KYBERNETIKA — VOLUME 21 (1985), NUMBER 5

STATISTICAL ANALYSIS OF MULTIPLE MOVING AVERAGE PROCESSES USING PERIODICITY

TOMÁŠ CIPRA

A method of parameter estimation in multiple moving average models is suggested using periodic models. Identification of these models is also discussed. The results are demonstrated by means of numerical simulations.

1. INTRODUCTION

Cipra [3] investigated so called periodic moving average process $\{X_t\}$ of the form

$$X_t = \varepsilon_t + \beta_1(t)\,\varepsilon_{t-1} + \ldots + \beta_{q_t}(t)\,\varepsilon_{t-q_t}\,, \tag{1.1}$$

where the coefficients $\beta_j(t)$ are periodic functions of time with a period d, i.e.

$$\beta_j(t) = \beta_j(t+d)\,, \quad q_t = q_{t+d} \tag{1.2}$$

and $\{\varepsilon_t\}$ is a normal white noise with zero mean value and a variance σ^2. Such process is natural analogy of the periodic autoregressive process (see e.g. [1], [5], [6], [7]) and therefore the idea originates to use the estimation technique described in [3] for the treatment of the multiple moving average models (the same approach to the multiple autoregressive models is presented in [5] and [6]).

Let us consider a d-dimensional moving average process $\{\boldsymbol{X}_t\}$ of an order q. The corresponding model can be written in the form

$$\boldsymbol{X}_t = \boldsymbol{\beta}_0\boldsymbol{\varepsilon}_t + \boldsymbol{\beta}_1\boldsymbol{\varepsilon}_{t-1} + \ldots + \boldsymbol{\beta}_q\boldsymbol{\varepsilon}_{t-q}\,, \tag{1.3}$$

where $\boldsymbol{\beta}_j$ are $d \times d$ matrices of parameters such that $\boldsymbol{\beta}_0$ is lower triangular with unities on the main diagonal and $\{\boldsymbol{\varepsilon}_t\}$ is a d-dimensional normal white noise with zero mean vector and a diagonal variance matrix having positive numbers $\sigma_1^2, \ldots, \sigma_d^2$ on the main diagonal. Indeed, if we consider the following more usual form of the model for $\{\boldsymbol{X}_t\}$

$$\boldsymbol{X}_t = \boldsymbol{\eta}_t + \boldsymbol{\gamma}_1\boldsymbol{\eta}_{t-1} + \ldots + \boldsymbol{\gamma}_q\boldsymbol{\eta}_{t-q} \tag{1.4}$$

with a normal white noise $\{\boldsymbol{\eta}_t\}$ such that var $(\boldsymbol{\eta}_t)$ is a general positive definite matrix we can set $\boldsymbol{\varepsilon}_t = \boldsymbol{T}^{-1}\boldsymbol{\eta}_t$, $\boldsymbol{\beta}_0 = \boldsymbol{T}$ and $\boldsymbol{\beta}_j = \gamma_j \boldsymbol{T}$, $j = 1, \ldots, p$. The matrix $\boldsymbol{T}$ is lower diagonal with unities on the main diagonal taken from so called Cholesky decomposition var $(\boldsymbol{\eta}_t) = \boldsymbol{T}\boldsymbol{\Lambda}\boldsymbol{T}'$, where $\boldsymbol{\Lambda}$ is a diagonal matrix with positive numbers on the main diagonal. Thus we obtain the prescribed model (1.3) (it is var $(\boldsymbol{\varepsilon}_t) = \boldsymbol{\Lambda}$).

If we define univariate processes $\{X_t\}$ and $\{\varepsilon_t\}$ by means of the relations

$$(1.5) \qquad X_{j+d(t-1)} = X_{jt}\,, \quad \varepsilon_{j+d(t-1)} = \varepsilon_{jt}\,, \quad j = 1, \ldots, d\,,$$

where $\boldsymbol{X}_t = (X_{1t}, \ldots, X_{dt})'$ and $\boldsymbol{\varepsilon}_t = (\varepsilon_{1t}, \ldots, \varepsilon_{dt})'$ then $\{X_t\}$ is the periodic moving average process (1.1) with the period d and the orders $q_1 = 1 + dq$, $q_2 = 2 + dq, \ldots$ $\ldots, q_d = d + dq$. The only difference consists in the fact that the variances var $(\varepsilon_t) =$ $= \sigma_t^2$ are periodic fulfilling

$$(1.6) \qquad \sigma_t^2 = \sigma_{t+d}^2$$

and are not constant as in (1.1). In spite of it we shall show in the paper that the estimation method for (1.1) described in [3] can be extended also for the case with (1.6) so that we shall have in our disposal a method of estimating the parameters of the multiple moving average models.

First the method is demonstrated for two-dimensional case (i.e. $d = 2$) in Section 2 but then the general case is considered in Section 3. The possibility of identifying a multiple moving average process in addition to the previous estimation method is discussed in Section 4. Finally the results of some numerical simulations are given in Section 5.

2. CASE WITH DIMENSION TWO

We shall deal with a two-dimensional moving average process

$$(2.1) \qquad \begin{pmatrix} X_{1t} \\ X_{2t} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ \beta_{21}^0 & 1 \end{pmatrix} \begin{pmatrix} \varepsilon_{1t} \\ \varepsilon_{2t} \end{pmatrix} + \begin{pmatrix} \beta_{11}^1 & \beta_{12}^1 \\ \beta_{21}^1 & \beta_{22}^1 \end{pmatrix} \begin{pmatrix} \varepsilon_{1,t-1} \\ \varepsilon_{2,t-1} \end{pmatrix} + \ldots$$
$$\ldots + \begin{pmatrix} \beta_{11}^q & \beta_{12}^q \\ \beta_{21}^q & \beta_{22}^q \end{pmatrix} \begin{pmatrix} \varepsilon_{1,t-q} \\ \varepsilon_{2,t-q} \end{pmatrix},$$

where $\{(\varepsilon_{1t}, \varepsilon_{2t})'\}$ is a normal white noise with a variance matrix

$$(2.2) \qquad \text{var}\,\{(\varepsilon_{1t}, \varepsilon_{2t})'\} = \begin{pmatrix} \sigma_1^2 & 0 \\ 0 & \sigma_2^2 \end{pmatrix} \; (\sigma_1^2 > 0,\ \sigma_2^2 > 0)\,.$$

According to (1.5) we can write (2.1) as the following periodic moving average process with the period two (in addition we simplify the denotation and use new simpler symbols α_i and β_j instead of the previous symbols for the coefficients of the model)

$$(2.3) \qquad \begin{aligned} X_{2t-1} &= \varepsilon_{2t-1} + \alpha_1 \varepsilon_{2t-2} + \ldots + \alpha_{q_1} \varepsilon_{2t-1-q_1}\,, \\ X_{2t} &= \varepsilon_{2t} \quad + \beta_1 \varepsilon_{2t-1} + \ldots + \beta_{q_2} \varepsilon_{2t-q_2}\,, \end{aligned}$$

where

$$\text{(2.4)} \qquad \operatorname{var}(\varepsilon_{2t-1}) = \sigma_1^2\,, \quad \operatorname{var}(\varepsilon_{2t}) = \sigma_2^2\,.$$

Although $q_1 = 1 + 2q$ and $q_2 = 2 + 2q$ it shows convenient in practical situations not to use these constraints explicitly. If e.g. the matrix $\boldsymbol{\beta}_q$ in (2.1) has zero upper row (i.e. $\beta_{11}^q = \beta_{12}^q = 0$) then it must be $q_1 \leqq 1 + 2(q - 1)$ which simplifies the model. This is one of the advantages of the treatment of the multiple moving average models through the periodicity: although it may be e.g. $q_1 \ll q_2$ in (2.3) one must keep q large in (2.1) when the principle of periodicity is not applied. Newton [5] and Pagano [6] discuss the same effect in the autoregressive case.

The estimation method suggested in [3] consists in combining Durbin's [4] efficient estimation procedure for the classical (i.e. nonperiodic) univariate moving average processes and Pagano's [6] results on the periodic autoregressions.

Let us start approximating (2.3) by the following periodic autoregression with the period two

$$\text{(2.5)} \qquad \begin{aligned} X_{2t-1} + \gamma_1 X_{2t-2} + \ldots + \gamma_{k_1} X_{2t-1-k_1} &= \varepsilon_{2t-1}\,, \\ X_{2t} \quad + \delta_1 X_{2t-1} + \ldots + \delta_{k_2} X_{2t-k_2} &= \varepsilon_{2t}\,, \end{aligned}$$

where the numbers k_1 and k_2 are sufficiently large. For (2.5) to be admissible one should suppose that all roots of the characteristic equations

$$\text{(2.6)} \qquad \begin{aligned} z^{q_1} + \alpha_1 z^{q_1-1} + \ldots + \alpha_{q_1} &= 0\,, \\ z^{q_2} + \beta_1 z^{q_2-1} + \ldots + \beta_{q_2} &= 0 \end{aligned}$$

are in the absolute value less than one (i.e. the assumption of the invertibility of the univariate moving average models with coefficients $\alpha_1, \ldots, \alpha_{q_1}$ and $\beta_1, \ldots, \beta_{q_2}$ which cannot be replaced by the invertibility of the multiple model (2.1) – see e.g. Example in Section 5). However, the practical experiences show that the method usually works well even without this assumption.

According to [6] the parameters $\gamma = (\gamma_1, \ldots, \gamma_{k_1})'$ and $\delta = (\delta_1, \ldots, \delta_{k_2})'$ fulfil the following two systems of Yule-Walker equations

$$\text{(2.7)} \qquad R_1 \gamma = -g\,, \quad R_2 \delta = -h\,,$$

where the $k_1 \times k_1$ and $k_2 \times k_2$ matrices R_1 and R_2 and the vectors $g = (g_1, \ldots, g_{k_1})'$ and $h = (h_1, \ldots, h_{k_2})'$ are defined as

$$\text{(2.8)} \qquad \begin{aligned} R_1 &= \operatorname{var}\{(X_{2t-2}, X_{2t-3}, \ldots, X_{2t-1-k_1})'\}\,, \\ R_2 &= \operatorname{var}\{(X_{2t-1}, X_{2t-2}, \ldots, X_{2t-k_2})'\}\,, \\ g_i &= \operatorname{cov}(X_{2t-1}, X_{2t-1-i})\,, \quad i = 1, \ldots, k_1\,, \\ h_j &= \operatorname{cov}(X_{2t}, X_{2t-j})\,, \qquad j = 1, \ldots, k_2\,. \end{aligned}$$

Let us have two-dimensional observations $\boldsymbol{X}_1, \ldots, \boldsymbol{X}_T$ of the process (2.1) or equivalently univariate observations $X_1, X_2, \ldots, X_{2T}$ according to (1.5) at our disposal.

If we replace all covariances of the type $\operatorname{cov}(X_u, X_v)$ in (2.7) by their estimates

$$R_T(u, v) = \frac{1}{m_2 - m_1 + 1} \sum_{k=m_1}^{m_2} X_{u+2k} X_{v+2k} \tag{2.9}$$

(the limits m_1 and m_2 are chosen so that all terms in the preceding sum are defined and their number is maximal) we obtain the estimators $c = (c_1, \ldots, c_{k_1})'$ and $d = (d_1, \ldots, d_{k_2})'$ of the vectors γ and δ which have asymptotically normal distributions with mean vectors γ and δ, variance matrices $\sigma_1^2 R_1^{-1}/T$ and $\sigma_2^2 R_2^{-1}/T$ and are mutually uncorrelated (see [6]).

In this moment we can make advantage of Durbin's procedure and maximize over $\alpha = (\alpha_1, \ldots, \alpha_{q_1})'$ and $\beta = (\beta_1, \ldots, \beta_{q_2})'$ the likelihood function derived from the asymptotic distribution of the estimated parameters in the autoregressive approximation to the original moving average model. In our case we shall have to maximize the function

$$Q = -\frac{T}{2} \{(c - \gamma)' R_1(c - \gamma)/\sigma_1^2 + (d - \delta)' R_2(d - \delta)/\sigma_2^2\} , \tag{2.10}$$

which is the argument of the exponential curve in the (normal) likelihood function of c and d.

Now there are two alternatives how to proceed. Firstly we can maximize (2.10) after replacing σ_1^2 and σ_2^2 by their estimates $\hat{\sigma}_1^2$ and $\hat{\sigma}_2^2$. These estimates can be obtained either by means of (2.5), e.g.

$$\hat{\sigma}_1^2 = \frac{1}{t_2 - t_1 + 1} \sum_{t=t_1}^{t_2} (X_{2t-1} + c_1 X_{2t-2} + \ldots + c_{k_1} X_{2t-1-k_1})^2 \tag{2.11}$$

(the limits t_1 and t_2 are again chosen so that all terms in the preceding sum are defined and their number is maximal) or one can use the relations given in [6], e.g.

$$\hat{\sigma}_1^2 = R_T(2t - 1, 2t - 1) + \sum_{i=1}^{k_1} c_i R_T(2t - 1, 2t - 1 - i) . \tag{2.12}$$

Or secondly to obtain more explicit formulas we can assume that the approximation is admissible in which we minimize the following function $\tilde{Q}$ over α and β (instead of the maximization of Q)

$$\tilde{Q} = \tfrac{1}{2}\{(c - \gamma)' R_1(c - \gamma) + (d - \delta)' R_2(d - \delta)\} , \tag{2.13}$$

i.e. we neglect the multipliers $1/\sigma_1^2$ and $1/\sigma_2^2$. The practical experiences show that the results may stay still acceptable even if the difference between σ_1^2 and σ_2^2 is significant (see e.g. Example from Section 5 in which the theoretical value σ_2^2 is greater than the double of σ_1^2) and the numerical simplification following from this approximation is essential. Since it holds

$$\gamma' R_1 \gamma = \operatorname{var}\big(\sum_{j=1}^{k_1} \gamma_j X_{2t-1-j}\big) = \operatorname{var}(X_{2t-1} - \varepsilon_{2t-1}) \sim \operatorname{var}\big(\sum_{j=1}^{q_1} \alpha_j \varepsilon_{2t-1-j}\big)$$

(and similarly for $\delta' R_2 \delta$) we can write approximately

$$\tilde{Q} = \tfrac{1}{2}\{c'R_1 c + 2c'g + \alpha_1^2\sigma_2^2 + \alpha_2^2\sigma_1^2 + \alpha_3^2\sigma_2^2 + \ldots + d'R_2 d + \\ + 2d'h + \beta_1^2\sigma_1^2 + \beta_2^2\sigma_2^2 + \beta_3^2\sigma_1^2 + \ldots\}, \tag{2.14}$$

where $\alpha_r = 0$ for $r > q_1$ and $\beta_s = 0$ for $s > q_2$. The estimators $a = (a_1, \ldots, a_{q_1})'$ and $b = (b_1, \ldots, b_{q_2})'$ of the parameters α and β are constructed by differentiating $\tilde{Q}$ with respect to α and β and equating the derivatives to zero to obtain the normal equations. If one uses the explicit form of the elements of R_1, R_2, g and h then it is not difficult to show that e.g.

$$\frac{\partial \tilde{Q}}{\partial \alpha_1} = \sigma_2^2\{(c_1 + d_1 d_2 + c_2 c_3 + \ldots) + (1 + d_1^2 + c_2^2 + \ldots)\alpha_1 + \\ + (d_1 + c_1 c_2 + d_2 d_3 + \ldots)\beta_2 + (c_2 + d_1 d_3 + c_2 c_4 + \ldots)\alpha_3 + \ldots\}.$$

The numerical advantage of this approximate procedure consists in the fact that the parameters σ_1^2 and σ_2^2 can be excluded (see e.g. the previous form of $\partial\tilde{Q}/\partial\alpha_1$) so that we obtain the same systems of normal equations for a and b as in [3]

$$\begin{aligned} &(1 + d_1^2 + c_2^2 + \ldots)\, a_1 + (d_1 + c_1 c_2 + d_2 d_3 + \ldots)\, b_2 + \\ &+ (c_2 + d_1 d_3 + c_2 c_4 + \ldots)\, a_3 + \\ &+ (d_3 + c_1 c_4 + d_2 d_5 + \ldots)\, b_4 + \ldots = -(c_1 + d_1 d_2 + c_2 c_3 + \ldots), \\ &(d_1 + c_1 c_2 + d_2 d_3 + \ldots)\, a_1 + (1 + c_1^2 + d_2^2 + \ldots)\, b_2 + \\ &+ (c_1 + d_1 d_2 + c_2 c_3 + \ldots)\, a_3 + \\ &+ (d_2 + c_1 c_3 + d_2 d_4 + \ldots)\, b_4 + \ldots = -(d_2 + c_1 c_3 + d_2 d_4 + \ldots), \\ &(c_2 + d_1 d_3 + c_2 c_4 + \ldots)\, a_1 + (c_1 + d_1 d_2 + c_2 c_3 + \ldots)\, b_2 + \\ &+ (1 + d_1^2 + c_2^2 + \ldots)\, a_3 + (d_1 + c_1 c_2 + d_2 d_3 + \ldots)\, b_4 + \ldots \\ &\ldots = -(c_3 + d_1 d_4 + c_2 c_5 + \ldots), \\ &\vdots \end{aligned} \tag{2.15}$$

$$\begin{aligned} &(1 + c_1^2 + d_2^2 + \ldots)\, b_1 + (c_1 + d_1 d_2 + c_2 c_3 + \ldots)\, a_2 + \\ &+ (d_2 + c_1 c_3 + d_2 d_4 + \ldots)\, b_3 + \\ &+ (c_3 + d_1 d_4 + c_2 c_5 + \ldots)\, a_4 + \ldots = -(d_1 + c_1 c_2 + d_2 d_3 + \ldots), \\ &(c_1 + d_1 d_2 + c_2 c_3 + \ldots)\, b_1 + (1 + d_1^2 + c_2^2 + \ldots)\, a_2 + \\ &+ (d_1 + c_1 c_2 + d_2 d_3 + \ldots)\, b_3 + \\ &+ (c_2 + d_1 d_3 + c_2 c_4 + \ldots)\, a_4 + \ldots = -(c_2 + d_1 d_3 + c_2 c_4 + \ldots), \\ &(d_2 + c_1 c_3 + d_2 d_4 + \ldots)\, b_1 + (d_1 + c_1 c_2 + d_2 d_3 + \ldots)\, a_2 + \\ &+ (1 + c_1^2 + d_2^2 + \ldots)\, b_3 + \\ &+ (c_1 + d_1 d_2 + c_2 c_3 + \ldots)\, a_4 + \ldots = -(d_3 + c_1 c_4 + d_2 d_5 + \ldots), \\ &\vdots \end{aligned} \tag{2.16}$$

(we define $a_r = 0$ for $r > q_1$, $b_s = 0$ for $s > q_2$, $c_i = 0$ for $i > k_1$ and $d_j = 0$ for $j > k_2$). The first equation in the system (2.15) corresponds to $\partial\tilde{Q}/\partial\alpha_1 = 0$, the second one to $\partial\tilde{Q}/\partial\beta_2 = 0$, etc. and the first equation in the system (2.16) corresponds to $\partial\tilde{Q}/\partial\beta_1 = 0$, the second one to $\partial\tilde{Q}/\partial\alpha_2 = 0$, etc. The number of the equations in (2.15) and (2.16) is equal to the number of the unknown variables in these systems.

The former "exact" procedure using $\hat{\sigma}_1^2$ and $\hat{\sigma}_2^2$ must treat more complicated systems of equations, e.g. the first equation of (2.15) will have the form

$$(2.17)\quad (\hat{\sigma}_2^2 + d_1^2\hat{\sigma}_1^2 + c_2^2\hat{\sigma}_2^2 + \dots)\, a_1 + (d_1\hat{\sigma}_1^2 + c_1c_2\hat{\sigma}_2^2 + d_2d_3\hat{\sigma}_1^2 + \dots)\, b_2 +$$
$$+ (c_2\hat{\sigma}_2^2 + d_1d_3\hat{\sigma}_1^2 + c_2c_4\hat{\sigma}_2^2 + \dots)\, a_3 + \dots = -(c_1\hat{\sigma}_2^2 + d_1d_2\hat{\sigma}_1^2 + c_2c_3\hat{\sigma}_2^2 + \dots)$$

(i.e. all terms formed by c_i's are multiplied by $\hat{\sigma}_2^2$ and all terms formed by d_j's are multiplied by $\hat{\sigma}_1^2$; the same holds for the other equations in both systems (2.15) and (2.16)).

The improved estimates of σ_1^2 and σ_2^2 in comparison with (2.11) or (2.12) can be obtained in this phase from the residuals $\hat{\varepsilon}_t$ calculated by means of (2.3) using the estimated parameters a and b and setting $\hat{\varepsilon}_0 = \hat{\varepsilon}_{-1} = \hat{\varepsilon}_{-2} = \dots = 0$.

Finally the asymptotic covariance structure of a and b can be estimated similarly as in [3]. If we introduce the following vectors (with appropriate finite dimensions)

$$(2.18)\qquad \xi_1 = (a_1, b_2, a_3, b_4, \dots)',$$
$$\xi_2 = (b_1, a_2, b_3, a_4, \dots)', \quad \xi = (\xi_1', \xi_2')'$$

then the approximate covariance matrix of ξ is block-diagonal with the blocks equal to the inversed matrices of the systems of equations (2.15) and (2.16) multiplied by $1/T$ (in the "exact" covariance matrix all terms formed by c_i's (including the corresponding unities) in the first block must be multiplied by $\hat{\sigma}_2^2/\hat{\sigma}_1^2$ and all terms formed by d_j's (including the corresponding unities) in the second block must be multiplied by $\hat{\sigma}_1^2/\hat{\sigma}_2^2$).

3. CASE WITH GENERAL DIMENSION

Let us consider a d-dimensional moving average model (1.3) with given observations $X_1, \dots, X_T$. Let $\{X_t\}$ be the corresponding periodic moving average process with the period d constructed according to (1.5). Then the model for $\{X_t\}$ can be written in the form

$$(3.1)\quad \begin{aligned} X_{1+d(t-1)} &= \varepsilon_{1+d(t-1)} + \beta_1(1)\,\varepsilon_{d(t-1)} + \dots + \beta_{q_1}(1)\,\varepsilon_{1-q_1+d(t-1)}\,; \\ X_{2+d(t-1)} &= \varepsilon_{2+d(t-1)} + \beta_1(2)\,\varepsilon_{1+d(t-1)} + \dots + \beta_{q_2}(2)\,\varepsilon_{2-q_2+d(t-1)}\,, \\ &\vdots \\ X_{dt} &= \varepsilon_{dt} + \beta_1(d)\,\varepsilon_{-1+dt} + \dots + \beta_{q_d}(d)\,\varepsilon_{-q_d+dt}\,, \end{aligned}$$

where

$$(3.2)\qquad \operatorname{var}(\varepsilon_{i+d(t-1)}) = \sigma^2(i)\,, \quad i = 1, \dots, d\,.$$

Our estimation procedure will be the direct generalization of the previous two-dimensional case. The autoregressive approximations (2.5) have the following form now

$$(3.3)\qquad \begin{aligned} &X_{1-d(t-1)} + \alpha_1(1) X_{d(t-1)} + \ldots + \alpha_{k_1}(1) X_{1-k_1+d(t-1)} = \varepsilon_{1+d(t-1)}\,,\\ &X_{2+d(t-1)} + \alpha_1(2) X_{1+d(t-1)} + \ldots + \alpha_{k_2}(2) X_{2-k_2+d(t-1)} = \varepsilon_{2+d(t-1)}\,,\\ &\vdots\\ &X_{dt} + \alpha_1(d) X_{-1+dt} + \ldots + \alpha_{k_d}(d) X_{-k_d+dt} = \varepsilon_{dt}\,. \end{aligned}$$

Let $a(1), \ldots, a(d)$ be the estimated vectors of parameters in (3.3) constructed generalizing (2.7). If we accept the same simplifying assumption as in Section 2 replacing the function Q by $\tilde{Q}$ then the estimators $b(1), \ldots, b(d)$ of the parameters in (3.1) can be obtained as the solutions of d systems of linear equations. The ith system ($i = 1, \ldots, d$) which produces the values $b_1(i), b_2(i + 1), b_3(i + 2), \ldots$ has the form

$$(3.4)\qquad \sum_{k=1}^{j-1}\{\sum_{r=1} a_{r-1}(i + j + r - 2)\, a_{j-k+r-1}(i + j + r - 2)\}\, b_k(k + i - 1) +$$
$$+ \sum_{k=j}\{\sum_{r=1} a_{r-1}(i + k + r - 2)\, a_{k-j+r-1}(i + k + r - 2)\}\, b_k(k + i - 1) =$$
$$= - \sum_{r=1} a_{r-1}(i + j + r - 2)\, a_{j+r-1}(i + j + r - 2)\,, \quad j = 1, 2, \ldots,$$

where we put $a_r(i) = 0$ for $r > k_i$, $a_0(i) = 1$, $a_r(i) = a_r(i + d)$, $b_k(i) = 0$ for $k > q_i$, $b_k(i) = b_k(i + d)$. The number of equations in the ith system (3.4) is equal to the number of its unknown variables $b_1(i), b_2(i + 1), b_3(i + 2), \ldots$.

In the more complicated procedure based on Q without the approximation by $\tilde{Q}$ the ith system (3.4) ($i = 1, \ldots, d$) has to be replaced by

$$(3.5)\qquad \sum_{k=1}^{j-1}\{\sum_{r=1} a_{r-1}(i + j + r - 2)\, a_{j-k+r-1}(i + j + r - 2) :$$
$$: \hat{\sigma}^2(i + j + r - 2)\}\, b_k(k + i - 1) +$$
$$+ \sum_{k=j}\{\sum_{r=1} a_{r-1}(i + k + r - 2)\, a_{k-j+r-1}(i + k + r - 2) :$$
$$: \hat{\sigma}^2(i + k + r - 2)\}\, b_k(k + i - 1) =$$
$$= - \sum_{r=1} a_{r-1}(i + j + r - 2)\, a_{j+r-1}(i + j + r - 2) :$$
$$: \hat{\sigma}^2(i + j + r - 2)\,, \quad j = 1, 2, \ldots,$$

where the estimates $\hat{\sigma}^2(i) = \hat{\sigma}^2(i + d)$ can be obtained analogously as in (2.11) or (2.12).

As the covariance structure of the estimators $b(1), \ldots, b(d)$ is concerned the vectors $b^*(1) = (b_1(1), b_2(2), b_3(3), \ldots)'$, $b^*(2) = (b_1(2), b_2(3), b_3(4), \ldots)'$, $\ldots$, $b^*(d) = (b_1(d), b_2(1), b_3(2), \ldots)'$ of the solutions of the particular systems (3.4) are asymptotically mutually uncorrelated and the asymptotic variance matrix of $b^*(i)$

is equal to the coefficient matrix on the left of (3.4) inversed and multiplied by $1/T$ (in the "exact" case the coefficient matrices on the left of (3.5) multiplied by $\hat{\sigma}^2(i-1)$ must be used).

4. IDENTIFICATION

The identification of the model (1.3) can be carried out conveniently in the framework of the previous estimation procedure. Let us confine ourselves only to the two-dimensional case (2.3) for simplicity (the identification in the d-dimensional case will be the natural generalization of it).

According to [6] the estimators $R_T(u, v)$ defined in (2.9) have asymptotically the normal distribution with the mean value $R(u, v)$ and the variance

$$(4.1) \qquad \frac{1}{T} \sum_{j=-\infty}^{\infty} [R(u, u + 2j) R(v, v + 2j) + R(u, v + 2j) R(v, u + 2j)] .$$

First let us consider the estimates $R_T(2t - 1, 2t - 1 - \tau)$ for $\tau > q_1$. Then obviously

$$(4.2) \qquad \mathsf{E}\{R_T(2t - 1, 2t - 1 - \tau)\} = 0 .$$

For τ even ($\tau > q_1$) the corresponding variance (4.1) can be rewritten to the form

$$(4.3) \qquad \frac{1}{T} \sum_{|2j| \leqq q_1} R(2t - 1, 2t - 1 + 2j) R(2t - 1 - \tau, 2t - 1 - \tau + 2j)$$

since all summands in the sum

$$\sum_{j=-\infty}^{\infty} R(2t - 1, 2t - 1 - \tau + 2j) R(2t - 1 - \tau, 2t - 1 + 2j)$$

are equal to zero. Analogously for τ odd ($\tau > q_1$) this asymptotic variance is

$$(4.4) \qquad \frac{1}{T} \sum_{|2j| \leqq q} R(2t - 1, 2t - 1 + 2j) R(2t - 1 - \tau, 2t - 1 - \tau + 2j) ,$$

where $q = \min(q_1, q_2)$.

The results (4.2)–(4.4) can be used for identification of the number q_1 in the same way as it is done in the classical methodology of Box and Jenkins [2]. In our case we can compare the values $R_T(2t - 1, 2t - 1 - \tau)$ for $\tau = 1, 2, \ldots$ with the critical values

$$(4.5) \quad u(v) \left\{\frac{1}{T} \sum_{|2j| \leqq q_1} R_T(2t - 1, 2t - 1 + 2j) R_T(2t - 1 - \tau, 2t - 1 - \tau + 2j)\right\}^{1/2}$$

for τ even and

$$(4.6) \quad u(v) \left\{\frac{1}{T} \sum_{|2j| \leqq q} R_T(2t - 1, 2t - 1 + 2j) R_T(2t - 1 - \tau, 2t - 1 - \tau + 2j)\right\}^{1/2}$$

for τ odd ($u(v)$ is the critical value of the standard normal distribution on the significance level v). Then the identified q_1 is chosen as the smallest number $\tau \geqq 0$ such that all $R_T(2t-1, 2t-1-(\tau+1)), R_T(2t-1, 2t-1-(\tau+2)), \ldots$ do not exceed in the absolute value the corresponding limits (4.5) or (4.6).

As the choice of the number q_2 is concerned the values $R_T(2t, 2t-\tau)$ must be compared with the critical values

$$(4.7) \qquad u'(v) \left\{ \frac{1}{T} \sum_{|2j| \leqq q_2} R_T(2t, 2t+2j)\, R_T(2t-\tau, 2t-\tau+2j) \right\}^{1/2}$$

for τ even and

$$(4.8) \qquad u(v) \left\{ \frac{1}{T} \sum_{|2j| \leqq q} R_T(2t, 2t+2j)\, R_T(2t-\tau, 2t-\tau+2j) \right\}^{1/2}$$

for τ odd.

5. SIMULATIONS

Example. The following two-dimensional moving average model of the order two was considered

$$(5.1) \qquad \begin{pmatrix} X_{1t} \\ X_{2t} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0{\cdot}75 & 1 \end{pmatrix} \begin{pmatrix} \varepsilon_{1t} \\ \varepsilon_{2t} \end{pmatrix} + \begin{pmatrix} 0{\cdot}6 & -0{\cdot}5 \\ 0{\cdot}4 & -0{\cdot}3 \end{pmatrix} \begin{pmatrix} \varepsilon_{1,t-1} \\ \varepsilon_{2,t-1} \end{pmatrix}$$

with the normal white noise such that

$$(5.2) \qquad \operatorname{var}\{(\varepsilon_{1t}, \varepsilon_{2t})'\} = \begin{pmatrix} 0{\cdot}64 & 0 \\ 0 & 1{\cdot}44 \end{pmatrix}.$$

The corresponding periodic moving average process has the form

$$(5.3) \qquad \begin{aligned} X_{2t-1} &= \varepsilon_{2t-1} - 0{\cdot}5\varepsilon_{2t-2} + 0{\cdot}6\varepsilon_{2t-3}\,, \\ X_{2t} &= \varepsilon_{2t} + 0{\cdot}75\varepsilon_{2t-1} - 0{\cdot}3\varepsilon_{2t-2} + 0{\cdot}4\varepsilon_{2t-3}\,, \end{aligned}$$

where $\sigma_1^2 = \operatorname{var}(\varepsilon_{2t-1}) = 0{\cdot}64$ and $\sigma_2^2 = \operatorname{var}(\varepsilon_{2t}) = 1{\cdot}44$. It is $q_1 = 2$ and $q_2 = 3$ in (5.3). One can easily verify that both roots of the polynomial $z^2 - 0{\cdot}5z + 0{\cdot}6 = 0$ lie inside the unit circle but this is not the case for the polynomial $z^3 + 0{\cdot}75z^2 - 0{\cdot}3z + 0{\cdot}4$ (the value of this polynomial for $z = -1$ is positive while it decreases to $-\infty$ for the real z going to $-\infty$) so that the assumption (2.6) does not hold (on the other hand the two-dimensional model (5.1) is invertible since all roots of the equation $\det(\boldsymbol{\beta}_0 z^q + \boldsymbol{\beta}_1 z^{q-1} + \ldots + \boldsymbol{\beta}_q) = 0$ lie inside the unit circle in this case).

Fifty simulations of the length $T = 100$ based on this model were performed on the computer ADT 4100 at the Department of Statistics of Charles University and the corresponding parameters were estimated by means of the approximate procedure (2.15) and (2.16). The observed means and standard deviations of these fifty results are

$$\begin{aligned} \bar{a}_1 &= -0{\cdot}498(s_{a_1} = 0{\cdot}076)\,, & \bar{a}_2 &= 0{\cdot}552(s_{a_2} = 0{\cdot}115)\,, \\ \bar{b}_1 &= 0{\cdot}715(s_{b_1} = 0{\cdot}163)\,, & \bar{b}_2 &= -0{\cdot}328(s_{b_2} = 0{\cdot}126)\,, \\ \bar{b}_3 &= 0{\cdot}370(s_{b_3} = 0{\cdot}145)\,. \end{aligned}$$

Table 1(a). Identification of the number q_1 in the model (5.3) (the level of significance $v = 5\%$).

τ	1	2	3	4	5	6	7	8	9
$q_1 \leqq 1,\ q_2 \geqq 0$	0·337	0·273	0·341	0·275	0·344	0·274	0·344	0·279	0·343
$2 \leqq q_1 \leqq 3,\ q_2 \leqq 1$	0·337	0·307	0·341	0·308	0·344	0·308	0·344	0·313	0·343
$2 \leqq q_1 \leqq 3,\ q_2 \geqq 2$	0·331	0·307	0·333	0·308	0·336	0·308	0·337	0·313	0·335
$4 \leqq q_1 \leqq 5,\ q_2 \leqq 1$	0·337	0·307	0·341	0·308	0·344	0·308	0·344	0·313	0·343
$4 \leqq q_1 \leqq 5,\ 2 \leqq q_2 \leqq 3$	0·331	0·307	0·333	0·308	0·336	0·308	0·337	0·313	0·335
$4 \leqq q_1 \leqq 5,\ q_2 \geqq 4$	0·333	0·307	0·334	0·308	0·337	0·308	0·338	0·313	0·336
⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮
$R_T(2t-1,\ 2t-1-\tau)$	0·470	−0·494	0·142	0·031	−0·116	−0·029	−0·209	−0·143	0·213

Table 1(b). Identification of the number q_2 in the model (5.3) (the level of significance $v = 5\%$).

τ	1	2	3	4	5	6	7	8	9
$q_1 \geqq 0,\ q_2 \leqq 1$	0·335	0·415	0·335	0·419	0·338	0·423	0·337	0·423	0·343
$q_1 \leqq 1,\ 2 \leqq q_2 \leqq 3$	0·335	0·416	0·335	0·421	0·338	0·424	0·337	0·424	0·343
$q_1 \geqq 2,\ 2 \leqq q_2 \leqq 3$	0·330	0·416	0·330	0·421	0·333	0·424	0·332	0·424	0·338
$q_1 \leqq 1,\ 4 \leqq q_2 \leqq 5$	0·335	0·423	0·335	0·429	0·338	0·431	0·337	0·432	0·343
$2 \leqq q_1 \leqq 3,\ 4 \leqq q_2 \leqq 5$	0·330	0·423	0·330	0·429	0·333	0·431	0·332	0·432	0·338
$q_1 \geqq 4,\ 4 \leqq q_2 \leqq 5$	0·331	0·423	0·332	0·429	0·334	0·431	0·335	0·432	0·340
⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮
$R_T(2t,\ 2t-\tau)$	−0·835	0·078	−0·370	0·287	0·171	−0·088	0·134	−0·523	−0·114

One can conclude that the approximate estimation procedure is acceptable here although $\sigma_2^2/\sigma_1^2 > 2$.

The identification procedure suggested in Section 4 is demonstrated in Table 1(a) and (b) for the first simulation. E.g. values $R_T(2t - 1, 2t - 1 - \tau)$ for $\tau = 1, \ldots, 9$ are given in the last row of Table 1(a). In the previous rows of this table the critical values according to (4.5) and (4.6) with $\nu = 5\%$ are presented for $\tau = 1, \ldots, 9$ and various possible ranges of q_1 and q_2. Table 1(b) was constructed analogously. The smallest number q_1 which is admissible according to Table 1(a) is $q_1 = 2$ ($q_1 < 2$ is not admissible since $R_T(2t - 1, 2t - 2) = 0{\cdot}470$ for $\tau = 1$ is greater than the critical value 0·337 from the first row of this table). Similarly the smallest number q_2 admissible according to Table 1(b) is $q_2 = 3$ if we consider the inflated value $R_T(2t, 2t - 8) = -0{\cdot}523$ to be a negligible outlier. It can be summarized that the correct values of q_1 and q_2 were identified from the considered simulation.

REFERENCES

[1] J. Anděl: Statistical analysis of periodic autoregression. Aplikace matematiky *28* (1983), 364—385.

[2] G. E. P. Box and G. M. Jenkins: Time Series Analysis. Forecasting and Control. Holden Day, San Francisco 1970.

[3] T. Cipra: Periodic moving average process. Aplikace matematiky *30* (1985), 218—229.

[4] J. Durbin: Efficient estimation of parameters in moving average models. Biometrika *46* (1959), 306—316.

[5] H. J. Newton: Using periodic autoregressions for multiple spectral estimation. Technometrics *24* (1982), 109—116.

[6] M. Pagano: On periodic and multiple autoregressions. Ann. Statist. *6* (1978), 1310—1317.

[7] B. M. Troutman: Some results in periodic autoregressions. Biometrika *66*(1979), 219—228.

RNDr. Tomáš Cipra, CSc., matematicko-fyzikální fakulta UK (Faculty of Mathematics and Physics — Charles University), Sokolovská 83, 186 00 Praha 8. Czechoslovakia.