for all} \quad u \in \mathcal{M}(J, U), v \in \mathcal{M}(J, V), \quad \text{and all} \quad w \in \mathcal{M}(J, W)\,.$$

Indeed, we have

$$\varrho(z(t), M) = \|\pi\, z(t)\| = \Big\|\pi z_0 + \int_0^t (\pi(P_0(z(s)) +$$
$$+ F(z(s), u(s), w(s), v(s))\, ds\Big\| \geqq \varrho(z_0, M) -$$

$$- \int_0^t \|\pi(P_0(z(s)) + F(z(s), u(s), w(s), v(s))\| \, ds \geqq \varrho(z_0, M) - \tfrac{1}{2} t C_{\bar{z}}.$$

If we suppose that $\varrho(z_0, M) \leqq 1$ for $z_0 \in V_{\bar{z}}$, then (22) implies the inequality

(23) $$\varrho(z(t), M) \geqq \tfrac{1}{2}\varrho(z_0, M)/2 \quad \text{for all} \quad t \in [0, \tfrac{1}{2}\varrho(z_0, M)/C_{\bar{z}}] \, .$$

For each $z_0 \in V_{\bar{z}}$ choose a positive number $\varepsilon(z_0)$ such that

(24) $$\tilde{K}_{\bar{z}} t^p \geqq \tfrac{1}{2} K_{\bar{z}} t^p + \varepsilon(z_0) \quad \text{for all} \quad t \geqq \tfrac{1}{2}\varrho(z_0, M)/C_{\bar{z}} \, .$$

Take the evasion strategy $\tilde{v}^u_{\bar{z}}(z_0, t)$ for the game $(\mathrm{EG})_{\mathrm{ex}}$ and apply Lemma 1 for $\varepsilon = \varepsilon(z_0)$ and $T = T_{\bar{z}}$. By this lemma there exists an evasion strategy $v^u_{\bar{z}}(z_0, t)$ of the game (FG) defined for all $z_0 \in V_{\bar{z}}$, $z_0 \notin M$ and $\|z(t) - \tilde{z}(t)\| < \varepsilon(z_0)$ for all $t \in [0, T_{\bar{z}}]$, where one can choose $\varepsilon(z_0)$ such that also $\|\pi\, z(t) - \pi\, \tilde{z}(t)\| < \varepsilon(z_0)$ ($z(t)$, $\tilde{z}(t)$ are solutions of (11) and (12), respectively, with the initial value z_0). Then we have $\varrho(z(t), M) = \|\pi\, \tilde{z}(t) + \pi\, z(t) - \pi\, \tilde{z}(t)\| \geqq \|\pi\, \tilde{z}(t)\| - \|\pi\, z(t) - \pi\, \tilde{z}(t)\| \geqq$ $\geqq \tilde{K}_{\bar{z}} t^p - \varepsilon(z_0) \geqq \tfrac{1}{2} K_{\bar{z}} t^p$ for all $t \in [(C_{\bar{z}})^{-1}\, \varrho(z_0, M), T_{\bar{z}}]$. Choose a positive number $K_{\bar{z}}$ such that $K_{\bar{z}} \leqq \tfrac{1}{2} C_{\bar{z}}$, $K_{\bar{z}} \leqq \tfrac{1}{2} K_{\bar{z}}$. Then

(25) $$\varrho(z(t), M) \geqq K_{\bar{z}} t^p$$

$$\varrho(z(t), M) \geqq K_{\bar{z}} \frac{\varrho(z_0, M)^p}{C^p_{\bar{z}}} \quad \text{for} \quad t \in [0, T_{\bar{z}}] \, ,$$

where $p = p(\bar{z})$.

We have constructed the local evasion strategy near the point $\bar{z}$. The process of globalization of this strategy and the construction of the functions T can be performed in the same way as in [5] and therefore we omit the rest of proof.

Example. Let the dynamic of the object O_1 be described by the equation

(29) $$\ddot{x} + (\alpha + \varphi(\dot{x}))\, \dot{x} = \varrho \tilde{u}$$

and the dynamic of the object O_2 is given by the equation

(30) $$\ddot{y} + \beta \dot{y} = \sigma \tilde{v} + \gamma \tilde{w} + p \, ,$$

where $x, y, p \in \mathbb{R}^m$, $\alpha, \beta, \varrho, \sigma, \gamma$ are positive constants, φ is a smooth function with real values and there exists a positive constant K such that $|\varphi(x)| \leqq K$ for all $x \in \mathbb{R}^m$. Assume that the control parameters $\tilde{u}, \tilde{v}, \tilde{w}$ be such that $\tilde{u} \in U = \{u \in \mathbb{R}^m : \|u\| \leqq 1\}$, $\tilde{v} \in V = \{v \in \mathbb{R}^m : \|v\| \leqq \tfrac{1}{2}\}$, $\tilde{w} \in W =^{\mathrm{def}} V$. Let the constraints be given by the inequality

(31) $$(q, y) \geqq 0 \, ,$$

where $q \in \mathbb{R}^m$ is a constant vector. If $z = (z_1, z_2, z_3) = (x - y, \dot{x}, \dot{z})$, then the system (29), (30) can be written in the form

(32) $$\begin{aligned} \dot{z}_1 &= z_2 - z_3 \\ \dot{z}_2 &= -(\alpha + \varphi(z_2))\, z_2 + \varrho \tilde{u} \\ \dot{z}_3 &= -\beta z_3 + \sigma \tilde{v} + \gamma \tilde{w} + p \end{aligned}$$

(compare it with the system (6)) and this system can be written in the form (11) (here we have (w, v) instead of v), where $P_0(z) = (z_2 - z_3, -(\alpha + \varphi(z_2)\, z_2, -\beta z_3)$ and $F(z, u, w, v) = {}'(0, \varrho\tilde{u}, \sigma\tilde{v} + \gamma\tilde{w} + p)$, Let $M = \{(z_1, z_2, z_3) \in \mathbb{R}^{3m}: z_1 = 0\}$. The orthogonal complement to M is $L = \{(z_1, z_2, z_3) \in \mathbb{R}^{3m}: z_2 = 0,\ z_3 = 0\}$. The orthogonal projection π be as in example of Section 1 and $C_0(z) =^{\text{def}} \text{id}$,

$$C_1(z) = D\, P_0(z) = \begin{bmatrix} 0 & I & -I \\ 0 & Q(z_2) & 0 \\ 0 & 0 & -\beta I \end{bmatrix},$$

I is the $m \times m$ unit matrix, $Q(z_1) = -(\alpha + \partial(\varphi_2(z_2)\, z_2)/\partial z_2)$. Obviously, $\pi\, C_0(z)\,.\, F(z, u, w, v) = 0$ for all $(z, u, w, v) \in \mathbb{R}^{3m} \times U \times W \times V$ and thus the evasion condition (E 1) – (i) for $p = 2$ is satisfied. The form of the system (32) implies that the assumptions (A), (B), (C) concerning the existence and uniqueness and the global existence of solutions of this system are satisfied. Obviously, $\pi\, C_1(z)\,.\, F(z, u, w, v) = (\varrho\tilde{u} + \sigma\tilde{v} + \gamma\tilde{w}, 0, 0)^* \in L$. If $\sigma > \varrho + \gamma$, then the set $\bigcap_{u \in U} \bigcap_{w \in W} \{\pi\, C_1(z)\,.\, F(z, u, w, v): v \in V\}$ is the ball in L with center at the origin and radius $\sigma - (\varrho + \gamma)$. Hence we have shown that the evasion condition (E 1) is satisfied. The system (32) can also be written in the form (9) (here we have (w, v) instead of v), where $x = (z_1, z_2), y = z_3, A(x, y) = (z_2 - z_3, -(\alpha + \varphi(z_2)\, z_2)), B(x, y, u, w, v) = (0, \varrho\tilde{u})^*$ (there we have $-\beta$ instead of β), $g(x, y, u, w, v) = \sigma\tilde{v}$ and $f(u, w) = \gamma\tilde{w} + p$. Obviously, $\Gamma = \bigcap_{u \in U} f(u, W) = \gamma W + p$ and $\omega =^{\text{def}} p \in \Gamma$. If $(p, q) > 0$ and $\sigma < (p, q) : \|q\|$, then $\|g(x, y, u, w, v)\| \leqq \sigma < (p, q)/\|q\|$ and hence under these conditions also the evasion condition (E 2) is satisfied. Therefore Theorem 2 implies that there exists an evasion strategy for our evasion game with constraints.

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