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ON THE BOUNDED INPUT-BOUNDED OUTPUT STABILITY OF A SECOND-ORDER LINEAR DIFFERENCE EQUATION

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The problem of transforming a second order linear difference equation in a special system of two linear difference equations of the first order and using this system to bounded input-bounded output (BIBO) stability testing is investigated.

1. INTRODUCTION

In [1], bounded input-bounded output (BIBO) stability conditions have been constructed for a system of first order linear difference equations with lower-triangular system matrix. For an arbitrary linear system, existence of a matrix transform converting this system in one with a lower-triangular system matrix is proven indirectly but not explicitly shown.

In this article another proof of stability conditions, based on known Toeplitz theorem [3] is given, the matrix transform is explicitly contructed for a system of two first order difference equations and especially for a second order difference equation. The connection of stability conditions with the known conditions for an equation with constant coefficients is shown. Finally, the results of application of the method to numerically testing the stability on an example are given.

2. A SPECIAL SYSTEM OF FIRST-ORDER LINEAR DIFFERENCE EQUATIONS AND ITS BIBO STABILITY CONDITIONS

Let the following system of linear difference equations be given (unless an exception is explicitly stated, $t = 0, 1, \ldots$ in what follows):

$$
\begin{aligned}
(1) \qquad y_1(t+1) &= a_{11}(t)\, y_1(t) && + x_1(t) \\
y_2(t+1) &= a_{21}(t)\, y_1(t) + a_{22}(t)\, y_2(t) && + x_2(t) \\
&\vdots \\
y_n(t+1) &= a_{n1}(t)\, y_1(t) + a_{n2}(t)\, y_2(t) + \ldots + a_{nn}(t)\, y_n(t) && + x_n(t)
\end{aligned}
$$

All occurring sequences can be considered as complex ones. Further, one will suppose

$$a_{ii}(t) \neq 0\,, \quad i = 1, 2, \ldots, n\,. \tag{2}$$

Given initial conditions

$$y_i(0)\,, \quad i = 1, 2, \ldots, n\,, \tag{3}$$

the system (1) possesses unique solution which can be computed recurrently in an obvious way.

Let us suppose that there exists a finite $X > 0$ such that

$$|x_i(t)| < X\,, \quad i = 1, 2, \ldots, n \tag{4}$$

and a finite $A > 0$ such that

$$|a_{ij}(t)| < A\,, \quad i, j = 1, 2, \ldots, n\,. \tag{5}$$

Theorem 1 (BIBO stability condition). Let (3) (4), (5) hold. Then the necessary and sufficient condition for the existence of a finite $Y > 0$ such that

$$|y_i(t)| < Y, \quad i = 1, 2, \ldots, n \tag{6}$$

is the existence of a finite $K > 0$ such that

$$\sum_{k=0}^{t} \prod_{l=k}^{t} |a_{ii}(l)| < K\,, \quad i = 1, 2, \ldots, n \tag{7}$$

([1], [3]).

Proof. By induction, one obtains for the first of the equations of (1)

$$\begin{aligned} y_1(t+1) &= a_{11}(t)\, a_{11}(t-1) \ldots a_{11}(0)\, y_1(0) + \\ &+ a_{11}(t)\, a_{11}(t-1) \ldots a_{11}(1)\, x_1(0) + a_{11}(t)\, a_{11}(t-1) \ldots \\ &\ldots a_{11}(2)\, x_1(1) + \ldots + a_{11}(t)\, x_1(t-1) + x_1(t) \end{aligned} \tag{8}$$

Let us define

$$y_1(0) = x_1(-1)\,, \quad a_{11}(-1) = 0\,. \tag{9}$$

Then, the sequences $\{y_1(t)\}$, $t = 0, 1, \ldots$ and $\{x_1(t)\}$, $t = -1, 0, \ldots$ are coupled by a matrix transformation

$$\begin{pmatrix} 1, & & \mathbf{0} \\ a_{11}(0), & 1, & \\ a_{11}(1) a_{11}(0), & a_{11}(1), & 1, \\ & \vdots & \end{pmatrix}. \tag{10}$$

But according to a theorem of Toeplitz (cf. [3]), the necessary and sufficient condition for $\{y_1(t)\}$ to be bounded if $\{x_1(t)\}$ is bounded is the existence of a $K_1 > 0$ such that for each t

$$A_{11}(t) = 1 + |a_{11}(t)| + |a_{11}(t)|\,|a_{11}(t-1)| + \ldots + |a_{11}(t)| \ldots |a_{11}(0)| < K_1\,. \tag{11}$$

Now, one can consider the sum $a_{21}(t)\, y_1(t) + x_2(t)$ as a new bounded input to the second equation of (1) and repeat the reasoning, then similarly for the third equation and so on. Finally, collecting all the conditions together, one gets (7), where $K - 1$ is the maximum of all K_i, $i = 1, 2, \ldots, n$.

From (11), one obtains easily

$$A_{11}(t + 1) = 1 + |a_{11}(t + 1)|\, A_{11}(t)\,, \qquad A_{11}(-1) = 1\,. \tag{12}$$

The same relation (a difference equation) holds also for the other $A_{ii}(t)$, $i = 2, \ldots, n$.

Further, there is clear from (11) and (2) that for $i = 1, 2, \ldots, n$

$$1 + |a_{ii}(t)| + \ldots + |a_{ii}(t)| \ldots |a_{ii}(0)| = \tag{13}$$
$$= |a_{ii}(t)| \ldots |a_{ii}(0)| \left(1 + \frac{1}{|a_{ii}(0)|} + \ldots + \frac{1}{|a_{ii}(t)| \ldots |a_{ii}(0)|}\right) < K_i\,. \qquad \square$$

Theorem 2. Let there hold (7). Then for each $i = 1, 2, \ldots, n$ there holds

$$\lim_{t \to \infty} |a_{ii}(0)| \ldots |a_{ii}(t)| = 0\,. \tag{14}$$

Proof. Firstly, the sequence in (14) cannot be unbounded. If it were so, then, since the expression in the parenthesis in (13) is greater than 1, (13) could not be fulfilled, which is impossible. Thus for every subsequence $\{t_j\}$, $j = 1, 2, \ldots$, $t_j \to \infty$, the corresponding subsequence of the sequence in (14) is bounded from above, say, by $\beta > 0$. Suppose now that for some subsequence an $\alpha > 0$ exists so that

$$\alpha < |a_{ii}(0)| \ldots |a_{ii}(t_j)| < \beta\,. \tag{15}$$

Then the expression in the middle of (13) is greater than $\alpha(1 + n_j/\beta)$, where $n_j \to \infty$ with $t_i \to \infty$. But $\alpha(1 + n_j/\beta) \to \infty$ is impossible due to (13). $\square$

Let (1) be BIBO stable. According to theorem of Toeplitz (cf. [3]), (14) is then necessary and sufficient for the following assertion to hold: For every $\{x_i(t)\}$ converging to 0, the corresponding $\{y_i(t)\}$ is also converging to 0.

It is useful to note that for equations with constant coefficients (or reducible to this case) (13) and (14) are equivalent.

3. A GENERAL SYSTEM OF TWO DIFFERENCE EQUATIONS OF THE FIRST ORDER

In [1] there has been proven in an indirect way that there is possible to convert a general system of difference equations of the first order in a system with lower-triangular matrix by a matrix transform. Now, such a transform for two difference equations of the first order will be chosen.

Let the equations be

$$z_1(t+1) = b_{11}(t)\,z_1(t) + b_{12}(t)\,z_2(t) + v_1(t)\,, \tag{16}$$
$$z_2(t+1) = b_{21}(t)\,z_1(t) + b_{22}(t)\,z_2(t) + v_2(t)\,.$$

Let each of the sequences $\{v_1(t)\}$, $\{v_2(t)\}$, $\{b_{ij}(t)\}$, $i = 1, 2, j = 1, 2$ be bounded and at least one of $\{b_{12}(t)\}$, $\{b_{21}(t)\}$ possesses only non zero terms. One can write (16) in the matrix form

$$\mathbf{Z}(t+1) = \mathbf{B}(t)\,\mathbf{Z}(t) + \mathbf{V}(t)\,. \tag{17}$$

Suppose there exists a nonsingular matrix sequence $\{\mathbf{G}(t)\}$ so that

$$\mathbf{Y}(t) = \mathbf{G}(t)\,\mathbf{Z}(t)\,, \tag{18}$$

where $\mathbf{Y}(t)$ is given by (1). Using (17) one obtains

$$\mathbf{Y}(t+1) = \mathbf{G}(t+1)\,\mathbf{B}(t)\,\mathbf{G}^{-1}(t)\,\mathbf{Y}(t) + \mathbf{G}(t+1)\,\mathbf{V}(t)\,. \tag{19}$$

Comparing (19) with (1) it is necessary and sufficient that

$$\mathbf{A}(t) = \mathbf{G}(t+1)\,\mathbf{B}(t)\,\mathbf{G}^{-1}(t) \tag{20}$$

is lower triangular. But

$$\mathbf{G}^{-1}(t) = \frac{1}{|\mathbf{G}(t)|}\begin{pmatrix} g_{22}(t), & -g_{12}(t) \\ -g_{21}(t), & g_{11}(t)\end{pmatrix}, \tag{21}$$

where $|\mathbf{G}(t)|$ denotes the determinant of $\mathbf{G}(t)$. The necessary and sufficient condition of lower triangularity obtained by elementary calculations is

$$0 = a_{12}(t) = -g_{11}(t+1)\,g_{11}(t)\,b_{12}(t) + g_{11}(t+1)\,g_{12}(t)\,b_{11}(t) - \tag{22}$$
$$- g_{12}(t+1)\,g_{11}(t)\,b_{22}(t) + g_{12}(t+1)\,g_{12}(t)\,b_{21}(t)\,.$$

It is seen that this recurrence relation is more complicated than the original equations, but also that some terms of $\mathbf{G}(t)$ can be freely chosen. Let us suppose

$$\mathbf{G}(t) = \begin{pmatrix} -1, & g_{12}(t) \\ 0, & 1 \end{pmatrix}. \tag{23}$$

In what follows, $g_{12}(t) = g(t)$ will be written.

Then, one gets from (22) a sufficient condition

$$g(t+1) = \frac{b_{12}(t) + g(t)\,b_{11}(t)}{b_{22}(t) + g(t)\,b_{21}(t)}\,. \tag{24}$$

Given the initial condition $g(0)$, the sequence $\{g(t)\}$ can be constructed provided the denominator in (24) differs from zero for every t.

Now, the matrix $\mathbf{A}(t)$ will be found. After some computations

$$\mathbf{A}(t) = \begin{pmatrix} b_{11}(t) - b_{21}(t)\,g(t+1), & 0 \\ -b_{21}(t), & b_{21}(t)\,g(t) + b_{22}(t) \end{pmatrix} \tag{25}$$

and according to (5) also the boundedness of $\{g(t)\}$ must be postulated.

For the term $a_{11}(t)$ one will find yet another expression. From (22) there follows

$$g(t)\left(-b_{11}(t) + b_{21}(t)\, g(t+1)\right) = b_{12}(t) - b_{22}(t)\, g(t+1)\,, \tag{26}$$

thus

$$a_{11}(t) = \frac{-b_{12}(t) + b_{22}(t)\, g(t+1)}{g(t)}\,, \tag{27}$$

this having sense for $g(t) \neq 0$.

4. ONE LINEAR DIFFERENCE EQUATION OF THE SECOND ORDER

Instead of analyzing the general case, we will further simplify the situation concerning only one difference equation of second order, which is sufficiently interesting per se.

Thus let the equation

$$u(t+2) = a(t)\, u(t+1) + b(t)\, u(t) + x(t) \tag{28}$$

with real bounded coefficients and real bounded $\{x(t)\}$ be given, let $b(t) \neq 0$. The usual equivalent system to (28) is

$$\begin{aligned} z_1(t+1) &= a(t)\, z_1(t) + b(t)\, z_2(t) + x(t)\,, \\ z_2(t+1) &= z_1(t)\,, \end{aligned} \tag{29}$$

by the substitution

$$z_1(t) = u(t+1)\,, \quad z_2(t) = u(t) \tag{30}$$

by which also the initial conditions are given

$$z_1(0) = u(1)\,, \quad z_2(0) = u(0)\,. \tag{31}$$

There is in (29)

$$\begin{pmatrix} b_{11}(t), & b_{12}(t) \\ b_{21}(t), & b_{22}(t) \end{pmatrix} = \begin{pmatrix} a(t), & b(t) \\ 1, & 0 \end{pmatrix}. \tag{32}$$

Thus, one gets from (24)

$$g(t+1) = a(t) + \frac{b(t)}{g(t)}\,, \tag{33}$$

having sense for $g(t) \neq 0$.

Further, one gets from (25), (27)

$$\mathbf{A}(t) = \begin{pmatrix} a(t) - g(t+1), & 0 \\ -1 &, \ g(t) \end{pmatrix} = \begin{pmatrix} -\dfrac{b(t)}{g(t)}, & 0 \\ -1 &, \ g(t) \end{pmatrix}. \tag{34}$$

Thus the system of the form (1), corresponding to (28), is (using also (19), (23))

$$\begin{aligned} y_1(t+1) &= -\frac{b(t)}{g(t)}\, y_1(t) \quad - x(t)\,, \\ y_2(t+1) &= -y_1(t) + g(t)\, y_2(t)\,. \end{aligned} \tag{35}$$

Under the supposition of $g(t) \neq 0$ and bounded for every t, the equation (28) with (33) determines (35) uniquely. Conversely, (35) and (33) determine (28) uniquely. (28) can be expressed with the aid of (34) as

$$u(t+2) = (a_{11}(t) + g(t+1))\, u(t+1) - a_{11}(t)\, g(t)\, u(t) + x(t). \tag{36}$$

And the initial conditions are

$$u(0) = y_2(0), \quad u(1) = -y_1(0) + g(0)\, y_2(0). \tag{37}$$

Necessary and sufficient stability conditions (7) are now from (11), (35)

$$|g(t)| + |g(t)|\,|g(t-1)| + \ldots + |g(t)| \ldots |g(0)| < K_1, \tag{38}$$

$$\left|\frac{b(t)}{g(t)}\right| + \left|\frac{b(t)}{g(t)}\right| \left|\frac{b(t-1)}{g(t-1)}\right| + \ldots + \left|\frac{b(t)}{g(t)}\right| \ldots \left|\frac{b(0)}{g(0)}\right| < K_2 \tag{39}$$

for every t.

Let us now consider in more detail the relation (33). One finds the sequence

$$\begin{aligned} g(1) &= a(0) + \frac{b(0)}{g(0)}, \\ g(2) &= a(1) + \cfrac{b(1)}{a(0) + \cfrac{b(0)}{g(0)}}, \\ g(3) &= a(2) + \cfrac{b(2)}{a(1) + \cfrac{b(1)}{a(0) + \cfrac{b(0)}{g(0)}}}, \\ &\vdots \end{aligned} \tag{40}$$

This is very similar (with only reversed order) to the known continued fraction expansion [2]. We will try to express the expansion with the aid of two sequences $\{P(t)\}, \{Q(t)\}$ so that for every t

$$g(t) = \frac{P(t)}{Q(t)}, \quad Q(t) \neq 0. \tag{41}$$

From (40) one gets

$$\begin{aligned} \frac{P(1)\, P(0)}{Q(1)} &= a(0)\, P(0) + b(0)\, Q(0), \\ \frac{P(2)\, P(1)}{Q(2)} &= a(1)\, P(1) + b(1)\, Q(1), \\ &\vdots \end{aligned} \tag{42}$$

Thus one may define

$$Q(t) = P(t-1) \tag{43}$$

under the condition $P(t) \neq 0$ and one gets for $P(t)$ the difference equation

$$P(t+1) = a(t)\,P(t) + b(t)\,P(t-1)\,, \tag{44}$$

which is the homogeneous equation corresponding to (28). Then (41) is

$$g(t) = \frac{P(t)}{P(t-1)}\,. \tag{45}$$

Although this seems to be of little value in practical applications, one can formulate from it and (38), (39) immediately an interesting result, namely:

Let a difference equation (28) be given. Knowing one particular solution $\{P(t)\}$ of the corresponding homogeneous equation such that $P(t) \neq 0$ and bounded for all t, one can obtain (with the aid of (45)) the solution of (33) needed in (38),(39).

An interesting theorem is the following one:

Theorem 3. Let $a(t) > 0$, $b(t) > 0$ or $a(t) < 0$, $b(t) > 0$ for all t. Then, a sequence in (33) can be found such that $g(t) \neq 0$ for all t.

Proof. For the first case, it suffices to choose $g(0) > 0$, for the second case $g(0) < 0$. In both cases the sequences $\{g(t)\}$ change no sign. □

5. ONE CONSTANT-COEFFICIENTS LINEAR DIFFERENCE EQUATION OF SECOND ORDER

Since the stability conditions and the character of solutions of a constant coefficients equation are well known, there is interesting to see what is the meaning of (33), (38), (39) in this case of constant a, b.

The equation

$$z^2 - az - b = 0 \tag{46}$$

is the characteristic equation and we suppose $b \neq 0$ as in the assumption after (27). The case of $a = 0$ can be also excluded since in this case the difference equation of the second order degenerates into two equations of the first order.

Now, (33) is

$$z_{t+1} = a + \frac{b}{z_t} \tag{47}$$

and there can be shown without difficulty that for (46) possessing two distinct real roots (47) represents the known recurrent method of finding the root ζ with greater absolute value.

Choosing especially $z_0 = \zeta$ in (47) and substituting in (38), one sees that (38) is fulfilled precisely if $|\zeta| < 1$ in accordance with the well known result. And $|b/\zeta|$ is the smaller absolute value of the remaining root and thus, for the equation with constant coefficients, (39) follows from (38). Unhappily, for real roots of the characteristic equation, the convergence of (47) depends on the choice of z_0, for complex roots (47) is not convergent.

For complex roots one obtains with the aid of (45) the formula

$$g(t + 1) = \varrho(\cos \alpha + \sin \alpha \operatorname{cotg} (\varphi + t\alpha)) \tag{48}$$

in which ϱ is the modulus, α is the argument of an arbitrarily chosen root and φ depends on the initial conditions of the equation. Note that there is $\sin \alpha \neq 0$ since real roots are now excluded.

From (48) there follows (since $\varrho \neq 0$)

$$g(t + 1) = 0 \tag{49}$$

for

$$\operatorname{cotg} (-\alpha) = \operatorname{cotg} (\varphi + t\alpha), \tag{50}$$

thus

$$(t + 1)\alpha + \varphi = k\pi, \quad k = 0, \pm 1, \ldots \tag{51}$$

There will be shown on an example that this can occur even in the case of stability.

Example 1. Let the characteristic equation be

$$z^2 - z + \tfrac{1}{3} = 0 \tag{52}$$

For its roots

$$\varrho = 1/\sqrt{3}, \quad \alpha = \pm\pi/6. \tag{53}$$

For (50), k in (51) can be chosen arbitrarily. Postulating e.g. $g(4) = 0$, then choosing $k = 1$ in (51), one gets

$$4\pi/6 + \varphi = \pi, \tag{54}$$

thus

$$\varphi = \pi/3. \tag{55}$$

Then from (48)

$$g(0) = \frac{1}{\sqrt{3}}\left(\frac{\sqrt{3}}{2} + \frac{1}{2}\sqrt{3}\right) = 1. \tag{56}$$

And 0 occurs in the sequence $\{g(t)\}$ after $t + 1 = 4$ with period 6, since in (51) $\pi/\alpha = 6$.

One can check these results using (33), beginning with $g(0) = 1$.

Naturally, such a situation can be expected to occur also for equations with variable coefficients and it is clear that in this case no system of the form equivalent to (28) can be constructed with the aid of (33) (briefly, the "case of nonequivalence"). Moreover, practical difficulties can arise also if (51) is only approximately satisfied.

6. APPLICATIONS

If a system (1) of the second order with general coefficients would be given, the expressions (7) could be computed at least in some noncomplicated cases.

However, for the case $|a_{ii}(t)| < \varepsilon < 1$ for every $i = 1, 2, \ldots, n$, it is clear that the system (1) is stable. But for an equation (28) with general coefficients, (33) is hopelessly complicated.

Nevertheless, for an equation with numerically given coefficients, the stability can be checked numerically with the aid of a computer, computing successively (33) and (38), (39) with the aid of (12).

This will be shown on an example with known stability boundary, taken from [4].

Eliminating the variable $v(t)$ from the equations systems (13), (14) of [4], one gets

$$(57) \quad u(2k) = \left((1-\alpha) + (1-\beta)\frac{1-a}{a}\right) u(2k-1) - (1-\alpha)\frac{1-a}{a}\, u(2k-2)\,,$$

$$u(2k+1) = \left((1-\alpha) + (1-\beta)\frac{a}{1-a}\right) u(2k) - (1-\alpha)\frac{a}{1-a}\, u(2k-1)\,,$$

$$k = 1, 2, \ldots$$

The meaning of α, β, a is given in [4]. The stability will be checked here for $a = 0{,}35$ (Fig. 4 in[4]). Only some typical results will be given for $\alpha = 0{,}25$ and $\alpha = 1$. For $\alpha = 0{,}25$, the values of β on the elliptical stability boundary are resp. 1,20644 and 2,59026.

The results of computing $A_{11}(t)$ by (12) (and analogously $A_{22}(t)$) have been arranged in successive hundreds by the computer and in each hundred the maximum and minimum has been displayed. The results (rounded off) are in the following tables.

Table 1. $\alpha = 0{\cdot}25$

$\beta = 1$				$\beta = 2$			
A_{11}		A_{22}		A_{11}		A_{22}	
Max	min	Max	min	Max	min	Max	min
32	1·54	180	1·93	2·6 E + 12	2·3	3·67	1·56
35	1·36	275	3·76	8·6 E + 23	1·2 E + 12	3·67	1·83
51	1·16	587	3·76				
293	1·02	4329	3·76				
109	1·11	834	3·76				
possibly stable				unstable			

For $\alpha = 1$, $\beta = 1$, the maximum of A_{11} and A_{22} was 1,7 E + 38 in the first hundred showing machine overflow and thus the case od nonequivalence (the equation can be both stable or unstable, in this concrete case it is stable).

Table 2. $\alpha = 1$

$\beta = 0$				$\beta = 0{\cdot}5$			
A_{11}		A_{22}		A_{11}		A_{22}	
Max	min.	Max	min	Max	min	Max	min
143	3	1	1	2·57	1·52	1	1
286	79	1	1	2·57	1·69	1	1
429	156	1	1				
571	233	1	1				
unstable				well behaved stable			

The cases of clear instability and clear (well behaved) stability and also the case of nonequivalence are easily distinguishable. The case (as $\alpha = 1$, $\beta = 0$), where A_{11} (or A_{22}) grows steadily in maximum and minimum values, although not too rapidly, seems also to pose no difficulty. Only the cases of oscillating behaviour (as $\alpha = 0{,}25$, $\beta = 1$) are difficult to discern from slow growth and are demanding more observations (due to general difficulty to discern practically slowly convergent and slowly divergent series).

7. CONCLUDING REMARKS

The described method can be useful in testing stability of numerically given linear difference equations of the second order, when more general methods are not available (this can occur quite often).

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