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A GENERALIZATION OF ENTROPY EQUATION: HOMOGENEOUS ENTROPIES

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In the present communication, we propose a generalization of Kamiński and Mikusiński's entropy equation, and characterize what may be called homogeneous entropies of degrees 1 and β.

1. INTRODUCTION

Shannon's entropy has been characterized in several ways. Kamiński and Mikusiński [5] simplified Fadeev's [3] approach by considering, what they called the entropy equation:

$$H(x, y, z) = H(x + y, 0, z) + H(x, y, 0), \quad (x \geqq 0, \; y \geqq 0, \; z \geqq 0, \; xy + yz + zx > 0). \tag{1.1}$$

A general continuous symmetric solution of (1.1) given by

$$H(x, y, z) = (x + y + z) \log (x + y + z) - x \log x - y \log y - z \log z, \tag{1.2}$$

was obtained by Kamiński and Mikusiński [5] under homogeneity of degree 1, viz.

$$H(\lambda x, \lambda y, \lambda z) = \lambda H(x, y, z), \quad \lambda > 0. \tag{1.3}$$

Aczél [1] solved (1.1) under weaker regularity conditions. Sharma and Singh [6] relaxed (1.3) in two different ways:

(i) by considering homogeneity of degree β, given by

$$H(\lambda x, \lambda y, \lambda z) = \lambda^{\beta} H(x, y, z), \quad \lambda > 0, \; \beta > 0, \; \beta \neq 1, \tag{1.4}$$

and (ii) by 'bi-homogeneity' of degree (α, β) given by

$$H(\lambda x, \lambda y, \lambda z) = A \lambda^{\alpha} H_{\alpha}(x, y, z) + B \lambda^{\beta} H_{\beta}(x, y, z), \quad \lambda > 0, \; \alpha \neq \beta, \; \beta > 0, \; \alpha \neq 1, \; \beta \neq 1, \tag{1.5}$$

where A and B are arbitrary constants, and obtained the generalized solutions of

(1.1), viz.

$$H(x, y, z) = C[(x + y + z)^\beta - x^\beta - y^\beta - z^\beta] \tag{1.6}$$

and

(1.7)

$$H(x, y, z) = C[(x + y + z)^\alpha - x^\alpha - y^\alpha - z^\alpha] + D[(x + y + z)^\beta - x^\beta - y^\beta - z^\beta]$$

respectively in the two cases.

In the present paper, we propose a generalization of (1.1). We consider equation in H, a homogeneous function of three variables, satisfying a certain difference relation containing change in two variables, the difference being free from the variable held fixed.

The generalized entropy equation, that we propose, is taken up in Section 2. In solving the equation, method of Kamiński and Mikusiński [5] has been adopted. It is possible to relax the regularity condition of continuity and work out the solution on the lines of Aczél [1]. Two solutions arise by considering homogeneity of order 1 and different from 1. In Section 3, entropy of a discrete probability distribution is defined in terms of these solutions. Under an additional boundary condition, these jointly characterize Shannon's entropy and type-β entropy of Havrda and Charvát [4].

2. THE GENERALIZED ENTROPY EQUATION

Let $H(x, y, z)$ be a homogeneous function of degree β (i.e. (1.4)) in the domain $\mathscr{D}$: $\{(x, y, z), x, y, z \geqq 0, xy + yz + zx > 0\}$ satisfying

$$H(x, y, z) - H(x + y, 0, z) = G(x, y)\,, \tag{2.1}$$

where $G(x, y)$ is also defined in $\mathscr{D}$. The function $G(x, y)$ is obviously homogeneous of degree β.

Equation (2.1) is a generalization of (1.1) in so much so that it involves two functions, one of two variables and the other of three variables.

In what follows, homogeneous degree β (>0) functions $H(x, y, z)$ i.e. (1.4) in $\mathscr{D}$, which satisfy (2.1) in region $\mathscr{D}^0$: $\{(x, y, z) : x, y, z \geqq 0, x + y + z = 1\}$ have been obtained. As is done by Kamiński and Mikusiński [5], to avoid the restriction of twice differentiable functions, the solutions of (2.1) are sought in the class of distributions. Thus we consider $H(x, y, z)$ as a distribution defined in an open set $\mathscr{O}$ including $\mathscr{D}$. The theorem may be stated as follows:

Theorem 1. If $H(x, y, z)$, a distribution defined in an open set $\mathscr{O}$ including $\mathscr{D}$, is symmetric, homogeneous of degree β in $\mathscr{D}^0$ and satisfies (2.1), where $G(x, y)$ is a distribution in $\mathscr{O}$, then H can have two forms viz. $H^1(x, y, z)$ when $\beta = 1$, and $H^\beta(x, y, z)$ when $\beta \neq 1$, where

$$H^1(x, y, z) = a[(x + y + z)\log(x + y + z) - x\log x - y\log y - z\log z] + b(x + y + z)\,, \tag{2.2}$$

and

$$(2.3)\qquad H^\beta(x, y, z) = c(x + y + z)^\beta - d(x^\beta + y^\beta + z^\beta), \quad \beta \neq 1,$$

a, b, c, d being arbitrary constants.

Note: The values of $G(x, y)$ corresponding to two solutions are $G^1(x, y)$ and $G^\beta(x, y)$ simply given by

$$(2.4)\qquad G^1(x, y) = a[(x + y)\log(x + y) - x \log x - y \log y]$$

and

$$(2.5)\qquad G^\beta(x, y) = d[(x + y)^\beta - x^\beta - y^\beta].$$

Proof. Differentiating (2.1) with respect to z, we get

$$(2.6)\qquad H_z(x, y, z) = H_z(x + y, 0, z)$$

where H_z denotes the distributional derivative of H with respect to z. Next differentiating (2.6) with respect to x and then with respect to y, we obtain

$$(2.7)\qquad H_{zx}(x, y, z) = H_{zx}(x + y, 0, z)$$

and

$$(2.8)\qquad H_{zy}(x, y, z) = H_{zy}(x + y, 0, z).$$

These give $H_{zx} = H_{zy}$.

Also invoking symmetry of $H(x, y, z)$, we get

$$(2.9)\qquad H_{xy} = H_{xz} = H_{yx} = H_{yz} = H_{zx} = H_{zy} = H'' \quad \text{(say)}$$

which means that $H''(x, y, z)$ is also symmetric. Thus from (2.7) and (2.8), we get

$$(2.10)\qquad H''(x, y, z) = H''(x + y, 0, z) = H''(y + z, 0, x).$$

Now, in view of Lemma 1 of Kamiński and Mikusiński [5], there exists a distribution M' of one variable such that

$$(2.11)\qquad H''(x, y, z) = M'(x + y + z)$$

since H'' can be taken as $H_{yz}(x, y, z)$ or $H_{xz}(x, y, z)$. On integrating these with respect to y and x respectively, we get

$$H_z(x, y, z) = M(x + y + z) + p(y, z)$$

and

$$H_z(x, y, z) = M(x + y + z) + q(x, z)$$

where M is a primitive distribution of M' and p, q are distributions of y, z and x, z respectively. On comparing these we find that $p(y, z)$ and $q(x, z)$ have to be same. Thus $p(y, z) = q(x, z) = -N(z)$ say, so that

$$H_z(x, y, z) = M(x + y + z) - N(z).$$

Similarly

$$H_x(x, y, z) = M(x + y + z) - N(x)$$

and

$$H_y(x, y, z) = M(x + y + z) - N(y).$$

Now using Euler's theorem for homogeneous distributions, (c.f. [2], pp. 71, (4.1.9)), we have

$$\beta H(x, y, z) = x H_x(x, y, z) + y H_y(x, y, z) + z H_z(x, y, z),$$

i.e.

$$\beta H(x, y, z) = (x + y + z) M(x + y + z) - x N(x) - y N(y) - z N(z). \tag{2.12}$$

Next step in the proof is to determine the distributions M and N of single variable. There arise two cases.

Case 1. When $\beta \neq 1$:
From homogeneity of degree β of $H(x, y, z)$ we get

$$\begin{aligned} H(x, y, z) = (x + y + z) \lambda^{1-\beta} M(\lambda(x + y + z)) - x \lambda^{1-\beta} N(\lambda x) - \\ - y\lambda^{1-\beta} N(\lambda y) - z \lambda^{1-\beta} N(\lambda z). \end{aligned} \tag{2.13}$$

Differentiating (2.13) with respect to λ and then simplifying, we get

$$\begin{aligned} (x + y + z)[(1-\beta) M(\lambda(x + y + z)) + (x + y + z) M'(\lambda(x + y + z))] = \\ = x[(1 - \beta) N(\lambda x) + x N'(\lambda x)] + y[(1 - \beta) N(\lambda y) + y N'(\lambda y)] + \\ + z[(1 - \beta) N(\lambda z) + z N'(\lambda z)]. \end{aligned} \tag{2.14}$$

Setting $\bar{x}, \bar{y}, \bar{z}$ for $\lambda x, \lambda y, \lambda z$ and then x, y, z for $\bar{x}, \bar{y}$ and $\bar{z}$ in (2.14), we obtain

$$\begin{aligned} (1 - \beta)(x + y + z) M(x + y + z) + (x + y + z)^2 M'(x + y + z) = \\ = x(1 - \beta) N(x) + x^2 N'(x) + y(1 - \beta) N(y) + y^2 N'(y) + \\ + z(1 - \beta) N(z) + z^2 N'(z), \end{aligned}$$

or

$$g(x + y + z) = f(x) + f(y) + f(z) \tag{2.15}$$

where

$$f(x) = (1 - \beta) x N(x) + x^2 N'(x) \tag{2.16}$$

and

$$g(x) = (1 - \beta) x M(x) + x^2 M'(x). \tag{2.17}$$

Now differentiating (2.15) with respect to x and y, we get

$$g'(x + y + z) = f'(x)$$

and

$$g'(x + y + z) = f'(y).$$

From these it follows that,

$$f'(x) = f'(y) = a \quad (\text{constant}) \quad (\text{say}).$$

Giving

$$f(x) = ax + b \tag{2.18}$$

where b is an arbitrary constant.

From (2.16) and (2.18), we then get

$$x^2 N'(x) + (1 - \beta) x N(x) = ax + b$$

which, as $\beta \neq 1$, is a linear differential equation of first order and first degree whose solution is

$$N(x) = dx^{\beta-1} - \frac{b}{\beta} x^{-1} + \frac{a}{1-\beta}, \tag{2.19}$$

where d is an arbitrary constant.

Next (2.15), in view of (2.18) gives

$$g(x + y + z) = a(x + y + z) + 3b,$$

which on setting $\bar{x} = x + y + z$, $\bar{y} = y$, $\bar{z} = z$ and then $\bar{x} = x$ gives

$$g(x) = ax + 3b.$$

With this expression for $g(x)$, (2.17) reduces to a differential equation of first order and first degree in $M(x)$, whose solution is

$$M(x) = cx^{\beta-1} - \frac{3b}{\beta} x^{-1} + \frac{a}{1-\beta}, \tag{2.20}$$

where c is an arbitrary constant.

For the expressions of $M(x)$ and $N(x)$ obtained in (2.19) and (2.20), $\beta H(x, y, z)$ in (2.12) is just $H^\beta(x, y, z)$ as given in (2.3).

Case 2. When $\beta = 1$:

This case is largely similar to what has been handled by Kamiński and Mikusiński, proceeding as there, we get

$$\begin{aligned} f(x) &= ax + b \\ N(x) &= a \log x - (b/x) + c \\ M(x) &= a \log x - (3b/x) + d \end{aligned}$$

and then from (2.12) $H(x, y, z) = H^1(x, y, z)$ with $d - c$ replaced by b in the final result.

This completes the proof of the theorem. □

3. MEASURES OF HOMOGENEOUS ENTROPY

Let us consider a discrete random variable x taking finite number of values $x_1, x_2, \ldots, x_n$ with an associated probability distribution $P = (p_1, p_2, \ldots, p_n)$, $\sum_{i=1}^{n} p_i = 1$.

The homogeneous entropy $H_n(p_1, p_2, \ldots, p_n)$ of the distribution $P = (p_1, p_2, \ldots, p_n)$, may now be defined in terms of $H(x, y, z)$ and $G(x, z)$ as follows:

Definition. For $n = 2$

$$H_2(p_1, p_2) = G(p_1, p_2)\,. \tag{3.1}$$

For $n = 3$

$$H_3(p_1, p_2, p_3) = H(p_1, p_2, p_3)\,. \tag{3.2}$$

For $n \geqq 4$

$$H_n(p_1, p_2, \dots, p_n) = H_{n-1}(p_1 + p_2, p_3, \dots, p_n) + G(p_1, p_2)\,. \tag{3.3}$$

The last recurrence relation may be used to express the homogeneous entropy in terms of $H(x, y, z)$ and $G(x, y)$ in the following way:

$$H_n(p_1, p_2, \dots, p_n) = H(s_{n-2}, p_{n-1}) + \sum_{i=1}^{n-2} G(s_i, p_{i+1}) \tag{3.4}$$

where $s_i = p_1 + p_2 + \dots + p_i$.

Corresponding to the two solutions obtained in Theorem 1, we have then the two forms of the homogeneous entropy.

Theorem 2. The homogeneous entropy of a generalized probability distribution $P = (p_1, p_2, \dots, p_n)$, $(\sum p_i \leqq 1)$ as defined above, can be only of one of the following forms:

$$H_n^1(p_1, \dots, p_n) = -a \sum_{i=1}^{n} p_i \log \frac{p_i}{\sum_{i=1}^{n} p_i} + b \sum_{i=1}^{n} p_i \tag{3.5}$$

with $b = 0$ for $n = 2$, or

$$H_n^\beta(p_1, \dots, p_n) = c\Big(\sum_{i=1}^{n} p_i\Big)^\beta - d \sum_{i=1}^{n} p_i^\beta\,, \quad \beta \neq 1 \tag{3.6}$$

where a, b, c, d are arbitrary constants.

The results follow immediately from definition of homogeneous entropy and the two sets of H and G functions in (2.2), (2.4) and (2.3), (2.5) respectively.

Note. The measures (3.5) and (3.6) clearly do not satisfy the condition

$$H_n(1, 0, \dots, 0) = 0\,. \tag{3.7}$$

In fact

$$H_n^1(1, 0, \dots, 0) = b$$

and

$$H_n^\beta(1, 0, \dots, 0) = c - d\,.$$

However if (3.7) is taken to hold, then the entropy has to be of only of one of the following two forms:

$$H_n^1(p_1, \dots, p_n) = -a \sum p_i \log \frac{p_i}{\sum p_i}$$

and

$$H_n^\beta(p_1, \ldots, p_n) = c[(\sum p_i)^\beta - \sum p_i^\beta], \quad \beta \neq 1, \quad \beta > 0$$

where a and c are arbitrary constants.

When $\sum p_i = 1$, these are in fact Shannon's entropy and type-β entropies. By defining the unit suitably, constants a and c can be specified further.

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