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SUFFICIENT CONDITIONS FOR THE SOLUTION EXISTENCE IN GENERAL COALITION GAMES

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The concept of the general coalition game was suggested in [2]. Here we are interested in a few sufficient conditions under which the strongly stable solution of such game exists. It is especially shown that the classical concept of the convex game can be also generalized for our coalition game model, and that it guarantees the existence of the solution. Besides that a few other sufficient conditions are presented. It is also shown that the convexity and superadditivity concepts defined here really generalize the classical ones known in the side-payments coalition games theory.

0. INTRODUCTION

As the general coalition game model includes some more special types of games as its modifications, it is useful to derive some conditions under which there exists a stable solution of a game in the sense of [2].

The main purpose of the presented paper is to show that the notion of the convex coalition game, known in the side-payments games theory, can be generalized up to the level of the general coalition game. Moreover, it keeps its most useful property, namely the ability to guarantee the existence of a strongly stable solution. This result is completed by a few other statements of rather similar type.

It is also useful to remember the classical Scarf's generalization of the balanced game concept. This is done in this paper, and it is noted that it can be used for a large class of the general coalition games.

The formal definition of the superadditivity and convexity used here substantially differs from the one known in the theory of the side-payments coalition games with the von Neumann characteristic function. It is shown in the last section of this paper that the dissimilarity is purely formal, and that both definitions are equivalent on the class of the side-payments games.

1. GENERAL COALITION GAME

The notion of general coalition game was suggested in [2], where also some of its elementary properties were derived. It was also investigated in some further papers among which [4] is relevant for this work. In this section we briefly remember the necessary definitions and present a few auxiliary results useful in the following parts of this paper.

In the whole paper we denote by $\mathbb{R}$ the set of all real numbers. The *general coalition game* (or briefly the *game*) is a pair (I, V), where I is a non-empty and finite set and V is a mapping associating any subset K of I with a subset of the real space $\mathbb{R}^I$ and preserving the following properties for any $K \subset I$.

(1.1) $V(K)$ *is a closed subset of* $\mathbb{R}^I$;

(1.2) *if* $x \in V(K)$, $y \in \mathbb{R}^I$, $y_i \leqq x_i$ *for all* $i \in K$, *then also* $y \in V(K)$;

(1.3) $V(K) \neq \emptyset$;

(1.4) $V(K) = \mathbb{R}^I$ *iff* $K = \emptyset$.

The elements of the set I are called *players*, its subsets are *coalitions* and its partitions into non-empty disjoint coalitions are called *coalition structures*. The real-valued vectors from $\mathbb{R}^I$ are called *imputations* and the sets $V(K)$, $K \subset I$, represent the sets of all imputations achievable by the coalition K. The mapping V of 2^I into the class of subsets of $\mathbb{R}^I$ is called the *generalized characteristic function*.

If $\boldsymbol{x} = (x_i)_{i \in I} \in \mathbb{R}^I$, $\boldsymbol{y} = (y_i)_{i \in I} \in \mathbb{R}^I$ are imputations and $K \subset I$ is a coalition then we say that $\boldsymbol{x}$ *dominates* $\boldsymbol{y}$ *via* K and write $\boldsymbol{x} \operatorname{dom}_K \boldsymbol{y}$ iff

$$x_i \geqq y_i \quad \text{for all} \quad i \in K ,$$

$$x_i > y_i \quad \text{for at least one} \quad i \in K .$$

It is useful for the further explanation to define for any coalition $K \subset I$ the set

(1.5)
$$V^*(K) = \\ = \{\boldsymbol{y} \in R^I : \textit{there does not exist any } \boldsymbol{x} \in V(K) \textit{ such that } x \operatorname{dom}_K y\} = \\ = \{\boldsymbol{y} \in R^I : \textit{for any } \boldsymbol{x} \in V(K) \textit{ there exists an } i \in K \textit{ such that } x_i < y_i, \\ \textit{or } x_j = y_j \textit{ for all } j \in K\} .$$

The sets $V(K) \cap V^*(K)$ for all $K \subset I$ represent the imputations that are achievable by the coalition K and that are not worse than any other achievable imputation. In this sense, they are important for the definition of the solution concept. The necessary and sufficient condition under which the intersection $V(K) \cap V^*(K)$ is non-empty was presented in [3]. More important for the existence of a solution is the sufficiency of $V(K) \cap V^*(K)$ briefly investigated in this section.

We say that *the set* $V(K) \cap V^*(K)$ *is sufficient* iff for any imputation $\boldsymbol{x} \in V(K) - V^*(K)$, $x \in V^*(\{i\})$ for all $i \in I$, there exists an imputation $\boldsymbol{y} \in V(K) \cap V^*(K)$ such that $\boldsymbol{y} \operatorname{dom}_K \boldsymbol{x}$.

Remark 1. Let $K \neq \emptyset$ be a coalition and let us denote by $\partial V(K)$ the boundary set of $V(K)$. Then always

$$V(K) \cap V^*(K) \subset \partial V(K) \quad \text{and} \quad V(K) \cup V^*(K) = \mathbb{R}^I .$$

Remark 2. If the coalition K is empty then obviously $V^*(K) = V(K) = \mathbb{R}^I$.

Lemma 1. If $K \neq \emptyset$ and $V(K) \cap V^*(K) = \partial V(K)$ then the set $V(K) \cap V^*(K)$ is sufficient.

Proof. The statement follows immediately from (1.1), (1.4), (1.5) and Remark 1. If $x \notin V^*(K)$ then there exists $y \in V^*(K)$ such that $y \operatorname{dom}_K x$, and it can be chosen in such way that $y \in \partial V(K)$. □

Lemma 2. Let $K \subset I$ be a coalition and let for any

$$x \in (V(K) - V^*(K)) \cap (\bigcap_{i \in I} V^*(\{i\}))$$

be the set

$$V(K) \cap \{y \in \mathbb{R}^I : y_i \geqq x_i \textit{ for all } i \in K\} \cap \mathbb{R}^K$$

bounded. Then the set $V(K) \cap V^*(K)$ is sufficient.

Proof. Let us choose an arbitrary $x \in V(K) - V^*(K)$, $x \in V^*(\{j\})$ for all $j \in I$, and an $i \in K$. Then there exists $z \in \partial V(K)$ such that $z_j = x_j$ for all $j \in K$, $j \neq i$, and $z_i \geqq x_i$, and such that for all $z' \in \mathbb{R}^I$, $z' = x_j$ for all $j \in K$, $j \neq i$, $z'_i > z_i$, the relation $z' \in \mathbb{R}^I - V(K)$ holds. If $z \notin V(K) \cap V^*(K)$ then we choose another $j \in K$, $j \neq i$, and repeat the procedure. After a finite number of steps, we construct $y \in \partial V(K)$ such that for all $i \in K$ the relations $y \in V^*(K)$ and $y_i \geqq x_i$ hold. As $x \notin V^*(K)$ there necessarilly exists $i \in K$ such that $x_i \neq y_i$ and consequently $y \operatorname{dom}_K x$. □

Lemma 3. Let $K \subset I$ and let us suppose that for any $x \in V(K)$, $x \in V^*(\{i\})$ for all $i \in I$, and for any $i \in K$ there exists $y \in V^*(K)$ such that $y_j = x_j$ for all $j \in I$, $j \neq i$. Then the set $V(K) \cap V^*(K)$ is sufficient.

Proof. It is not difficult to verify that the set $V(K)$ satisfying the assumptions of this lemma fulfils also the assumptions of Lemma 2. □

One of the important notions of the coalition games theory is the notion of superadditivity. For the general coalition games it was formulated and investigated in [4],

We say that a game (I, V) is *superadditive* iff for every pair of disjoint coalitions $K, L \subset I$, $K \cap L = \emptyset$, the following inclusion holds

$$V(K \cup L) \supset V(K) \cap V(L) . \tag{1.6}$$

Lemma 4. If $V^*(K \cup L) \subset V^*(K) \cap V^*(L)$ for every pair of disjoint coalitions $K, L \subset I$, $K \cap L = \emptyset$, then the game (I, V) is superadditive.

Proof. Let us suppose that there exists an imputation $x \in V(K) \cap V(L)$ such that $x \notin V(K \cup L)$. Then $x \in V^*(K \cup L)$ by Remark 1. As the set $V(K \cup L)$ is closed, its complement $\mathbb{R}^I - V(K \cup L)$ is open and there exists an open neighbourhood $U(x)$ of x such that $U(x) \subset \mathbb{R}^I - V(K \cup L) \subset V^*(K \cup L)$. Let us choose $y \in U(x)$ such that $y_j \leqq x_j$ for all $j \in K \cup L$ and $y_i < x_i$ for some $i \in K \cup L$. Then at least one of the relations $x \operatorname{dom}_K y$ and $x \operatorname{dom}_L y$ holds. As $x \in V(K) \cap V(L)$ then $y \notin V^*(K)$ or $y \notin V^*(L)$. Hence, $y \notin V^*(K) \cap V^*(L)$ and $y \in V^*(K \cup L)$ by its construction. It is a contradiction with the assumed inclusion, and the game (I, V) is necessarily superadditive. □

As a solution of the general coalition game we shall define any imputation achievable by some coalition structure and undominated by any other achievable imputation via any coalition. Exactly formulated; an imputation $x \in \mathbb{R}^I$ is said to be *strongly stable* iff there exists a coalition structure $\mathscr{K}$ such that

$$x \in \bigcap_{K \in \mathscr{K}} V(K) \tag{1.7}$$

and $x \in V^*(L)$ for all coalitions $L \subset I$. The coalition structure $\mathscr{K}$ is called a strongly stable coalition structure iff there exists a strongly stable imputation $x \in \mathbb{R}^I$ such that (1.7) holds.

Lemma 5. If (I, V) is a superadditive game and if $x \in \mathbb{R}^I$ is a strongly stable imputation then $x \in V(I)$ and the coalition structure $\{I\}$ containing exactly the all players coalition is strongly stable.

Proof. The statement follows from the definition of superadditivity immediately if we take into account that any coalition structure $\mathscr{K}$ is a subpartition of I into disjoint coalitions and hence

$$\bigcap_{K \in \mathscr{K}} V(K) \subset V(I) . \qquad \square$$

The existence of the strongly stable imputations represents the main subject of this paper. A few elementary results of that kind were already obtained in [2], and some other concerning rather special types of games were presented in [4].

2. CONVEX GAMES

Convexity is an important property of some coalition games with side-payments. Its importance follows from the fact that it implies the existence of the non-empty core (c.f. [8]), and that it also represents a natural game theoretical model of some propertirs of markets (c.f. [6]). In this section we define the convexity concept for the general coalition game model and prove that it implies the existence of strongly stable imputations that are a generalization of the elements of core as it is shown in [2].

We say that a general coalition game (I, V) is *convex* iff for every pair of coalitions

$K, L \subset I$ and every pair of imputations

$$x = (x_i)_{i\in I} \in V(K) \cap V^*(K \cap L), \quad y = (y_i)_{i\in I} \in V(L) \cap V^*(K \cap L),$$

any imputation $z \in \mathbb{R}^I$ such that

$$z_i = x_i, \quad \text{for} \quad i \in K,$$
$$z_i = y_i, \quad \text{for} \quad i \in L - K,$$

belongs to the set $V(K \cup L)$.

This definition is not graphically very similar to the classical definition of the convexity, known for example from [8] or [5]. The logical equivalence of both concepts for the side-payments games is shown in Section 5 of this paper. The following auxiliary results make these two definitions at least partially more similar.

Lemma 6. If a game (I, V) is convex then for every pair of coalitions $K, L \subset I$ the following inclusion holds

$$V(K) \cap V(L) \cap V^*(K \cap L) \subset V(K \cup L).$$

Proof. If we chose $x = y \in V(K) \cap V(L) \cap V^*(K \cap L)$ and construct $z \in \mathbb{R}^I$ according to the definition of convexity then $y_i = x_i = z_i$ for $i \in K \cup L$, and $z \in V(K \cup L)$ as follows from the convexity assumption. □

Lemma 7. Every convex game is superadditive.

Proof. The statement immediately follows from Lemma 6. If the coalitions K and L are disjoint then $V^*(K \cap L) = \mathbb{R}^I$ by (1.4), and the superadditivity inclusion is obvious. □

It is possible, now, to introduce the main result of this section.

Theorem 1. Let (I, V) be a convex game and let all the sets $V(K) \cap V^*(K)$, $K \subset I$, be sufficient. Then there exists at least one strongly stable imputation in (I, V).

Proof. We shall prove the theorem by induction. The assumptions as well as the statement are obviously fulfilled for any one-player game. Let us suppose that the theorem is true for a game (J, V), where $J \subset I$ and V is naturally reduced on the class of subsets of J. Let us denote by $x' \in \mathbb{R}^J$ the strongly stable imputation in (J, V). As the game is convex and then also superadditive, Lemma 5 implies that $x' \in V(J)$. Moreover,

$$x' \in V(J) \cap V^*(J) \quad \text{and} \quad x' \in V^*(L) \quad \text{for all} \quad L \subset J.$$

Let us choose $i \in I - J$, denote $M = J \cup \{i\}$ and construct $x \in \mathbb{R}^M$ such that $x_j = x'_j$ for all $j \in J$, and x_i is such that $x \in V(M)$. Such construction is possible as the game is superadditive by Lemma 7, and $V(M) \supset V(J) \cap V(\{i\})$. Then $x \in V(M)$ and $x \in V^*(L)$ for all $L \subset J$. Let us suppose that $x \notin V^*(\{i\})$. Then there exists $x'' \in V(\{i\}) \cap V^*(\{i\})$ such that $x'' \operatorname{dom}_{\{i\}} x$, i.e. $x''_i > x_i$. It means that $x'' \in V(\{i\})$, $x'' \in V^*(\emptyset) = V^*(J \cap \{i\})$, and $x \in V(J)$, $x \in V^*(\emptyset) = V^*(J \cap \{i\})$ (c.f. Remark 2).

Let us construct $z' \in \mathbb{R}^M$ such that

$$z'_j = x_j \quad \text{for all} \quad j \in J\,,$$
$$z'_i = x''_i\,.$$

Then, according to the convexity assumption, $z' \in V(M)$ and, moreover, $z'_j \geqq x_j$ for all $j \in M$. If $z' \notin V^*(M)$ then by the sufficiency assumption there exists $z \in V(M) \cap \cap V^*(M)$ such that $z \operatorname{dom}_M z'$. Then

$$z \in V(M) \cap V^*(M)\,, \quad z \in V^*(L) \quad \text{for all} \quad L \subset J\,, \quad z \in V^*(\{i\})\,.$$

Let us prove now that $z \in V^*(K)$ for all $K \subset M$, too. Let us suppose that $z \notin V^*(K)$ for some $K \subset M$. Then necessarily $i \in K$. As $z \in V(K) - V^*(K)$ there exists $y \in V(K)$ such that $y \operatorname{dom}_K z$, i.e. $y_j \geqq z_j$ for all $j \in K$. It means that $y \in V^*(L)$ for all $L \subset J$ including $y \in V^*(J \cap K)$. The convexity assumption implies that the imputation $u \in \mathbb{R}^M$ such that

$$u_j = z_j \quad \text{for} \quad j \in J - K\,, \quad u_j = y_j \quad \text{for} \quad j \in K\,,$$

belongs to the set $V(M)$ and moreover $u \operatorname{dom}_M z$. It is a contradiction with the fact that $z \in V^*(M)$, hence, $z \in V^*(K)$ for all $K \subset M$. As also $z \in V(M)$, it is strongly stable in the game (M, V), where V is considered on the class of subcoalitions of M. □

Theorem 2. If. (I, V) is a convex game where all sets $V(K) \cap V^*(K)$, $K \subset I$, are sufficient, then the coalition structure $\{I\}$ containing exactly the coalition of all players is strongly stable.

Proof. The statement follows from Theorem 1 and Lemma 5 immediately. □

3. OTHER SUFFICIENT CONDITIONS

In this section we shall prove two sufficient conditions for the existence of strongly stable imputations. As it is obvious also from the technique of their proofs there exists certain similarity between them and the convexity condition presented in the previous section.

Theorem 3. Let (I, V) be a game, let the sets $V(K) \cap V^*(K)$ be sufficient for all coalitions $K \subset I$, $K \neq \emptyset$, and let for any triple of coalitions H, M, $N \subset I$ such that $N \supset M$, $N \cap H = \emptyset$, the following inclusion holds

$$V(M \cup H) \cap V^*(M) \cap V^*(N) \subset V(N \cup H)\,.$$

Let further for every $x \in V(M)$ there exists $y \in V(N) \cap V^*(N)$ such that $y_i = x_i$ for all $i \in M$. Then there exists a strongly stable imputation in (I, V).

Proof. The statement will be proved by induction. It is obviously true for any one-player game. Let us suppose now that it is true for a game (J, V) where $J \subset I$

and V is considered on the class of subcoalitions of J. Let us suppose further that $x \in \mathbb{R}^J$ is the strongly stable imputation in (J, V), and choose $i \in I - J$, $G = J \cup \{i\}$ and $y \in \mathbb{R}^G$ such that $y_j = x_j$ for all $j \neq i$. As $x \in V(J)$ then the last assumption of the theorem implies that y can be chosen in such a way that $y \in V^*(G) \cap V(G)$. Because $x \in V(J)$ and $x \in V^*(L)$ for all $L \subset J$, the same is true for y, as well. It is necessary to show that $y \in V^*(K)$ for $K \subset G$ such that $i \in K$. Let us suppose that there exists $K \subset G$ such that $y \notin V^*(K)$. It means that there exists $z \in V(K)$ such that $z \operatorname{dom}_K y$ and, moreover, $z_j = y_j$ for $j \in K$. Then $z \in V^*(L) \cap V(K) \cap V^*(J)$ where $L = K - \{i\}$ and, by assumption, $z \in V(G)$ where we have put $H = \{i\}$, $N = J$, $M = L$. But it means that $z \operatorname{dom}_G y$ and there is a contradiction with $y \in V^*(G)$. It means that y must be a strongly stable imputation in the game (G, V). □

Theorem 4. Let (I, V) be a game, let for every pair of coalitions $M, N \subset I$, such that $M \subset N$, the relation $V(M) \cap V^*(N) \subset V(N)$ holds. Let further for every $x \in \in V(M)$ there exists $y \in V(N) \cap V^*(N)$ such that $y_i = x_i$ for all $i \in M$. Then there exists a strongly stable imputation in the considered game.

Proof. The proof is rather analogous to the one of Theorem 3. The statement is obviously valid for any one-player game. Let us suppose that it is true for a game (J, V) where $J \subset I$, and that x is the strongly stable imputation in (J, V). Let us choose $i \in I - J$ and denote $G = J \cup \{i\}$ and $y \in \mathbb{R}^G$ such that $y_j = x_j$ for all $j \in J$. It is possible to construct y in such a way that $y \in V(G) \cap V^*(G)$ as follows from the second assumption of the theorem. Then $y \in V(G) \cap V^*(G)$ and $y \in V^*(L)$ for all $L \subset J$. Let us suppose that $y \notin V^*(K)$ for some $K \subset G$ where $i \in K$. Then there exists $z \in V(K)$ such that $z \operatorname{dom}_K y$ and moreover $z_j = y_j$ for all $j \in K$. Then $z \in V^*(G) \cap V(K)$ and, by assumption, $z \in V(G)$ as well. It contradicts the fact that $y \in V^*(G)$. Hence, $y \in V^*(K)$ for all $K \subset G$. □

Remark 3. The last assumption of the previous Theorem 4 implies that every set $V(K) \cap V^*(K)$, $K \subset I$, is sufficient in the considered game. It can be easily verified if we put $M = N$.

4. A FEW NOTES ON THE BALANCE

The concept of the balanced sets and balanced games is the crucial one in the theory of the side-payments coalition games, as follows from [1] or [7]. Every side-payments coalition game is balanced if and only if it has non-empty core, i.e. iff there exists a strongly stable imputation in it. Even if this equivalence is not generally true in case of more general games, the condition of the balancedness is still sufficient for the existence of strongly stable imputations in a wide scale of games as follows from Scarf's paper [6]. The Scarf's result is of a principal character also for the general coalition games investigated here, and it is useful to mention a few comments on it.

Scarf had investigated the coalition games without side-payments described by the pair (I, V) where I is a finite set of players and V is a characteristic function fulfilling conditions (1.1), (1.2) and also the following one

(4.1) *For any coalition structure $\mathcal{K}$ the set*

$$\left(\bigcap_{i \in I} V^*(\{i\})\right) \cap \left(\bigcap_{K \in \mathcal{K}} V(K)\right)$$

is bounded.

Remark 4. It is easy to prove analogously to Lemma 2 that for every $x \in R^I$ such that $x \in V^*(\{i\})$ for all $i \in I$, and for every $K \subset I$ such that $x \in V(K) - V^*(K)$ there exists an imputation $y \in V(K) \cap V^*(K)$ fulfilling the relation $y \operatorname{dom}_K x$, whenever (4.1) is valid.

When our considerations are concentrated to the strong stability of imputations then we are interested in the imputations that belog to all sets $V^*(K)$, $K \subset I$, only. It means that condition (4.1) and Remark 4 do not really limit our possibilities to derive the properties of the strongly stable imputations.

Assumptions (1.3) and (1.4) are not mentioned in the Scarf's paper explicitly. However, they are completely natural and they are implicitely presumed also in [6]. Namely the condition $V(K) \neq R^I$ for $K \neq \emptyset$ follows from (4.1), the validity of (1.3) for the Scarf's game is selfevident from the sense of [6], and empty coalitions are not relevant for the problems solved by Scarf.

It means that the Scarf's result is applicable for a wide class of general coalition games fulfilling (4.1). The result represents a generalization of the Bondareva's classical theorem on balanced games with side-payments proved in [1]. However, in case of games without side-payments the balancedness is only sufficient but not necessary condition for the existence of the strongly stable solution.

According to Scarf, a class of coalitions $\mathcal{M}, \mathcal{M} \subset 2^I$, is *balanced* iff there exist non-negative constants δ_K for $K \in \mathcal{M}$ such that for each $i \in I$

$$\sum_{K \in \mathcal{M}_i} \delta_K = 1, \quad \text{where} \quad \mathcal{M}_i = \{K \in \mathcal{M} : i \in K\}.$$

The general coalition game (I, V) is *balanced* iff for every balanced set of coalitions $\mathcal{M}$ the inclusion

$$\bigcap_{K \in \mathcal{M}} V(K) \subset V(I)$$

is fulfilled. The main result of Scarf's paper [6] is valid also for general coalition games in our sense.

Statement (Scarf). There exist strongly stable imputations in every balanced coalition game (I, V) fulfilling (1.1), (1.2) and (4.1).

5. GAMES WITH SIDE-PAYMENTS

Some of the concepts used in the previous sections, namely the superadditivity and convexity of games, were originally introduced in the theory of the side-payments coalition games. As their formal definitions used in this paper rather differ from the classical ones, it is useful to prove their equivalence.

We say that a game (I, V) is a *side-payments game* iff for every coalition $K \subset I$, $K \neq \emptyset$, there exists a real constant v_K such that

$$V(K) = \{x \in \mathbb{R}^I : \sum_{i \in K} x_i \leqq v_K\} . \tag{5.1}$$

For $K = \emptyset$ we define $v_K = 0$, and it is obvious that the equality $V(\emptyset) = \mathbb{R}^I$ does not contradict (5.1).

It is easy to verify that for any $K \subset I, K \neq \emptyset$,

$$V^*(K) = \{x \in \mathbb{R}^I : \sum_{i \in K} x_i \geqq v_K\} , \tag{5.2}$$

and an imputation $x \in \mathbb{R}^I$ is strongly stable in the side-payments game (I, V) iff there exists a coalition structure $\mathscr{K}$ such that

$$\sum_{i \in K} x_i \leqq v_K \quad \text{for all} \quad K \in \mathscr{K} , \tag{5.3}$$

and

$$\sum_{i \in L} x_i \geqq v_L \quad \text{for all} \quad L \subset I . \tag{5.4}$$

If the game (I, V) is also superadditive then (5.3) turns into

$$\sum_{i \in I} x_i \leqq v_I \tag{5.5}$$

as follows from Lemma 5.

Lemma 8. In the coalition games with side-payments the sets $V(K) \cap V^*(K)$ are sufficient for all $K \subset I, K \neq \emptyset$.

Proof. Relations (5.1) and (5.2) imply that for any non-empty coalition $K \subset I$

$$V(K) \cap V^*(K) = \partial V(K) = \{x \in \mathbb{R}^I : \sum_{i \in K} x_i = v_K\} .$$

Then the statement follows immediately from Lemma 1. □

The equivalence of the classical definitions of the superadditivity and convexity and the corresponding ones used in this paper is proved by the following two theorems.

Theorem 5. If (I, V) is a coalition game with side-payments and if $K, L \subset I$, $K \cap L = \emptyset$, then

$$V(K) \cap V(L) \subset V(K \cup L) \Leftrightarrow v_K + v_L \leqq v_{K \cup L} .$$

Proof. Let $V(K) \cap V(L) \subset V(K \cup L)$ and let $x \in \mathbb{R}^I$ be such that

$$\sum_{i \in K} x_i = v_K, \quad \sum_{i \in L} x_i = v_L.$$

Then $x \in V(K) \cap V(L)$ and, consequently, $x \in V(K \cup L)$. It means that

$$v_{K \cup L} \geqq \sum_{i \in K \cup L} x_i = \sum_{i \in K} x_i + \sum_{i \in L} x_i = v_K + v_L.$$

Let $v_K + v_L \leqq v_{K \cup L}$ and let $x \in V(K) \cap V(L)$. Then

$$\sum_{i \in K \cup L} x_i = \sum_{i \in K} x_i + \sum_{i \in L} x_i \leqq v_K + v_L \leqq v_{K \cup L},$$

and $x \in V(K \cup L)$. □

Theorem 6. Let (I, V) be a coalition game with side-payments and let $K, L \subset I$ be coalitions. Then the following three statements are equivalent.

(a) The convexity condition is fulfilled.

(b) $V(K) \cap V(L) \cap V^*(K \cap L) \subset V(K \cup L)$.

(c) $v_K + v_L - v_{K \cap L} \leqq v_{K \cup L}$.

Proof. The implication (a) ⇒ (b) follows from Lemma 6. Now, we shall prove (b) ⇒ (c). Let us suppose that

$$v_K + v_L - v_{K \cap L} > v_{K \cup L}. \tag{5.6}$$

Coalitions K and L are necessarily such that $K - L \neq \emptyset$ and $L - K \neq \emptyset$, as in the opposite case, e.g. in case $K \subset L$, $v_K = v_{K \cap L}$ and $v_L = v_{K \cup L}$ and (5.6) cannot be true. It means that it is possible to choose $x \in \mathbb{R}^I$ such that

$$\sum_{i \in L} x_i = v_L, \quad \sum_{i \in K} x_i = v_K, \quad \sum_{i \in K \cap L} x_i = v_{K \cap L}.$$

Then

$$\sum_{i \in K \cup L} x_i = \sum_{i \in L} x_i + \sum_{i \in K} x_i - \sum_{i \in K \cap L} x_i = v_L + v_K - v_{K \cap L} > v_{K \cup L}.$$

It means that

$$x \in V(K) \cap V(L) \cap V^*(K \cap L) \quad \text{and} \quad x \notin V(K \cup L).$$

Consequently, if (c) is not true then (b) cannot be true as well, and the implication (b) ⇒ (c) is proved. The last implication to be proved is (c) ⇒ (a). Let us choose imputations

$$x \in V(K) \cap V^*(K \cap L), \quad y \in V(L) \cap V^*(K \cap L),$$

and construct $z \in \mathbb{R}^I$ such that

$$z_i = x_i \quad \text{for} \quad i \in K, \quad z_i = y_i \quad \text{for} \quad i \in L - K.$$

Then $z \in V^*(K \cap L)$,

$$\sum_{i \in L} z_i - \sum_{i \in L \cap K} z_i = \sum_{i \in L - K} z_i = \sum_{i \in L - K} y_i = \sum_{i \in L} y_i - \sum_{i \in L \cap K} y_i,$$

$$\sum_{i \in K} x_i \leqq v_K, \quad \sum_{i \in L} y_i \leqq v_L, \quad \sum_{i \in K \cap L} y_i \geqq v_{K \cap L},$$

and consequently

$$\sum_{i \in K \cup L} z_i = \sum_{i \in K} z_i + \sum_{i \in L} z_i - \sum_{i \in K \cap L} z_i =$$

$$= \sum_{i \in K} x_i + \sum_{i \in L} y_i - \sum_{i \in K \cap L} y_i \leqq \boldsymbol{v}_K + \boldsymbol{v}_L - \boldsymbol{v}_{K \cap L} \leqq \boldsymbol{v}_{K \cup L} .$$

It means that $z \in \boldsymbol{V}(K \cup L)$ and the game is convex. □

6. CONCLUSIONS

A few sufficient conditions for the existence of the strongly stable imputations in general coalition games were presented in the preceding sections. Together with the more elementary results introduced in [2] they cover quite a wide class of coalition games in which the existence of the strongly stable imputations and consequently of the strongly stable coalitions can be tested.

The general coalition games represent an adequate mathematical model of a rich class of cooperative situations. Consequently, the strongly stable solutions correspond to the rational outcome of the behaviour in such situations, and it is useful to know some conditions under which such rationality may be achieved.

The applicability of the results presented above is supported by the fact that many of the assumptions used here (e.g. the sufficiency of $\boldsymbol{V}(K) \cap \boldsymbol{V}^*(K)$, the superadditivity or the balance) characterize coalition games derived from real applications of the general theory (cf. [5], [7] and others).

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