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REPRESENTABLE P. MARTIN-LÖF TESTS

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In some recent papers [2, 3] the problem of representability of P. Martin-Löf tests [5] by Kolmogorov's concept of program complexity [4] has been considered. Here we derive some simple combinatorial properties of representable P. Martin-Löf tests which enable us to solve several problems which remained open in [3]. Moreover by the help of these conditions we rederive and generalize some statements (theorems) of [2] and [3] in a manner which makes them more transparent and avoids cumbersome constructions.

1. PRELIMINARIES

Let $\mathbb{N} = \{0, 1, 2, \ldots\}$ denote the set of natural numbers, and let $\mathbb{N}_+ =_{df} \{1, 2, \ldots\}$. For any finite alphabet $\boldsymbol{X}$, card $\boldsymbol{X} = \boldsymbol{p} \geqq 2$, let $\boldsymbol{X}^*$ be the set of words on $\boldsymbol{X}$ including the empty word e. For $v, w \in \boldsymbol{X}^*$ their concatenation is denoted by vw, and $|w|$ is the *length* of the word w.

Throughout this paper let

$$x_1^{(0)} = e\,, \quad x_1^{(1)}, \ldots, x_{\boldsymbol{p}}^{(1)}\,, \quad x_1^{(2)}, \ldots, x_{\boldsymbol{p}^2}^{(2)}\,; \quad x_1^{(3)}, \ldots, x_{\boldsymbol{p}^3}^{(3)}\,; \quad x_1^{(4)}, \ldots,$$

be a *quasilexicographic ordering* of $\boldsymbol{X}^*$. Consequently $x_1^{(n)}, \ldots, x_{\boldsymbol{p}^n}^{(n)}$ is a *lexicographic ordering* of $\boldsymbol{X}^n = \{w : w \in \boldsymbol{X}^* \;\&\; |w| = n\}$.

According to [5] we introduce the following notion.

A subset $\boldsymbol{V} \subseteq \boldsymbol{X}^* \times \mathbb{N}_+$ is called *P. Martin-Löf test* (*M-L test*) provided

(0) $\boldsymbol{V}$ is recursively enumerable ,

(1) for all $m \in \mathbb{N}_+$, $V_{m+1} \subseteq V_m$, where $V_j =_{df} \{w : (w, j) \in \boldsymbol{V}\}$, and

(2) $$\text{card } V_m \cap \boldsymbol{X}^n \leqq \frac{\boldsymbol{p}^{n-m} - 1}{\boldsymbol{p} - 1}$$

In particular, we have

$$(3) \qquad \begin{aligned} &V_m \cap X^n = \emptyset\,, \quad \text{if} \quad m \geqq n \\ &\text{card } V_{n-1} \cap X^n \leqq 1\,, \quad \text{and} \\ &\text{card } V_{n-2} \cap X^n \leqq p + 1\,. \end{aligned}$$

Since $V_1 \supseteq V_m$ for all $m \in \mathbb{N}_+$, and $V_m \cap X^n = \emptyset$ for $m \geqq n$, the function

$$\boldsymbol{m}_V(w) =_{\text{df}} \begin{cases} \max\{m : w \in V_m\}\,, & \text{if} \quad w \in V_1 \\ 0\,, & \text{otherwise} \end{cases}$$

is well-defined, and it is referred to as the *critical level function* of the test $\boldsymbol{V}$.

As a further function connected with M-L tests we introduce the *extent* $\boldsymbol{\beta}_V$ of the test $\boldsymbol{V} \subseteq \boldsymbol{X}^* \times \mathbb{N}_+$:

$$(4) \qquad \boldsymbol{\beta}_V(m, n) =_{\text{df}} \text{card}\,\{w : w \in X^n \,\&\, \boldsymbol{m}_V(w) = m\}\,.$$

Since $w \in V_m$ iff $\boldsymbol{m}_V(w) \geqq m$. we obtain

$$(5) \qquad \text{card } V_m \cap X^n = \sum_{i=m}^{n-1} \boldsymbol{\beta}_V(i, n)\,.$$

A particular case of M-L tests are the *recursive* tests $\boldsymbol{V}$ investiged in [3], i.e. tests $\boldsymbol{V} \subseteq X^* \times \mathbb{N}_+$ fot which an algorithm deciding whether $(w, m) \in \boldsymbol{V}$ exists.

Lemma 1. Let $\boldsymbol{V}$ be an M-L test. Then the following conditions are equivalent:
(a) $\boldsymbol{V}$ is recursive subset of $\boldsymbol{X}^* \times \mathbb{N}_+$.
(b) $\boldsymbol{m}_V$ is a recursive function.
(c) $\boldsymbol{\beta}_V$ is a recursive function.

Proof. (a) → (b) is shown in [3].
(b) → (c) is easily verified by the defining equation (4).
(c) → (a) In view of Eq. (5) an algorithm deciding $(w, m) \in \boldsymbol{V}$ is described as follows. Compute $n = |w|$ and enumerate $\boldsymbol{V}$ up to $\sum_{i=m}^{n} \boldsymbol{\beta}_V(i, n)$ distinct pairs (v, m) with $|v| = n$ appear. Check, whether (w, m) appeared in the enumeration. □

We define still another subclass of M-L tests. An M-L test $\boldsymbol{V}$ is called *weakly recursive* provided the set

$$\mathfrak{C}_V =_{\text{df}} \{(w, \boldsymbol{m}_V(w)) : w \in V_1\}$$

is recursively enumerable. $\mathfrak{C}_V$ is the graph of the *partial critical level* function

$$\boldsymbol{m}'_V(w) =_{\text{df}} \begin{cases} \max\{m : w \in V_m\}\,, & \text{if} \quad w \in V_1 \\ \text{undefined}\,, & \text{otherwise}\,. \end{cases}$$

Hence an M-L test $\boldsymbol{V}$ is weakly recursive iff its partial critical level function $\boldsymbol{m}'_V$ is partial recursive. Clearly, every recursive M-L test is also weakly recursive.

2. REPRESENTABLE M-L TESTS

To the concept of M-L test one can relate in some sense the concept of Kolmogorov program complexity, though both concepts are not equivalent [7, 8].

For a partial recursive function $\varphi : X^* \times \mathbb{N} \to X^*$ the *Kolmogorov complexity function* [4] K_φ induced by φ is defined by

$$K_\varphi(w/n) =_{\mathrm{df}} \begin{cases} \min\{|\pi| : \pi \in X^* \,\&\, \varphi(\pi, n) = w\}, & \text{if} \quad |w| = n \,\&\, \exists\pi(\varphi(\pi, n) = w) \\ \text{undefined}, & \text{otherwise}. \end{cases}$$

If $w = \varphi(\pi, |w|)$, the word π is referred to as a program computing w when given $|w|$.

Since there are at most p^k programs of lengt k, we have

$$\text{(6)} \qquad \operatorname{card}\{w : |w| = n \,\&\, K_\varphi(w/n) = k\} \leqq p^k .$$

For every partial recursive function $\varphi : X^* \times \mathbb{N} \to X^*$ the set

$$\text{(7)} \qquad V(\varphi) =_{\mathrm{df}} \{(w, m) : w \in X^* \,\&\, m \in \mathbb{N}_+ \,\&\, m < |w| - K_\varphi(w/|w|)\}$$

is an M-L test (see Example 10 of [1]).

As in [2] we call a Martin-Löf test $W \subseteq X^* \times \mathbb{N}$ representable over X provided there is a partial recursive function $\varphi : X^* \times \mathbb{N} \to X^*$ such that $W = V(\varphi)$. If $W = V(\varphi)$ is a representable Martin-Löf test then its critical level function m_W and the Kolmogorov complexity function K_φ induced by φ are strongly related via

$$\text{(8)} \qquad m_W(w) = |w| - K_\varphi(w/|w|) - 1 \quad \text{for} \quad w \in W_1 ,$$

i.e. to every $w \in W_1$ there is a shortest program π of length $|w| - m_W(w) - 1$ for which φ computes w when given $|w|$.

From Eqs. (6) and (8) we obtain the following necessary condition (cf. also Theorem 3 of [3]).

Proposition 2. If W is an M-L test representable over X, $m \in \mathbb{N}_+$, then

$$\text{(2')} \qquad \beta_W(m, n) \leqq p^{n-m-1} \quad \text{for all} \quad m, n \geqq 1 .$$

Eq. (2') explains also Example 2 of [2] where it is shown that the Martin-Löf test $V = \{(000, 1), (010, 1), (111, 1)\}$ is not representable over $X = \{0, 1\}$. The condition (2'), however, is not sufficient for a Martin-Löf test $V \subseteq X^* \times \mathbb{N}_+$ to be representable over X.

Before proceeding to a counterexemple, we mention the following easily derived property of representable Martin-Löf tests.

Proposition 3. If $W = V(\varphi)$ is an M-L test representable over X and $\beta_V(m, n) = \operatorname{card}\{w : w \in X^n \,\&\, m_W(w) = m\} = p^{n-m-1}$ for some $n, m \in \mathbb{N}_+$ then φ maps $X^{n-m-1} \times \{n\}$ in a one-to-one manner onto $\{w : w \in X^n \,\&\, m_W(w) = m\}$.

Proof. Since $W = V(\varphi)$ is representable over X, to every $w \in X^n$ with $m_W(w) = m$

there is a program π of length $n - m - 1$ for which φ computes w when given n. But there are exactly p^{n-m-1} programs of length $n - m - 1$. $\square$

Example 1. (A nonrepresentable M-L test.) Let $M \subseteq \mathbb{N}_+$ $(1, 2, \notin M)$ be a nonrecursive recursively enumerable set.

Define $V \subseteq X^* \times \mathbb{N}_+$ via $V_1 \cap X = V_1 \cap X^2 =_{\mathrm{df}} \emptyset$,

$$V_{n-1} \cap X^n =_{\mathrm{df}} \begin{cases} \{x_1^{(n)}\}, & \text{if } n \in M \\ \emptyset, & \text{otherwise}, \end{cases}$$

and for $n \geqq 3$

$$V_{n-2} \cap X^n = \dots = V_1 \cap X^n =_{\mathrm{df}} \begin{cases} \{x_1^{(n)}, x_2^{(n)}, \dots, x_{p+1}^{(n)}\}, & \text{if } n \in M \\ \{x_1^{(n)}, x_2^{(n)}, \dots, x_p^{(n)}\}, & \text{otherwise}. \end{cases}$$

Clearly, V is a P. Martin-Löf test which satisfies (2′). Moreover card $\{w : w \in X^n \,\&\, m_V(w) = n - 2\} = p$ for all $n \geqq 3$.

If $V = V(\varphi)$ for some partial-recursive $\varphi : X^* \times \mathbb{N} \to X^*$ by Proposition 3 to each $w \in X^n$ with $m_V(w) = n - 2$ there is a program π of length 1 for which φ computes w when given n. Hence

$$\varphi(X, \{n\}) = \begin{cases} \{x_2^{(n)}, \dots, x_{p+1}^{(n)}\} & \text{if } n \in M \\ \{x_1^{(n)}, \dots, x_p^{(n)}\} & \text{if } n \notin M. \end{cases}$$

Define for $n \geqq 3$

$$f(n) =_{\mathrm{df}} \begin{cases} p + 1, & \text{if } \exists x (x \in X \,\&\, \varphi(x, n) = x_{p+1}^{(n)}) \\ 1, & \text{if } \exists x (x \in X \,\&\, \varphi(x, n) = x_1^{(n)}). \end{cases}$$

Since φ is partial recursive and either $x_{p+1}^{(n)} \in \varphi(X, \{n\})$ or $x_1^{(n)} \in \varphi(X, \{n\})$, the thus defined function f is recursive. Now, $M = f^{-1}(p + 1)$ is also recursive which contradicts our assumption. $\square$

Though Eq. (2′) is not sufficient for the representability of an M-L test V, an additional assumption on the test V will make it representable when satisfying Eq. (2′).

Theorem 4. If $V \subseteq X^* \times \mathbb{N}_+$ is a weakly recursive M-L test satisfying Eq. (2′) then V is representable over X.

Proof. We describe an algorithm computing a function φ such that $V = V(\varphi)$.

Let be given the inputs π and n. If $|\pi| \geqq n - 1$ then output $\varphi(\pi, n) =_{\mathrm{df}} \pi$.

For $|\pi| \leqq n - 2$ estimate the position $g(\pi)$ of π in the lexicographical ordering of $X^{|\pi|}$ i.e. $\pi = x_{g(\pi)}^{(|\pi|)}$. Then enumerate $\mathfrak{C}_V$ up to $g(\pi)$ distinct elements of the form (w, m) with $m = n - |\pi| - 1$ appear (if $\beta_V(m, n) < g(\pi)$, $\varphi(\pi, n)$ remains undefined), and output the first component of this ith element.

Since (w, m), $(w, m') \in \mathfrak{C}_V$ implies $m = m'$, by the above construction to every word w belongs at most one program π of length $|\pi| \leqq |w| - 2$ for which π computes

w when given $|w|$. Moreover, this very program π satisfies

$$|\pi| = |w| - m_V(w) - 1\,, \quad \text{hence} \quad \boldsymbol{m}_V(w) = |w| - \boldsymbol{K}_\varphi(w/|w|) - 1$$

whenever $\boldsymbol{K}_\varphi(w/|w|) \leqq |w| - 2$.

Finally, the condition (2′) $\boldsymbol{\beta}_V(m, n) \leqq p^{n-m-1}$ guarantees that to every w with $\boldsymbol{m}_V(w) \geqq 1$ (i.e. $(w, \boldsymbol{m}_V(w)) \in \mathfrak{C}_V$) there is a program π of length $|w| - \boldsymbol{m}_V(w) - 1$ such that $\varphi(\pi, |w|) = w$. □

Corollary 5. Not every M-L test is weakly recursive, and not every weakly recursive M-L test is recursive.

Proof. The first assertion follows immediately from Example 1 and Theorem 4, and the second one is readily seen by the example

$$V =_{\text{df}} \{(x_1^{(n)}, 1) : n \in M\}$$

where $M \subseteq \mathbb{N}_+$ $(1, 2 \notin M)$ is a nonrecursive recursively enumerable set. □

For recursive M-L tests we obtain the following strengthening of the Theorems 3 and 9 in [3].

Corollary 6. Let $V \subseteq X^* \times \mathbb{N}_+$ be an M-L test. Then V is recursive and satisfies Eq. (2′) if and only if there is a recursive function $\varphi : X^* \times N \to X^*$ such that $V = V(\varphi)$.

Proof. Let V be recursive. We proceed as in the proof of Theorem 4. Since $\boldsymbol{\beta}_V$ is also recursive, the condition $\boldsymbol{\beta}_V(m, n) < g(\pi)$ can be checked, and if $\boldsymbol{\beta}_V(m, n) < g(\pi)$ we set $\varphi(\pi, n) =_{\text{df}} \pi$.

Conversely, let $\varphi : X^* \times \mathbb{N} \mapsto X^*$ be recursive. Then the condition $\boldsymbol{K}_\varphi(w/|w|) \leqq k$ is equivalent to $\exists\pi(|\pi| \leqq k \,\&\, \varphi(\pi, |w|) = w)$ and is recursively decidable. Now, Eq. (7) yields $(w, m) \in V(\varphi)$ iff $\boldsymbol{K}_\varphi(w/|w|) \leqq |w| - m - 1$, which proves the assertion. □

3. EMBEDDING OF M-L TESTS

In [3] (cf. Theorem 2) it has been shown that every recursive M-L test $V \subseteq X^* \times \mathbb{N}_+$ is embeddable into an M-L test $V(\varphi)$ representable over X satisfying $(w, 1) \in V$ iff $(w, 1) \in V(\varphi)$. In fact, studying the results of [3] more thoroughly, one could even prove the following assertion: For every recursive M-L test $V \subseteq X^* \times \mathbb{N}_+$ there is a recursive M-L test W representable over X such that $V \subseteq W$ and $(w, 1) \in V$ iff $(w, 1) \in W$.

In this section we solve that question which remained open in [3] whether an arbitrary M-L test $V \subseteq X^* \times \mathbb{N}_+$ can be embedded into a representable one.

To this end we derive the following auxiliary result.

Proposition 7. Let $W \subseteq X^* \times \mathbb{N}_+$ be an M-L test such that

$$\text{card } W_m \cap X^n = \frac{p^{n-m} - 1}{p - 1}$$

for some $m, n \in \mathbb{N}_+$. If there is a partial recursive function $\varphi : \boldsymbol{X}^* \times \mathbb{N} \to \boldsymbol{X}^*$ such that $\boldsymbol{W} \subseteq \boldsymbol{V}(\varphi)$, then φ maps the set

$$\{(\pi, n) : |\pi| \leqq n - m - 1\}$$

in a one-to-one manner onto $W_m \cap \boldsymbol{X}^n$.

Proof. Since $\boldsymbol{W} \subseteq \boldsymbol{V}(\varphi)$ we have $\boldsymbol{m}_{\boldsymbol{W}}(w) \leqq \boldsymbol{m}_{V(\varphi)}(w) = |w| - \boldsymbol{K}_\varphi(w/|w|) - 1$ for all $w \in W_1$. Hence for every $w \in W_m \cap \boldsymbol{X}^n$ (i.e. $\boldsymbol{m}_{\boldsymbol{W}}(w) \geqq m$) there is a program π_w of length $|\pi_w| \leqq n - m - 1$ such that $\varphi(\pi_w, n) = w$. Since there are at most $\sum_{i=0}^{n-m-1} \boldsymbol{p}^i = (\boldsymbol{p}^{n-m} - 1)/(\boldsymbol{p} - 1)$ programs of length $\leqq n - m - 1$ and since card $V_m \cap \boldsymbol{X}^n = (\boldsymbol{p}^{n-m} - 1)/(\boldsymbol{p} - 1)$, the assertion follows. □

Now we can construct an M-L test $\boldsymbol{V} \subseteq \boldsymbol{X}^* \times \mathbb{N}_+$ which cannot be embedded into any M-L test representable over $\boldsymbol{X}$.

Example 2. (A nonembeddable M-L test.) Let $A, B \subseteq \mathbb{N}_+$ $(1, 2 \notin A \cup B)$ be a pair of recursively inseparable sets (cf. [6]), i.e. a pair of disjoint recursively enumerable sets such that any function $f : \mathbb{N} \mapsto \mathbb{N}$ satisfying $A \subseteq f^{-1}(1)$ and $B \subseteq f^{-1}(2)$ is not recursive.

We define our M-L test $\boldsymbol{W} \subseteq \boldsymbol{X}^* \times \mathbb{N}_+$ as follows:

$$W_m \cap \boldsymbol{X}^n = \emptyset\,, \quad \text{if} \quad n \leqq 2$$

$$W_{n-2} \cap \boldsymbol{X}^n = \ldots = W_1 \cap \boldsymbol{X}^n = \{x_1^{(n)}, \ldots, x_{\boldsymbol{p}+1}^{(n)}\}\,, \quad \text{if} \quad n \geqq 3\,,$$

and

$$W_{n-1} \cap \boldsymbol{X}^n =_{\text{df}} \begin{cases} \{x_1^{(n)}\}\,, & \text{if} \quad n \in A \\ \{x_2^{(n)}\}\,, & \text{if} \quad n \in B \\ \emptyset & \text{otherwise}\,. \end{cases}$$

Since card $W_{n-2} \cap \boldsymbol{X}^n = \boldsymbol{p} + 1$, Proposition 7 implies that $\varphi(e, n)$ is defined for all $n \geqq 3$ if $\boldsymbol{W} \subseteq \boldsymbol{V}(\varphi)$ for some partial recursive function φ. In this case, according to the definition of W_{n-1}, we have $\varphi(e, n) = x_1^{(n)}$ if $n \in A$ and $\varphi(e, n) = x_2^{(n)}$ if $n \in B$. Set

$$f(n) =_{\text{df}} \begin{cases} i, & \text{if} \quad \varphi(e, n) = x_i^{(n)} \quad \text{and} \quad n \geqq 3 \\ 0, & \text{otherwise}\,. \end{cases}$$

Then, since $\varphi(e, n)$ is defined for all $n \geqq 3$, the function f is recursive and satisfies $f^{-1}(1) \supseteq A$ and $f^{-1}(2) \supseteq B$, a contradition to our assumption. □

The test of Example 2 can be shown to be not weakly recursive. Thus, it is an open problem whether weakly recursive M-L tests can be embedded into representable ones. We conjecture that the following more general (cf. Theorem 4) statement be true.

Conjectured statement. Let $\boldsymbol{W} \subseteq \boldsymbol{X}^* \times \mathbb{N}_+$ be a weakly recursive M-L test. Then there is a weakly recursive M-L test $\boldsymbol{V} \subseteq \boldsymbol{X}^* \times \mathbb{N}_+$ satisfying Eq. (2') such that $\boldsymbol{W} \subseteq \boldsymbol{V}$.

4. A SUFFICIENT CONDITION

In this section we explain why we have stressed the term representability over $\boldsymbol{X}$. In [2], (cf. Theorem 3) it has been shown that every M-L test $\boldsymbol{V} \subseteq \boldsymbol{X}^* \times \mathbb{N}_+$ is representable over a larger alphabet $Y \supset \boldsymbol{X}$, i.e. if we admit a larger quantity of programs of every length $\geqq 1$.

A slight modification of the proof of Theorem 4 yields a simple combinatorial explanation of the above quoted fact and moreover, yields some interesting consequences.

Lemma 8. Let $\boldsymbol{W}$ be a P. Martin-Löf test over $\boldsymbol{X}$ which satisfies

$$(2'') \qquad \text{card } W_m \cap \boldsymbol{X}^n \leqq p^{n-m-1} .$$

Then $\boldsymbol{W}$ is representable over $\boldsymbol{X}$.

Proof. We describe an algorithm computing a partial recursive function $\varphi : \boldsymbol{X}^* \times \mathbb{N} \to \boldsymbol{X}^*$ representing $\boldsymbol{W}$.

The algorithm computing φ operates as follows:

Given a program π and an output-length n it estimates $m = n - |\pi| - 1$ and the position $g(\pi)$ of π in the lexicographic ordering of $\boldsymbol{X}^{|\pi|}$. Then it enumerates W_m up to $g(\pi)$ distinct elements of length n appear, and outputs this $g(\pi)$th element.

From (2″) it follows that every word $w \in W_m \cap \boldsymbol{X}^n$ has a program π of length $n - m - 1$ for which φ computes w when given $|w| = n$, and by construction only a word $w \in W_m \cap \boldsymbol{X}^n$ can have a program π of length $n - m - 1$ for which φ computes w when given $|w| = n$. □

The condition of Lemma 4 is however not necessary. To this end consider *full P. Martin-Löf tests* (cf. [3]), i.e. tests satisfying Eq. (2) with equality. Consequently, a full P. Martin-Löf test $\boldsymbol{V}$ also satisfies Eq. (2′) with equality, i.e. $\beta_V(m, n) = p^{n-m-1}$, hence $\boldsymbol{V}$ cannot satisfy Eq. (2″) unless $n = m + 1$. Thus, according to Lemma 1 every full P. Martin-Löf test is recursive and by Corollary 6 also representable over $\boldsymbol{X}$.

An example of a full M-L test $\boldsymbol{V}$ is the following:

$$V_m \cap \boldsymbol{X}^n =_{\text{df}} \left\{ x_j^{(n)} : 1 \leqq j \leqq \frac{p^{n-m} - 1}{p - 1} \right\} .$$

Although being an easily derived sufficient condition for representability, Lemma 8 gives simple explanations why an increase of the program resources (cf. Theorem 3 of [2]) or a limitation of the set to be tested makes Martin-Löf tests representable: Since

$$\frac{p^{n-m} - 1}{p - 1} = \sum_{i=0}^{n-m-1} p^i \leqq (p + 1)^{n-m-1} ,$$

every Martin-Löf test $V \subseteq X^* \times \mathbb{N}_+$ will satisfy Eq. (2″) when we regard V as a Martin-Löf test over a larger alphabet $Y \supset X$. This yields Theorem 3 of [2].

Corollary 9. Let $V \subseteq X^* \times \mathbb{N}_+$ be an M-L test over X. Then for any larger alphabet $Y \supset X$ the set V is an M-L test representable over Y.

Define for $u \in X^*$ and a set $V \subseteq X^* \times \mathbb{N}$ their concatenation $uV =_{\mathrm{df}} \{(uv, m) : : (v, m) \in V\}$. Clearly, if V is a Martin-Löf test over X and $u \in X^*$ then uV is also a Martin-Löf test over X.

Corollary 10. Let $u \in X^*$, $|u| \geqq 1$. Then uV is an M-L test representable over X whenever $V \subseteq X^* \times \mathbb{N}_+$ is an M-L test over X.

Proof. Since $k =_{\mathrm{df}} |u| \geqq 1$, we have

$$\operatorname{card}(uV_m \cap X^n) = \operatorname{card} V_m \cap X^{n-k} \leqq \frac{p^{n-k-m} - 1}{p - 1} \leqq p^{n-m-1},$$

and the assertion follows from Lemma 8. □

It is interesting to note that Corollary 10 yields the well-known (cf. [5]) relation

$$(9) \qquad m_V(w) \leqq |w| - K(w/|w|) + c_V \quad \text{for all} \quad w \in X^*$$

between the critical level function of a Martin-Löf test V and a universal Kolmogorov complexity function K (cf. [4]) not utilizing the existence of a universal Martin-Löf test. Let V be a Martin-Löf test over X, and let $u \in X$. Following Corollary 10, there is a partial recursive function φ such that $uV = V(\varphi)$. Consequently

$$(10) \qquad m_{uV}(uw) = |uw| - K_\varphi(uw/|uw|) - 1$$

whenever $uw \in uV_1$, i.e. $w \in V_1$. Clearly,

$$(11) \qquad m_{uV}(uw) = m_V(w), \quad \text{for all} \quad w \in X^*.$$

Since K is a universal Kolmogorov complexity function, there is a c_φ depending only on φ such that

$$(12) \qquad K_\varphi(w/|w|) \geqq K(w/|w|) - c_\varphi \quad \text{for all} \quad w \in X^*.$$

Moreover (cf. [8]), there is a c satisfying

$$(13) \qquad K(uw/|uw|) \geqq K(w/|w|) - c - 2 \log |u|$$

for all $u, w \in X^*$.
Now, substituting Eqs. (11), (12) and (13) into Eq. (19) and utilizing $|u| = 1$ we get

$$(9') \qquad m_V(w) \leqq |w| - K(w/|w|) + c_\varphi + c$$

for $w \in V_1$, where $c_\varphi + c$ depends only on V. If $w \notin V_1$, $m_V(w) = 0$ and (9′) is trivially satisfied.

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