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# CONTROLLABILITY OF NONLINEAR DELAY SYSTEMS WITH DELAY DEPENDING ON STATE VARIABLE

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A theorem is proved on the existence of solution of a certain type of nonlinear delay systems with implicit derivative by Darbo's fixed point theorem. Sufficient conditions are established for global controllability of such systems.

## 1. INTRODUCTION

Controllability of nonlinear systems with various types of delays in control variables has been studied by several authors [2, 9, 10, 11, 14] by means of Schauder's fixed point theorem. Using the same principle Mirza and Womack [12] and Dauer and Gahl [7] have discussed the controllability of nonlinear delay systems. In [8] Gahl investigated this problem for nonlinear systems of neutral type. Dacka [4] introduced a new method of analysis to study the controllability of nonlinear systems with implicit derivative with aid of Darbo's fixed point theorem. This method is extended by Dacka [5, 6] to nonlinear delay systems having implicit derivative and Balachandran and Somasundaram [1, 3, 13] to systems with different types of delays in control variables. In this paper we shall study the existence of solution and controllability of nonlinear delay systems in which the delay depends on the state variable.

This paper is similar in spirit to the authors' paper [1, 3] and hence the mathematical preliminaries relating to the measure of noncompactness of a set and Darbo's fixed point theorem are omitted. For details the reader can refer to [3, 4].

## 2. BASIC ASSUMPTIONS AND DEFINITIONS

Consider the following nonlinear delay system with implicit derivative as represented by the differential equation

(1) $$\dot{x}(t) = f(u(t), x(t - h(x(t), t)), \dot{x}(t), t) + B(t)\, u(t), \quad t \geqq t_0$$
$$x(t) = \phi(t) \quad \text{for} \quad t \leqq t_0$$

where $x \in \mathbb{R}^n$, $u \in \mathbb{R}^m$, $f$ is an $n$-vector function, $B(t)$ is $n \times m$ matrix and $h(x(t), t) \geqq 0$. Let the initial function $\phi(t)$ and the delay $h(x(t), t)$ be continuous. Set $\alpha(t) = t - h(x(t), t)$, $a = \inf \alpha(t)$ and $-\infty < a < t_0$ therefore $x(t) = \phi(t)$ on $[a, t_0]$.

Assume that the function $f(u, x, y, t)$ is continuous and satisfies the following condition

$$(2) \qquad |f(u, x, y, t)| \leqq M \quad \text{for} \quad u \in \mathbb{R}^m, \quad x, y \in \mathbb{R}^n, \quad t \in [t_0, t_1]$$

and for each $y, \bar{y} \in \mathbb{R}^n$ and $u \in \mathbb{R}^m$, $x \in \mathbb{R}^n$, $t \in [t_0, t_1]$

$$(3) \qquad |f(u, x, y, t) - f(u, x, \bar{y}, t)| \leqq k|y - \bar{y}|$$

where $M$ and $k$ are positive constants such that $0 \leqq k < 1$. Moreover assume that $\|B(t)\| \leqq N$, where $N$ is positive constant. Define

$$W(t_0, t_1) = \int_{t_0}^{t_1} B(s)\, B'(s)\, ds$$

where the prime indicates the matrix transpose. The norm of a continuous matrix valued function is taken as in [2, 3].

Let $\phi(t)$ be a continuous $R^n$-valued function defined on $[a, t_0]$ and let $u(t)$ be a fixed continuous function on $[t_0, t_1]$.

**Definition 1.** The solution of (1) is the function $x(t)$ such that:

(i) $x(t)$ is defined and continuous on the interval $[a, t_1]$ and of class $C^1$ on $[t_0, t_1]$ such that at the point $t_0$ the right side derivative only is taken into account.

(ii) Equation (1) is satisfied by the function $x(t)$ in the interval $[t_0, t_1]$ where on the interval $[a, t_0]$ the function $x(t) = \phi(t)$.

In this paper we shall use the following definition of controllability due to Mirza and Womack [12].

**Definition 2.** For system (1), an initial function $\phi \in C_n[a, t_0]$ at time $t_0$ is said to be controllable to the origin if, for some $t_1 > t_0$, there exists a control function $u: [t_0, t_1] \to \mathbb{R}^m$ such that the solution $x(t)$ of (1) exists and satisfies $x(t_1) = 0$. If this is true for all $\phi \in C_n[a, t_0]$ then the system is globally controllable to the origin.

We shall prove that the solution (1) exists under certain conditions. If $u(t)$ is considered as fixed for the instant then we may consider the equation

$$(4) \qquad \begin{aligned} \dot{x}(t) &= f(x(t - h(x(t), t)), \dot{x}(t), t), \quad t_0 \leqq t \leqq t_1 \\ x(t) &= \phi(t), \quad a \leqq t \leqq t_0 \end{aligned}$$

where the initial function $\phi(t)$ is continuous. Here the whole dependence on $u$ is concealed in the dependence on $t$. Therefore it is enough to give sufficient conditions for the existence of a solution of (4).

## 3. EXISTENCE THEOREM

**Theorem 1.** If the function $f(x, y, t)$ satisfies the conditions (2) and (3) with $h(x(t), t) \geqq 0$ then (4) has at least one solution for any initial function $\phi \in C_n[a, t_0]$.

Proof. Consider the Banach space $C_n^1[t_0, t_1]$ and the subset

$$H = [x : x \in C_n^1[t_0, t_1] , \quad x(t_0) = \phi(t_0)] .$$

For any function $x \in H$, we shall mean by $x(\alpha(t))$, the function defined in such a way that if $\alpha(t) < t_0$ for $t \in [t_0, t_1]$ then

$$x(\alpha(t)) = \phi(\alpha(t)) .$$

Define the mapping $T$ by

$$T(x)(t) = \phi(t_0) + \int_{t_0}^{t} f(x(s - h(x(s), s)) , \dot{x}(s), s) \, \mathrm{d}s .$$

Moreover, consider the bounded closed set $B$ in $H$ by

$$B = [x \in H : \|x\| \leqq r , \quad \|\dot{x}\| \leqq M]$$

where $M$ and $r$ are positive constants such that

$$r = |\phi(t_0)| + M(t_1 - t_0) .$$

Since $f$ is continuous, $T$ is continuous and maps $B$ into itself. Using similar argument as in [1, 3] we obtain

$$w(DTx) \leqq k \, w(Dx, h) + \beta(h).$$

Therefore, we have

$$\mu(TE) \leqq k \, \mu(E)$$

for any bounded set $E \subset B \subset H$. Hence by Darbo's fixed point theorem, the mapping $T$ has a fixed point $x \in C_n^1[t_0, t_1]$ such that

$$x(t) = T(x)(t).$$

Clearly the extension of this function to the interval $[a, t_0]$ by means of the function $\phi$ is a solution of equation (4) of the following form

$$x(t) = \phi(t_0) + \int_{t_0}^{t} f(x(s - h(x(s), s)), \dot{x}(s), s) \, \mathrm{d}s \quad t \geqq t_0$$

$$x(t) = \phi(t) , \quad a \leqq t \leqq t_0 .$$

**Remark 1.** It should be observed that if we assume that the function $f$ and $h$ satisfy also Lipschitz condition with respect to the state variables, then the uniqueness of the solution of (4) can be established by standard techniques used in proving uniqueness theorems.

## 4. MAIN RESULT

**Theorem 2.** Given the system (1) with conditions (2) and (3) and the matrix $W(t_0, t_1)$ is nonsingular for some $t_1 > t_0$. Then the system (1) is globally controllable to the origin.

Proof. Let $\phi \in C_n[a, t_0]$ be an arbitrary initial function. Consider the Banach space $Q = C_m[t_0, t_1] \times C_n^1[t_0, t_1]$. Define the following nonlinear transformation of the space $Q$ by

$$T([u, x])(t) = [T_1([u, x])(t), T_2([u, x])(t)]$$

where $T_1$ and $T_2$ are defined as follows

(5)
$$T_1([u, x])(t) = \\ = -B'(t)\, W^{-1}(t_0, t_1)\left[\phi(t_0) + \int_{t_0}^{t_1} f(u(s), x(\cdot), \dot{x}(s), s)\, \mathrm{d}s\right]$$

and

(6)
$$T_2([u, x])(t) = \\ = \phi(t_0) + \int_{t_0}^{t} B(s)\, T_1([u, x])(s)\, \mathrm{d}s + \int_{t_0}^{t} f(T_1([u, x])(s), x(\cdot), \dot{x}(s), s)\, \mathrm{d}s$$

where if $\alpha(t) < t_0$, then we assume that

$$x(\alpha(t)) = \phi(\alpha(t)).$$

By the definition of $T$, the operator $T$ is continuous and maps the space $Q$ into $Q$.

Let us consider the closed convex subset of $Q$

$$H = \{[u, x] \in Q: \|u\| \leqq K_1, \|x\| \leqq K_2, \|Dx\| \leqq K_3\}$$

where the positive constants $K_1, K_2$ and $K_3$ are as follows

$$K_1 = N\|W^{-1}(t_0, t_1)\|\,[|\phi(t_0)| + (t_1 - t_0)\, M]$$
$$K_2 = |\phi(t_0)| + (t_1 - t_0)\, NK_1 + (t_1 - t_0)\, M$$
$$K_3 = NK_1 + M\,.$$

It is easily seen that $T$ transforms $H$ into $H$ and for each pair $[u, x] \in H$, we have

$$w(T_1([u, x]), h) \leqq w(B', h)\, q$$

where

$$q = \sup_{[u,x]\in H} \|W^{-1}(t_0, t_1)\| \left[\phi(t_0) + \int_{t_0}^{t_1} f[u(s), x(\cdot), \dot{x}(s), s)\, \mathrm{d}s\right].$$

Since the function $B$ does not depend on the choice of the points in $H$, all the functions $T_1([u, x])(t)$ have a uniformly bounded modulus of continuity, therefore, they are equicontinuous. Further the function $f$ is bounded, all the functions $T_1([u, x])(t)$

are equicontinuous and uniformly bounded, hence they form a compact subset in the space $C_m[t_0, t_1]$. Note that all the functions $T_2([u, x])(t)$ are equicontinuous and by similar argument we can find the modulus of continuity of $DT_2([u, x])(t)$ for $t, s \in [t_0, t_1]$ as

$$|DT_2([u, x])(t) - DT_2([u, x])(s)| \leqq k|\dot{x}(t) - \dot{x}(s)| + \beta(|t - s|)$$

hence

$$w(DT_2([u, x]), h) \leqq k\, w(Dx, h) + \beta(h).$$

Thus we have for any set $E \subset H$

$$w_0(T_1E) = 0 \quad \text{and} \quad w_0(DT_2E) \leqq k\, w_0(DE_2)$$

where $E_2$ is the natural projection of the set $E$ on $C_n^1[t_0, t_1]$. Hence, it follows that

$$\mu(TE) \leqq k\, \mu(E).$$

By the Darbo fixed point theorem the mapping $T$ has at least one fixed point; therefore, there exist functions $u \in C_m[t_0, t_1]$ and $x \in C_n^1[t_0, t_1]$ such that

$$u(t) = T_1([u, x])(t) \tag{7}$$

$$x(t) = T_2([u, x])(t). \tag{8}$$

Extending the function (8) by the function $\phi(t)$ to $a \leqq t \leqq t_0$ and hence we have

$$u(t) = -B'(t)\, W^{-1}(t_0, t_1) \left[ \phi(t_0) + \int_{t_0}^{t_1} f(u(s), x(\cdot), \dot{x}(s), s)\, ds \right] \tag{9}$$

and

$$x(t) = \phi(t_0) + \int_{t_0}^{t} f(u(s), x(\cdot), \dot{x}(s), s)\, ds + \int_{t_0}^{t} B(s) u(s)\, ds, \quad t \in [t_0, t_1] \tag{10}$$

$$x(t) = \phi(t) \quad \text{for} \quad t \in [a, t_0].$$

Substituting (9) into (10) we see that $x(t_1) = 0$. Hence there exists a control $u(t)$ which satisfies the condition of Definition 2. Since $\phi$ was arbitrary, the system (1) is globally controllable. □

**Remark 2.** Using the method parallel to that of Theorem 2, we can establish the sufficient conditions for local relative controllability of the system (1).

**Example.** Consider the system

$$\dot{x}_1(t) = \frac{x_1(t) + x_2(t - (1 - \sin x_1(t)))}{1 + x_1^2(t) + x_2^2(t - (1 - \sin x_1(t)))} + \cos t\, u_1(t) + \sin t\, u_2(t)$$

$$x_2(t) = \frac{x_1(t)}{1 + x_2^2(t)} + \tfrac{1}{4} \sin(\dot{x}_1(t) + \dot{x}_2(t)) - \sin t\, u_1(t) + \cos t\, u_2(t)$$

then $W(0, t_1) = t_1 I$, where $I$ is the identity matrix, is nonsingular if $t_1 > 0$. The function $f$ is continuous and bounded and satisfies the Lipschitz condition with respect to $\dot{x}$ with constant $k = \frac{1}{2}$. Hence, by Theorem 2, the system is globally controllable.

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