KYBERNETIKA — VOLUME 22 (1986), NUMBER 2

# A NOTE ON ESTIMATION IN CONTROLLED DIFFUSION PROCESSES

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A system described by a stochastic differential equation is considered. Its evolution is affected by a nonanticipative control. The drift function involves an unknown parameter. The local asymptotic normality is proved for the corresponding family of distributions. The control, providing good quality of the identification, is exhibited.

## 1. INTRODUCTION

Let us consider a system $\mathscr{S}$. Its evolution is described by a stochastic process $X = \{X_t, t \geqq 0\}$ with the state space equal to the real line $\mathbb{R}$. We assume that the process $X$ is a diffusion one but depends on a control process $U = \{U_t, t \geqq 0\}$. The states of the control process, control parameters, range over a set $\mathscr{U} \subset \mathbb{R}^q$. To keep the presentation concise and simple we limit ourselves to the family of $X$ given by the following stochastic differential equation

$$(1) \qquad \mathrm{d}X_t = [a_1(X_t, U_t) + \alpha\, a_2(X_t)]\, \mathrm{d}t + \sigma(X_t)\, \mathrm{d}W_t\,, \quad X_0 = x_0 \in \mathbb{R}\,,$$

where $\alpha$ is an unknown parameter taking its values in $A$. Let us assume $A \subset \mathbb{R}$ is open, $\{W_t, t \geqq 0\}$ is a standard Wiener process.

The control parameter is chosen in the dependence on the past trajectory. Its value at time $t$ is $U_t = u_t\,(X_s, 0 \leqq s \leqq t)$. The control $U$ is stationary (homogeneous and Markovian) if $U_t = u(X_t)$, $t \geqq 0$. The set of all stationary controls is denoted by $\mathscr{U}^0$.

Let us introduce basic hypotheses concerning the equation (1).

**Assumption 1.** Let the diffusion coefficient $\sigma(x)$ be positive and Lipschitz continuous on $\mathbb{R}$.

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Denote by $\mathscr{C}_T$ the space of all continuous functions on $[0, T]$, by $\mathscr{C}_T$ the Kolmogorov $\sigma$-field on $\mathscr{C}_T$ and by $P_0$ the distribution of $Y$, satisfying $\mathrm{d}Y_t = \sigma(Y_t)\,\mathrm{d}W_t$, $Y_0 = x_0$, on $(\mathscr{C}_T, \mathscr{C}_T)$. Under Assumption 1 $P_0$ exists and it is unique.

**Assumption 2.** For every $K < \infty$ $\sup_{|x|<K, u\in\mathscr{U}} |a_1(x, u)| < \infty$. The function $a_1(x, u)$ is continuous in $u$ uniformly with respect to $x$ and $a_2(x) \not\equiv 0$ is continuous on $\mathbb{R}$.

**Assumption 3.** For $a(x, \alpha) = \sup_{u\in\mathscr{U}} |a_1(x, u) + \alpha\, a_2(x)|$ there holds

$$\mathsf{E}_0 \exp\left\{\frac{1}{2}\int_0^T \left[\frac{a(X_t, \alpha)}{\sigma(X_t)}\right]^2 \mathrm{d}t\right\} < \infty\,, \quad \alpha \in A\,,$$

where $\mathsf{E}_0$ denotes the mathematical expectation under $P_0$.

Under Assumptions 1, 2, 3 there exists a weak solution of (1). The detailed proof can be found in [1].

## 2. LOCAL ASYMPTOTIC NORMALITY

First, the definition of the local asymptotic normality is recalled. Let $U = \{U_t, t \geqq 0\}$ be a nonanticipative control. The probability distributions of the process $\{X_t, 0 \leqq t \leqq T\}$ form the family

$$\{P_\alpha^{T,U}, \alpha \in A, T \geqq 0\}\,. \tag{2}$$

The initial position $x_0$ is supposed to be independent of $\alpha$. Due to Assumption 1, 2, 3 the mutual probability density function of the processes corresponding to $\alpha, \alpha' \in A$ equals

$$\frac{\mathrm{d}P_{\alpha'}^{T,U}}{\mathrm{d}P_\alpha^{T,U}} = \exp\left\{(\alpha' - \alpha)\int_0^T \frac{a_2(X_t)}{\sigma^2(X_t)}\,\mathrm{d}X_t - (\alpha' - \alpha)\int_0^T \frac{a_1(X_t, U_t)\,a_2(X_t)}{\sigma^2(X_t)}\,\mathrm{d}t - \right.$$
$$\left. - (\alpha'^2 - \alpha^2)/2\int_0^T \left[\frac{a_2(X_t)}{\sigma(X_t)}\right]^2 \mathrm{d}t\right\}. \tag{3}$$

**Definition 1.** (2) is locally asymptotically normal at $\alpha \in A$ (LAN at $\alpha \in A$) if for any $h \in \mathbb{R}$ (3) conforms to

$$\frac{\mathrm{d}P_{\alpha+h/\sqrt{T}}^{T,U}}{\mathrm{d}P_\alpha^{T,U}} = \exp\left\{h\,.\,\Delta_\alpha^{T,U} - \frac{h^2}{2}\,\Gamma_\alpha^U + Q_\alpha^{T,U}(h)\right\}, \tag{4}$$

where for underlying distribution $P_\alpha^{T,U}$ $\Delta_\alpha^{T,U}$ has asymptotically normal distribution $N(0, \Gamma_\alpha^U)$, $\Gamma_\alpha^U > 0$, and $Q_\alpha^{T,U}(h)$ tends to zero in probability as $T \to \infty$. If (2) is LAN at $\alpha \in A$ for all $\alpha \in A$ then (2) is LAN at $A$.

**Definition 2.** The control $U$ is asymptotically stationary ($U \sim u^\alpha$) if to any $\alpha \in A$

there exists $u^\alpha \in \mathscr{U}^0$ such that

$$(5) \qquad \lim_{t\to\infty} \|U_t - u^\alpha(X_t)\| = 0 \quad \text{a.s.} \quad P_\alpha^{t,U} .$$

Substituting $\alpha + h/\sqrt{T}$ for $\alpha'$ in (3) we have

$$\exp\left\{\frac{h}{\sqrt{T}}\int_0^T \frac{a_2(X_t)}{\sigma(X_t)}\,\mathrm{d}W_t - \frac{h^2}{2T}\int_0^T \left[\frac{a_2(X_t)}{\sigma(X_t)}\right]^2 \mathrm{d}t\right\}.$$

To verify LAN for (2) it is sufficient to prove that

$$(6) \qquad \lim_{T\to\infty} T^{-1}\int_0^T \left[\frac{a_2(X_t)}{\sigma(X_t)}\right]^2 \mathrm{d}t = \Gamma_\alpha^U > 0 \quad \text{in probability} \quad P_\alpha^{T,U} .$$

It is known, e.g. [1], that if (6) is true then for

$$\Delta_\alpha^{T,U} = T^{-1/2}\int_0^T \frac{a_2(X_t)}{\sigma(X_t)}\,\mathrm{d}W_t$$

the central limit theorem holds, it means, that $\Delta_\alpha^{T,U}$ has asymptotically $N(0, \Gamma_\alpha^U)$, as $T \to \infty$. Simultaneously

$$Q_\alpha^{T,U}(h) = \frac{h^2}{2}\left[\Gamma_\alpha^U - T^{-1}\int_0^T \left[\frac{a_2(X_t)}{\sigma(X_t)}\right]^2 \mathrm{d}t\right]$$

tends to zero in probability $P_\alpha^{T,U}$. To ensure the validity of (6) we follow the procedure introduced in [1].

Using the notation of [1] or [4] let

$$a^+(x, \alpha) = \sup_{u\in\mathscr{U}} (a_1(x, u) + \alpha\, a_2(x)) ,$$

$$a^-(x, \alpha) = \inf_{u\in\mathscr{U}} (a_1(x, u) + \alpha\, a_2(x)) ,$$

$$I_+(x, \alpha) = 2\int_0^x a^+(y, \alpha)\, \sigma^{-2}(y)\, \mathrm{d}y ,$$

$$m_+(x, \alpha) = 2\int_0^x \mathrm{e}^{I_+(y,\alpha)}\, \sigma^{-2}(y)\, \mathrm{d}y ,$$

$$p_+(x, \alpha) = \int_0^x \mathrm{e}^{-I_+(y,\alpha)}\, \mathrm{d}y ,$$

$I_-(x, \alpha)$, $m_-(x, \alpha)$, $p_-(x, \alpha)$ are defined similarly. The integrals are assumed to exist.

The constant $\Gamma_\alpha^U$ is in fact determined by $u^\alpha(x)$. Were $U$ identically stationary control defined by $u^\alpha(x)$ the procedure how to get $\Gamma_\alpha^U$ would be as follows. $\Gamma_\alpha$ is the unique number such that the following differential equation

$$(7) \qquad \frac{\sigma^2(x)}{2}\, v''(x) + [a_1(x, u^\alpha(x)) + \alpha\, a_2(x)]\, v'(x) + \left[\frac{a_2(x)}{\sigma(x)}\right]^2 - \Gamma_\alpha = 0$$

has a solution satisfying the boundary conditions

$$\lim_{x \to \pm\infty} v'(x) \exp\left\{2 \int_0^x \frac{a_1(y, u^\alpha(y)) + \alpha\, a_2(y)}{\sigma^2(y)}\, \mathrm{d}y\right\} = 0\,.$$

Some questions concerning the solvability of the equation (7) were treated in [6]. The auxiliary function $v(x)$ will be an efficient instrument in our further proofs. From Assumption 2 and (7) it follows that $\Gamma_\alpha$ is positive.

Let us denote

$$\varphi(x, u) = \frac{\sigma^2(x)}{2}\, v''(x) + [a_1(x, u) + \alpha\, a_2(x)]\, v'(x) + \left[\frac{a_2(x)}{\sigma(x)}\right]^2 - \Gamma_\alpha\,, \tag{8}$$

$$L_t = \int_0^t \left[\frac{a_2(X_s)}{\sigma(X_s)}\right]^2 \mathrm{d}s - \Gamma_\alpha \,.\, t + v(X_t) - v(x_0) - \int_0^t \varphi(X_s, U_s)\, \mathrm{d}s\,. \tag{9}$$

**Assumption 4.** Let the following integrals be finite, $B(s) = 1 + \sigma^2(s)\, v'(s)^2$.

$$\begin{aligned} &\int_0^x \int_y^\infty B(s)\, \mathrm{d}m_+(s, \alpha)\, \mathrm{d}p_+(y, \alpha) = w_{1+}(x)\,, \\ &2\int_0^x \int_y^\infty B(s)\, w_{1+}(s)\, \mathrm{d}m_+(s, \alpha)\, \mathrm{d}p_+(y, \alpha) = w_{2+}(x)\,, \quad x \geqq 0\,, \end{aligned} \tag{10}$$

$$\begin{aligned} &\int_x^0 \int_{-\infty}^y B(s)\, \mathrm{d}m_-(s, \alpha)\, \mathrm{d}p_-(y, \alpha) = w_{1-}(x)\,, \\ &2\int_x^0 \int_{-\infty}^y B(s)\, w_{1-}(s)\, \mathrm{d}m_-(s, \alpha)\, \mathrm{d}p_-(y, \alpha) = w_{2-}(x)\,, \quad x < 0\,, \end{aligned} \tag{11}$$

while $w_{1+}(\infty) = w_{1-}(-\infty) = \infty$.

**Lemma 1.** Under Assumption 1–4 $\{X_t, t \geqq 0\}$ given by (1) is bounded in probability and $\{L_t, t \geqq 0\}$ is a local martingal fulfilling the law of large numbers.

Proof. The process $X$ is bounded in probability if

$$\lim_{R \to \infty} \sup_{t \geqq 0} P_\alpha^{t,U}(|X_t| > R) = 0\,. \tag{12}$$

We use the method introduced in [1] (the proof of Theorem 4). Applying Itô's formula to $\mathrm{d}w_{1+}(X_s)$ resp. $\mathrm{d}w_{1-}(X_s)$ and the differential operators

$$D_{m_+} D_{p_+} = \frac{\sigma^2(x)}{2} \frac{\mathrm{d}^2}{\mathrm{d}x^2} + a^+(x, \alpha) \frac{\mathrm{d}}{\mathrm{d}x}$$

resp.

$$D_{m_-} D_{p_-} = \frac{\sigma^2(x)}{2} \frac{\mathrm{d}^2}{\mathrm{d}x^2} + a^-(x, \alpha) \frac{\mathrm{d}}{\mathrm{d}x}$$

on (10) resp. (11), we find the majorant for $\mathsf{E}_\alpha^{t,U} w_{1+}(X^t)$ resp.. $\mathsf{E}_\alpha^{t,U}\, w_{1-}(X^t)$. Thus,

$$\sup_{t \geqq 0} P_\alpha^{t,U}(|X_t| > R) = \sup_{t \geqq 0} \int_{|X_t| > R} \mathrm{d}P_\alpha^{t,U} \leqq \text{const.}\, (w_{1+}^{-1}(R) + w_{1-}^{-1}(R))\,,$$

and (12) is a straighforward consequence of it. Substituting (8) into (9) we obtain

$$L_t = \int_0^t \sigma(X_s)\, v'(X_s)\, \mathrm{d}W_s\,, \quad t \geqq 0\,,$$

and that is as generally known a local martingale. A sufficient condition for the validity of the law of large numbers is according to [1] the following one

$$\lim_{t\to\infty} t^{-2} \operatorname{loglog} t \int_0^t [\sigma(X_s)\, v'(X_s)]^2 \,\mathrm{d}s = 0 \quad \text{in probability}\,. \tag{13}$$

It suffices to prove that

$$\limsup_{t\to\infty} t^{-1} \int_0^t [\sigma(X_s)\, v'(X_s)]^2 \,\mathrm{d}s < \infty \quad \text{a.s.}$$

Using the method of the proof introduced in [1] (Theorem 5) simultaneously with (10) and (11), (13) is proved without difficulties. □

**Assumption 5.** The function $\varphi(x, u)$ defined by (8) has the following property.

$$\varphi(x, u) = \varphi_1(x) \,.\, \varphi_2(x, u) \tag{14}$$

where $\varphi_2(x, u)$ is continuous in $u$ uniformly with respect to $x$, $\varphi_2(x, u^\alpha(x)) = 0$ and $\mathsf{E}_\alpha^{t,U}|\varphi_1(X_t)| \leqq \text{const.} < \infty$.

**Theorem 1.** Under Assumptions 1–5 the family (2) is LAN at $\alpha$ for the asymptotically stationary control $U$.

Proof. As it was mentioned in advance it remains to prove (6). Let $\varepsilon > 0$. Denote

$$A = \{|\varphi_2(X_t, U_t)| < \varepsilon^2, t \geqq t_0\}\,.$$

Since $|\varphi_2(X_t, U_t)| = |\varphi_2(X_t, U_t) - \varphi_2(X_t, u^\alpha(X_t))|$, then $P_\alpha^{t_0 U}(A) > 1 - \varepsilon$ for $t_0$ sufficiently large. Thus

$$\mathsf{E}_\alpha^{t,U} \chi_A |\varphi(X_t, U_t)| \leqq \varepsilon^3 \text{ cost.}\,, \quad t \geqq t_0$$

and

$$\limsup_{t\to\infty} \mathsf{E}_\alpha^{t,U} \chi_A t^{-1} \int_0^t |\varphi(X_s, U_s)| \,\mathrm{d}s \leqq \varepsilon^2 \text{ const.}$$

Therefore

$$\lim_{t\to\infty} t^{-1} \int_0^t \varphi(X_s, U_s) \,\mathrm{d}s = 0 \quad \text{in prob.}$$

Then using expression (9) and Lemma 1 we obtain (6). □

**Example 1.** Consider a linear case,

$$\mathrm{d}X_t = -(U_t + \alpha X_t)\,\mathrm{d}t + \mathrm{d}W_t\,,$$

where $\alpha \in A = (0, \infty)$ and $U_t = z(\tilde{\alpha}_t)\, X_t$, $z(y)$ is positive, continuous and bounded on $\mathbb{R}$. If $\tilde{\alpha}_t$ is a strongly consistent estimate of the unknown parameter $\alpha$, then $U = \{U_t, t \geqq 0\}$ is the asymptotically stationary control.

The problem of LAN in controlled Markov chains was treated in [5]. The conditions for a finite state space and also countable state space chains were set up.

## 3. OPTIMAL CONTROL

For the locally asymptotically normal families of distributions Theorem of J. Hájek [2] or [3] holds. Its special case is the asymptotic Rao-Cramér inequality

$$\liminf_{t \to \infty} \sup_{|a - \alpha| < \delta} T \mathsf{E}_a^{T,U} (\tilde{\alpha}_T - a)^2 \geqq 1/\Gamma_\alpha^U , \quad \delta > 0 , \tag{15}$$

for any family of estimates $\{\tilde{\alpha}_T, T \geqq 0\}$ of the unknown parameter $\alpha$. (15) provides a bound on the quality of the identification of the system $\mathcal{S}$. So, during the identification stage the control $\hat{U}$ is to be used for which

$$\lim_{T \to \infty} T^{-1} \int_0^T \left[ \frac{a_2(X_t)}{\sigma(X_t)} \right]^2 \mathrm{d}t = \sup_U \Gamma_\alpha^U = \Gamma_\alpha^{\hat{U}} = \hat{\Gamma}_\alpha . \tag{16}$$

The Bellman equation for the criterion (16) has the following form

$$\max_{u \in \mathcal{U}} \left\{ \frac{\sigma^2(x)}{2} w''(x) + [a_1(x, u) + \alpha\, a_2(x)]\, w'(x) + \left[ \frac{a_2(x)}{\sigma(x)} \right]^2 - \hat{\Gamma}_\alpha \right\} = 0 . \tag{17}$$

Some questions concerning a solution of (17) are discussed in [6]. The stationary control achieving $\hat{\Gamma}_\alpha$ is given by maximizer $\hat{u}(x, \alpha)$ of the expression in curly brackets in (17). A control that suits to (16) is obtained according to [1] by substituting a strong consistent estimate $\hat{\alpha}_t$ for $\alpha$ in the optimal stationary control given by $\hat{u}(x, \alpha)$. Thus $\hat{U} = \{\hat{U}_t, t \geqq 0\}$ corresponds to $\hat{U}_t = \hat{u}(X_t, \hat{\alpha}_t)$.

The maximum likelihood estimate of $\alpha$ equals see e.g., [1]

$$\hat{\alpha}_t = \frac{\int_0^t \frac{a_2(X_s)}{\sigma^2(X_s)} \mathrm{d}X_s - \int_0^t \frac{a_1(X_s U_s)\, a_2(X_s)}{\sigma^2(X_s)} \mathrm{d}s}{\int_0^t \left[ \frac{a_2(X_s)}{\sigma(X_s)} \right]^2 \mathrm{d}s} . \tag{18}$$

Under Assumptions 1–5 the estimate (18) has asymptotically normal distribution $N(\alpha, 1/t \,.\, \hat{\Gamma}_\alpha)$, $t \to \infty$, i.e. the best lower bound in (15) is reached.

**Example 2.** Take again

$$\mathrm{d}X_t = -(U_t + \alpha X_t)\, \mathrm{d}t + \sqrt{(2)}\, \mathrm{d}W_t , \quad t \geqq 0 ,$$

$\alpha \in (0, \infty)$, $U_t = z(\hat{\alpha}_t)\, X_t$, $0 < K_1 \leqq z(y) \leqq K_2 < \infty$, continuous on $\mathbb{R}$. Then (17) equals

$$\max_{K_1 \leqq z \leqq K_2} \{ w''(x) - (z + \alpha)\, x\, w'(x) + x^2/2 - \hat{\Gamma}_\alpha \} = 0 .$$

Maximum is achieved for $z = K_1$ independently of $x$ and

$$w(x) = \hat{\Gamma}_\alpha \, . \, x^2/2 \, , \quad \hat{\Gamma}_\alpha = [2(K_1 + \alpha)]^{-1} \, . \qquad \square$$

If condition (16) is used the expenses of the measurement are expressed through the duration of the experiment. Considering a more general evaluation, e.g., the costs of the measurement are taken into account, (16) should be replaced by the following expression

$$\lim_{T\to\infty} \frac{C_T}{D_T}\,, \tag{19}$$

where

$$D_T = \int_0^T \left[\frac{a_2(X_s)}{\sigma(X_s)}\right]^2 \mathrm{d}s \quad \text{and} \quad C_T = \int_0^T c(X_s, U_s, \alpha)\, \mathrm{d}s \, .$$

The function $c(x, u, \alpha)$ is supposed to be nonnegative and to fulfil assumptions analogous to Assumptions 4, 5, where $[a_2(x_t)/\sigma(X_t)]^2$ is replaced by $c(X_t, U_t, \alpha)$. The aim is to minimize (19). The minimal value is denoted by $\hat{\Theta}(\alpha)$ and the corresponding optimal control by $\hat{U}$. The Bellman equation for (19) reads

(20)

$$\min_{u\in\mathscr{U}} \left\{\frac{\sigma^2(x)}{2}\, w''(x) + [a_1(x, u) + \alpha\, a_2(x)]\, w'(x) + c(x, u, \alpha) - \left[\frac{a_2(x)}{\sigma(x)}\right]^2 \hat{\Theta}(\alpha)\right\} = 0 \, .$$

**Proposition 1.** Let $U$ be an asymptotically stationary control, $\{\tilde{\alpha}_t, t \geqq 0\}$ be any family of estimates of unknown parameter $\alpha$. Then an analogy of (15) holds for LAN at $\alpha$ families of distribution. Namely,

$$\lim_{\delta\to 0} \liminf_{T\to\infty} \sup_{|a-\alpha|<\delta} \mathsf{E}_a^{T,U} C_T \mathsf{E}_a^{T,U} (\tilde{\alpha}_T - a)^2 \geqq \hat{\Theta}(\alpha) \, . \tag{21}$$

Proof. Let $U \sim u^\alpha$. Then in probability

$$\lim_{T\to\infty} T^{-1} C_T = \hat{\Theta}_1(\alpha) \leqq \liminf_{T\to\infty} T^{-1} \mathsf{E}_\alpha^{T,U} C_T \quad \text{and}$$

$$\lim_{T\to\infty} T^{-1} D_T = \Theta_2(\alpha) = \Gamma_\alpha^U \, .$$

From the optimality of $\Theta(\alpha)$ with respect to (19) follows

$$\Theta_1(\alpha)/\Theta_2(\alpha) \geqq \hat{\Theta}(\alpha) \, .$$

Thus

$$\lim_{\delta\to 0} \liminf_{T\to\infty} \sup_{|a-\alpha|<\delta} \mathsf{E}_a^{T,U} C_T \mathsf{E}_a^{T,U} (\tilde{\alpha}_T - a)^2 \geqq$$

$$\geqq \lim_{\delta\to 0} \big( \inf_{|a-\alpha|<\delta} \Theta_1(a)\big) \liminf_{T\to\infty} \sup_{|a-\alpha|<\delta} T \mathsf{E}_a^{T,U} (\tilde{\alpha}_T - a)^2 \geqq \Theta_1(\alpha)/\Theta_2(\alpha) \geqq \hat{\Theta}(\alpha) \, . \qquad \square$$

The relation (21) gives asymptotically the minimal mean value of costs needed to achieve the given variance of $\tilde{\alpha}_T$, i.e., to attain the variance $V(\alpha)$ the minimal mean costs required are

$$\hat{\Theta}(\alpha)/V(\alpha) \, .$$

**Example 3.** We assume the linear case, i.e.,

$$\mathrm{d}X_t = -(aU_t + \alpha X_t)\,\mathrm{d}t + \sqrt{(2)}\,\mathrm{d}W_t\,, \quad \alpha \in (0, \infty)$$

is unknown. The criterion has the form (19), particularly, $c(x, u, \alpha) = c_0 + c_1 u^2$, where $a$, $c_0$, $c_1$ are known positive constants such that

$$2\alpha c_1 - c_0 a^2 > 0\,. \tag{22}$$

Substituting into (20) we have

$$\min_u \{w''(x) - (au + \alpha x)\,w'(x) + c_0 + c_1 u^2 - \hat{\Theta}(\alpha)\,x^2/2\} = 0\,.$$

The minimum is achieved for $u(x) = (a/2c_1)\,w'(x)$, $w(x)$ is a quadratic function and $w'(x) = -c_0 x$. The minimal value of the criterion equals

$$\hat{\Theta}(\alpha) = 2\alpha c_0 - \frac{a^2 c_0^2}{2c_1}\,.$$

It is positive as (22) holds. Thus, the process corresponding to the optimal control is described by the following stochastic differential equation

$$\mathrm{d}\hat{X}_t = -\left(\alpha - \frac{a^2 c_0}{2c_1}\right)\hat{X}_t\,\mathrm{d}t + \sqrt{(2)}\,\mathrm{d}W_t\,.$$

The optimal strategy during the identification stage $\hat{U}_t = -(ac_0/2c_1)\,.\,X_t$ is that one, under which the system has 'minimum stability'. □

If we have instead of (16) the criterion additively composed of two parts

$$J_T = \int_0^T c(X_s, U_s, \alpha)\,\mathrm{d}s + \int_0^T \left[\frac{a_2(X_s)}{\sigma(X_s)}\right]^2 \mathrm{d}s = \int_0^T j(X_s, U_s, \alpha)\,\mathrm{d}s\,,$$

then the theory presented in [1] can be used without any modification. The method of the insertion the strongly consistent estimate into the optimal stationary control gives the optimal procedure as it was mentioned formerly.

**Example 4.** Let $c(x, u, \alpha) = -(x^2 + \frac{1}{2}\lambda(\alpha)\,u^2)$, therefore

$$J_T = -\frac{1}{2}\int_0^T (X_s^2 + \lambda(\alpha)\,U_s^2)\,\mathrm{d}s\,,$$

where $\lambda(\alpha)$ is continuously differentiable with respect to $\alpha$, $0 < K \leqq \lambda(\alpha) < \infty$, $\mathrm{d}X_t = -(U_t + \alpha X_t)\,\mathrm{d}t + \sqrt{(2)}\,\mathrm{d}W_t$. This case was treated in [1], so only results are mentioned. It holds $\hat{\Theta}(\alpha) = \lambda(\alpha)\,(\alpha - \sqrt{(\alpha^2 + \lambda(\alpha)^{-1})})$ and the appropriate control is $\hat{U}_t = (\hat{\alpha}_t - \sqrt{[(\hat{\alpha}_t^2 + \lambda(\hat{\alpha}_t)^{-1})]})\,X_t$, where $\hat{\alpha}_t$ is a strongly consistent estimate of $\alpha$, e.g., MLE. □

## 4. CONCLUDING REMARKS

The stated theory can be generalized for processes satisfying instead of (1) a non-linear equation

$$\mathrm{d}X_t = a(X_t, U_t, \alpha)\,\mathrm{d}t + \sigma(X_t)\,\mathrm{d}W_t\,. \tag{23}$$

The modifications of Assumptions 2, 3 are straighforward. The likelihood function (3) has the expression

$$\frac{\mathrm{d}P^{T,U}_{\alpha+h/\sqrt{T}}}{\mathrm{d}P^{T,U}_{\alpha}} = \exp\left\{\int_0^T \frac{a(x_s, U_s, \alpha + h/\sqrt{(T)}) - a(X_s, U_s, \alpha)}{\sigma(X_s)}\,\mathrm{d}W_s - \right.$$

$$\left. - \frac{1}{2}\int_0^T \left[\frac{a(X_s, U_s, \alpha + h/\sqrt{(T)}) - a(X_s, U_s, \alpha)}{\sigma(X_s)}\right]^2 \mathrm{d}s\right\},$$

and therefore to prove the relation (4) Taylor's expansion and some additional conditions are needed. They are more complicated than Assumptions 4, 5, but they also concern the validity of the law of large numbers and the central limit theorem for local martingales. The reason for developing the theory for (1) instead of (23) was to keep the presentation coincise, illustrative and as simple as possible. The restriction to the one-dimensional unknown parameter follows the same reason.

(Received February 25, 1985.)

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