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# ON THE EQUIVALENCE CONDITIONS OF TWO-STAGE AND DIRECT IDENTIFICATION METHODS

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The paper deals with the parametric problem of multiple-input multiple-output (MIMO) system identification under deterministic conditions.

Two different identification procedures: two-stage [12] and a direct one are discussed and the equivalence conditions of these algorithms — in the sense of the obtained identification result — are investigated with a special emphasis given to the linear class of models. For this class the respective a priori and a posteriori equivalence conditions are formulated and the usefulness of the former for the experiment design purposes is examined.

The considerations are devoted to static as well as to dynamic systems for different kinds of experiment.

## 1. INTRODUCTION

In practice most of the systems being the subject of control are multiple-input, multiple-output (MIMO) systems. To formulate suitable control algorithms for such a systems it is necessary to establish their mathematical models.

Problems related to the MIMO systems identification lead, generally, to computationally complex identification algorithms, the realization of which is connected with great difficulties even when modern computational techniques are used. Moreover, the computational complexity of the identification algorithms grows, in general, faster than the dimension of problems. For this reason the search for such methods which would allow us to reduce the computational efforts connected with MIMO systems identification is justified. So far the possibilities of simplification of MIMO systems identification have been studied by several authors.

The first papers in this field were based on the concept of the system order reduction using the perturbation techniques and sensitivity analysis (e.g. [1]) and on the concept of its approximation by a system of a lower order (e.g. [2–8]). Another, widely developing direction in the literature is that connected with decomposition of the identification problem leading to hierarchical identification algorithms for

MIMO systems. Such algorithms are obtained from the decomposition of the system to be identified into the set of subsystems of a lower order and the coordination of the partial identification algorithms formulated for these subsystems. Typical examples in this respect are the papers [9–10].

In order to reduce the computational difficulties related to the MIMO system identification some multi-stage identification method is proposed in [11, 12] for the case of static systems which is next considered in two-stage version in [13]. In the literature the term two-stage multi-stage identification is understood in different ways ([14–22]).

Generally, the identification algorithms discussed in [14–22] and referred to as the two-stage ones can be divided into two subclasses. In the first subclass ([14–19]) on the 1st stage some quantities (e.g. the weighting sequence or the step response) are obtained, which on the 2nd stage are used as data for identification of the unknown parameters in the system description. In the second subclass ([20–22]) on the 1st stage a rough model is determined which then is improved in one step on the 2nd stage.

The main idea of the two-stage identification method proposed in [11, 12] and discussed in this paper can be briefly described as follows. In the first stage a model parameter $x$ is determined using the results of the measurements of some part of inputs $u^{(1)}$ and output $y$ for remaining inputs $u^{(2)}$ being fixed. In the second stage the model of the relationship between $x$ and $u^{(2)}$ (constant in the first stage) is determined. With such an approach a fictious object with input $u^{(2)}$ and output $x$ is considered. The model of the whole system is obtained as a composition of the models resulting in consecutive stages. The precise description of the method is given in Section 2.

Since according to the concept of [11, 12], the results of identification in the 1st stage constitute the data for the identification in the second stage, the two-stage approach proposed in [11, 12] should be classified among the first subclass (the parameter $x$ of the model from the 1st stage is datum for identification in the 2nd stage).

The above idea of MIMO system identification includes the usual experimentation restriction raising for such systems and consisting in the fact that in case of high number of system inputs only some part of them can be in practice manipulated by the experiment at the same time. Other justifications of this method can be found in [13].

The system model identification in two stages is expected to give some computational advantages, especially for a great number of system inputs (the first promising attempt in the direction of the computational complexity estimation has been made in [18]), but – in general – it is connected with the deterioration of the model quality, if compared with the model obtained in one stage, i.e. by the Direct Identification (DI) method. The experimental investigations of the method are actually in course. In this paper the attention is concentrated on some theoretical aspects of the method, exclusively. In particular, we shall formulate general conditions of the equivalence of two-stage and direct identification methods, i.e. the conditions assuring the identity of the models obtained by these two methods. In such a case the more convenient for MIMO systems two-stage approach can be realized without the deterioration of the model quality.

The considerations will be focused on the case when the system identification consists in choosing the best model (in the sense of some quality index) from the given parametric class of models, i.e. it is reduced to the determination of the parameter of the best model. The considerations are confined to the deterministic case of the input-output systems and refer to static as well as to dynamic systems and different kinds of experiments (continuous or discrete-time).

## 2. TWO-STAGE IDENTIFICATION (2SI) METHOD

Consider the MIMO system with the inputs $u^1, \ldots, u^s$ $(s \geqq 2)$ and the outputs $y^1, \ldots, y^l$. The collection of the inputs will be denoted by a vector $u \triangleq [u^1, \ldots, u^s]^T$. Similarly, $y \triangleq [y^1, \ldots, y^l]^T$. We shall assume that the results of the measurements of the input $u$ and output $y$ of the system are elements of linear normed spaces $(U, \|\cdot\|_U)$ and $(Y, \|\cdot\|_Y)$, respectively. They are in general the spaces of vector-valued functions of the real variable $t$: continuous in a continuous-time experiment and discrete in a discrete-time experiment.

The two-stage approach being the subject of our considerations is the following ([11, 12]).

*1st stage* (1*S*)

For the fixed decomposition $U^{(1)} \times U^{(2)} = U$ of the space $U$ and for the fixed value $u_i^{(2)}$ of the vector $u^{(2)} \in U^{(2)}(u^{(2)} = u_i^{(2)})$ the best model from the parametric class of models

$$y_M = F_1(u^{(1)}, x), \quad x \in X$$

(generated by the given mapping $F_1$: $U^{(1)} \times X \to Y$ and the linear normed space $(X, \|\cdot\|_X)$) is obtained, by using the results of the measurements $(u_{ij}^{(1)}, y_{ij})$, $j = 1, \ldots$ $\ldots, n_1$, of the pair $(u^{(1)}, y)$, from determination of a parameter $\tilde{x}_i \in X$ minimizing the value of the assumed identification quality criterion:

$$Q_1(x_i) = \|[\|y_{ij} - F_1(u_{ij}^{(1)}, x_i)\|_Y]_{j=1}^{n_1}\|_{R^{n_1}}^p, \tag{1}$$

$[\cdot]_{j=1}^{n_1}$ denotes the column vector in the space $R^n$, $p > 0$. Such a procedure is repeated for the successively fixed values of the vector $u^{(2)} = u_i^{(2)}$, $i = 1, \ldots, n_2$.

*2nd stage* (2*S*)

Using the pairs $(u_i^{(2)}, \tilde{x}_i)$, $i = 1, \ldots, n_2$, resulting from the first stage, the best model from the parametric class of models:

$$x_M = F_2(u^{(2)}, a), \quad a \in \mathscr{A}$$

(generated by the given mapping $F_2$: $U^{(2)} \times \mathscr{A} \to X$ and the set $\mathscr{A}$ of admissible parameters of the model) is obtained for the chosen identification quality criterion

$$(2) \qquad Q_2(a) = \| [\| \tilde{x}_i - F_2(u_i^{(2)}, a) \|_X]_{i=1}^{n_2} \|_{R^{n_2}}^{p}$$

i.e. the parameter $\tilde{a} \in \mathscr{A}$ minimizing the value of the functional $Q_2$ is determined.

The model of the whole identification system (Fig. 1) is composed in the following way:

$$y_M = F_1(u^{(1)}, F_2(u^{(2)}, \tilde{a})) = F(u, \tilde{a}) .$$

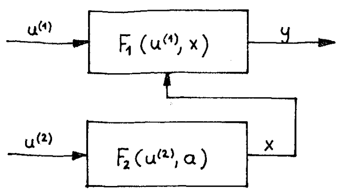


**Fig. 1.** Model of the system for two-stage identification concept.

From the above description it is evident that the two-stage approach is connected with a suitable organization of the experiment, in which for each successive fixed value of the subvector $u^{(2)}$ the value of the subvectors $u^{(1)}$ is changed $n_1$ times (two-stage structure of the experiment [11, 13]).

In the next section the general estimation of the quality of the model obtained by the two-stage approach is given and the equivalence conditions of 2SI and DI methods are investigated with special attention paid to the most frequent case of linear, with respect to the parameters, classes of models.

## 3. THE EQUIVALENCE CONDITIONS OF 2SI AND DI METHODS

Assume the following class of models:

$$(3) \qquad y_M = F_1(u^{(1)}, F_2(u^{(2)}, a)) , \quad a \in \mathscr{A} ,$$

defined by the mappings $F_1$: $U^{(1)} \times X \to Y$ and $F_2$: $U^{(2)} \times \mathscr{A} \to X$, where $U^{(2)} \times \times U^{(2)} = U$. We shall assume that the spaces $U$, $Y$, $X$ and the set of the parameters $\mathscr{A}$ are linear normed spaces.

Let us notice that each model

$$(4) \qquad y_M = F(u, a) , \quad a \in \mathscr{A}$$

can be represented in the form (3) for an arbitrary partition $u^T = [u^{(1)T}, u^{(2)T}]$

of the vector $u \in U$. In particular it suffices to take $F_1 = F$ and assume that $F_2$ is an identity mapping.

DI problem consists in choosing the best, in the sense of the taken quality criterion $Q$, model from the assumed class of models (3), using the results of the input and output measurements, i.e. in determining the model parameter $a^* \in \mathscr{A}$ minimizing the value of $Q$. If, in particular, the data used for the identification purposes are obtained in the two-stage experiment, then in general the quality index takes the form

$$Q(a) = \|\{\|y_{ij} - F_1(u_{ij}^{(1)}, F_2(u_i^{(2)}, a))\|_Y]_{j=1}^{n_1}\}_{i=1}^{n_2}\|_{R^{n_1 n_2}}^p , \tag{5}$$

where $\{\xi_i\}_{i=1}^n \triangleq [\xi_1^T, \xi_2^T, \ldots, \xi_n^T]^T$. The definite form of quality criterion (5) depends on the choice of the norm $\|\cdot\|_{R^{n_1 n_2}}$. In particular, if this norm satisfies the following condition

$$\|\{[\xi_{ij}]_{j=1}^{n_1}\}_{i=1}^{n_2}\|_{R^{n_1 n_2}}^p = \sum_{i=1}^{n_2} \|[\xi_{ij}]_{j=1}^{n_1}\|_{R^{n_1}}^p \tag{6}$$

then

$$Q(a) = \sum_{i=1}^{n_2} \|[\|y_{ij} - F_1(u_{ij}^{(1)}, F_2(u_i^{(2)}, a))\|_Y]_{j=1}^{n_1}\|_{R^{n_1}}^p . \tag{7}$$

Generally the model established in the result of two-stage identification differs from that obtained in a direct fashion ([12]), i.e. $Q(\tilde{a}) > Q(a^*)$.

Now we shall examine the quality of an identification problem introduced by two-stage approach. Further considerations will be restricted to the case when for each $u^{(1)} \in U^{(1)}$ the mapping $F_1$ is Lipschitz with respect to $x$, i.e.

$$\|F_1(u^{(1)}, x_1) - F_1(u^{(1)}, x_2)\|_Y \leqq \alpha(u^{(1)})\| x_1 - x_2\|_X, u^{(1)} \in U^{(1)} ,$$

where $\alpha(u^{(1)}) > 0$. Such an assumption is not too restrictive, since it is typical of many modelling problems.

Consider the "distance" between the model determined by 2SI method and that obtained by DI method in the sense of the index:

$$\varrho(\tilde{a}, a^*) \triangleq Q(\tilde{a}) - Q(a^*) .$$

By virtue of (7) we obtain:

$$\varrho(\tilde{a}, a^*) = \sum_{i=1}^{n_2} \{\|[\|y_{ij} - F_1(u_{ij}^{(1)}, F_2(u_i^{(2)}, \tilde{a}))\|_Y]_{j=1}^{n_1}\|_{R^{n_1}}^p + \\ - \|[\|y_{ij} - F_1(u_{ij}^{(1)}, F_2(u_i^{(2)}, a^*))\|_Y]_{j=1}^{n_1}\|_{R^{n_1}}^p\} .$$

For Lipschitz $F_1$ it may be easily shown that:

$$\varrho(\tilde{a}, a^*) = \sum_{i=1}^{n_2} \|[\|F_1(u_{ij}^{(1)}, \tilde{x}_i) - F_1(u_{ij}^{(1)}, F_2(u_i^{(2)}, a^*))\|_Y]_{j=1}^{n_1}\|_{R^{n_1}}^p + \\ + \sum_{i=1}^{n_2} \|[\alpha(u_{ij}^{(1)}) \|\tilde{x}_i - F_2(u_i^{(2)}, \tilde{a})\|_X]_{j=1}^{n_1}\|_{R^{n_1}}^p .$$

Consider the expression:

$$\delta \triangleq \sum_{i=1}^{n_2} \left\| \left[ \alpha(u_{ij}^{(1)}) \left\| \tilde{x}_i - F_2(u_i^{(2)}, \tilde{a}) \right\|_X \right]_{j=1}^{n_1} \right\|_{R^{n_1}}^p =$$

$$= \sum_{i=1}^{n_2} \left\| \left[ \alpha(u_{ij}^{(1)}) \right]_{j=1}^{n_1} \right\|_{R^{n_1}}^p \cdot \left\| \tilde{x}_i - F_2(u_i^{(2)}, \tilde{a}) \right\|_X^p .$$

By virtue of the equivalence of the norms in $R^{n_2}$

$$\delta \leqq c \left\| \left[ \left\| \tilde{x}_i - F_2(u_i^{(2)}, \tilde{a}) \right\|_X \right]_{i=1}^{n_2} \right\|_{R^{n_2}}^p ,$$

where $\|\cdot\|_{R^{n_2}}$ is an arbitrary norm in $R^{n_2}$ and $c$ a certain positive constant independent of $\tilde{a}$ and $u_i^{(2)}$. Hence

$$\varrho(\tilde{a}, a^*) \leqq \sum_{i=1}^{n_2} \left\| \left[ \left\| F_1(u_{ij}^{(1)}, \tilde{x}_i) - F_1(u_{ij}^{(1)}, F_2(u_i^{(2)}, a^*)) \right\|_Y \right]_{j=1}^{n_1} \right\|_{R^{n_1}}^p + c \left\| \left[ \left\| \tilde{x}_i - F_2(u_i^{(2)}, \tilde{a}) \right\|_X \right]_{i=1}^{n_2} \right\|_{R^{n_2}}^p . \tag{8}$$

From the above estimation it follows, in particular, that the difference between the model of the system obtained in two-stages and that obtained by DI method (in the sense of the value of the quality criterion $Q$) is the smaller, the better is the quality of identification in the second stage. In particular, if $Q_2(\tilde{a}) = 0$ and hence taking account of (1) and (7) $F_1(u_{ij}^{(1)}, \tilde{x}_i) = F_1(u_{ij}^{(1)}, F_2(u_i^{(2)}, a^*))$ hold, we see that the right hand side of the estimation (8) vanishes; therefore the quality of the model obtained by 2SI method is the same as that obtained by DI method.

Let us consider the equivalence conditions of 2SI and DI methods. Denote

$\bar{u}^{(1)} \triangleq \{\{u_{ij}^{(1)}\}_{j=1}^{n_1}\}_{i=1}^{n_2}$ – vector of measurements of the input $u^{(1)}$,

$\bar{u}^{(2)} \triangleq \{u_i^{(2)}\}_{i=1}^{n_2}$ – vector of measurements of the input $u^{(2)}$,

$\bar{y} \triangleq \{\{y_{ij}\}_{j=1}^{n_1}\}_{i=1}^{n_2}$ – vector of measurements of the output $y$,

$\bar{x} \triangleq \{x_i\}_{i=1}^{n_2}$ – vector of values of the parameter $x$,

$$\hat{Q}_1(\bar{x}) \triangleq \sum_{i=1}^{n_2} Q_1(x_i) \tag{9}$$
– aggregated identification quality criterion in the first stage in 2SI method,

$$\bar{y}_M = \bar{F}_1(\bar{u}^{(1)}, \bar{x}) \triangleq \{\{F_1(u_{ij}^{(1)}, x_i)\}_{j=1}^{n_1}\}_{i=1}^{n_2}, \quad (\bar{u}^{(1)}, \bar{x}) \in \bar{U}^{(1)} \times \bar{X}, \tag{10}$$

$$\bar{x}_M = \bar{F}_2(\bar{u}^{(2)}, a) \triangleq \{F_2(u_i^{(2)}, a)\}_{i=1}^{n_2}, \quad (\bar{u}^{(2)}, a) \in \bar{U}^{(2)} \times \mathscr{A},$$

where $\bar{U}^{(1)} = \mathop{\times}_{i=1}^{n_1 n_2} U^{(1)}$, $\bar{U}^{(2)} = \mathop{\times}_{i=1}^{n_2} U^{(2)}$, $\bar{X} = \mathop{\times}_{i=1}^{n_2} X$, $\bar{Y} = \mathop{\times}_{i=1}^{n_1 n_2} Y$,

and let

$$\|\bar{y}\|_{\bar{Y}} \triangleq \left\| \left\{ \left[ \|y_{ij}\|_Y \right]_{j=1}^{n_1} \right\}_{i=1}^{n_2} \right\|_{R^{n_1 n_2}}$$

$$\|\bar{x}\|_{\bar{X}} \triangleq \left\| \left[ \|x_i\|_X \right]_{i=1}^{n_2} \right\|_{R^{n_2}} .$$

In the notations assumed the identification quality criterion $Q(a)$ ((5)) for the DI method takes the form:

$$(11) \qquad Q(a) = \|\bar{y} - \bar{F}_1(\bar{u}^{(1)}, \bar{F}_2(\bar{u}^{(2)}, a))\|_{\bar{Y}}^p$$

and for the 2SI method the quality criteria $\hat{Q}_1(\bar{x})$ (9) and $Q_2(a)$ (2) on 1st and 2nd stages, respectively, can be rewritten as follows:

$$(12) \qquad \begin{cases} \hat{Q}_1(\bar{x}) = \|\bar{y} - \bar{F}_1(\bar{u}^{(1)}, \bar{x})\|_{\bar{Y}}^p \\ Q_2(a) = \|\tilde{\bar{x}} - \bar{F}_2(\bar{u}^{(2)}, a)\|_{\bar{X}}^p . \end{cases}$$

Further we shall assume that there exist the solutions of the DI problem and in both stages of 2SI problem and that they are unique. We introduce the following

**Definition 1.** For the given matrix of measurements $[\bar{u}^{(1)}, \bar{u}^{(2)}, \bar{y}]$, classes of models $(F_1, X), (F_2, \mathscr{A})$ and identification quality criteria (11) and (12), the methods 2SI and DI are equivalent, if the resulting models are identical, i.e. $a^* = \tilde{a}$.

Assume, that $\bar{X}$, $\bar{Y}$ and $\mathscr{A}$ are real Hilbert spaces. It is the case if $X$, $Y$ (see Section 2) are Hilbert spaces and the norms $\|\cdot\|_{R^{n_2}}$ and $\|\cdot\|_{R^{n_1 n_2}}$ in (2) and (5), respectively, are Euclidean norms (such norms fulfill (6) with $p = 2$). Taking $p = 2$ the criteria (11) and (12) can be rewritten in the following equivalent form:

$$(13) \qquad Q(a) = \langle \bar{y} - \bar{F}_1(\bar{u}^{(1)}, \bar{F}_2(\bar{u}^{(2)}, a)), \bar{y} - \bar{F}_1(\bar{u}^{(1)}, \bar{F}_2(\bar{u}^{(2)}, a)) \rangle$$

$$(14) \qquad \hat{Q}_1(\bar{x}) = \langle \bar{y} - \bar{F}_1(\bar{u}^{(1)}, \bar{x}), \bar{y} - \bar{F}_1(\bar{u}^{(1)}, \bar{x}) \rangle$$

$$(15) \qquad Q_2(a) = \langle \tilde{\bar{x}} - \bar{F}_2(\bar{u}^{(2)}, a), \tilde{\bar{x}} - \bar{F}_2(\bar{u}^{(2)}, a) \rangle .$$

Denote

$$\begin{aligned} R_1 &\triangleq \bar{F}_1(\bar{u}^{(1)}, \bar{X}) \subset \bar{Y}, \\ R_2 &\triangleq \bar{F}_1(\bar{u}^{(1)}, \bar{F}_2(\bar{u}^{(2)}, \mathscr{A})) \subset \bar{Y}, \\ S &\triangleq \bar{F}_2(\bar{u}^{(2)}, \mathscr{A}) \subset \bar{X} . \end{aligned}$$

The direct identification problem reduces to the problem of finding the best approximation $\bar{y}^* \in R_2$ to the element $\bar{y}$ of the space $\bar{Y}$ in the set $R_2$. Similarly, in two-stage approach the best approximation $\tilde{\bar{y}} \in R_1$ to the element $\bar{y} \in \bar{Y}$ in the set $R_1$ is determined in the 1st stage and next in the 2nd stage the best approximation $\hat{\bar{x}} \in S$ to the element $\tilde{\bar{x}} \in \bar{X}$, such that

$$(16) \qquad \tilde{\bar{y}} = \bar{F}_1(\bar{u}^{(1)}, \tilde{\bar{x}})$$

is obtained in the set $S$.

If, in particular, the sets $R_1$, $R_2$ and $S$ are closed subspaces, then the elements $\bar{y}^*$, $\tilde{\bar{y}}$ and $\hat{\bar{x}}$ are unique and are orthogonal projections of $\bar{y}$, $\bar{y}$ and $\tilde{\bar{x}}$ onto subspaces $R_2$, $R_1$ and $S$, respectively, i.e.

$$(17) \qquad \bar{y}^* = P_{R_2}\bar{y}$$

$$(18) \qquad \tilde{\bar{y}} = P_{R_1}\bar{y}$$

$$(19) \qquad \hat{\bar{x}} = P_S\tilde{\bar{x}} ,$$

where $P_{R_2}$, $P_{R_1}$ and $P_S$ denote the orthogonal projection operators onto $R_2$, $R_1$ and $S$. In such a case – taking into account that the operation $\bar{F}_1(\bar{u}^{(1)}, \cdot) \triangleq \bar{F}_{1,\bar{u}^{(1)}}(\cdot)$ has an inverse at the point $\tilde{\bar{y}}$ provided that the solution of the 2SI problem is unique – we get from (16) and (18):

$$\tilde{\bar{x}} = \bar{F}^{-1}_{1,\bar{u}^{(1)}}(\{\tilde{\bar{y}}\}) = \bar{F}^{-1}_{1,\bar{u}^{(1)}}(\{P_{R_1}\bar{y}\}) . \tag{20}$$

Hence, and from (19), we have

$$\hat{\bar{y}} \triangleq \bar{F}_1(\bar{u}^{(1)}, \hat{\bar{x}}) = \bar{F}_{1,\bar{u}^{(1)}}(P_S\tilde{\bar{x}}) = \bar{F}_{1,\bar{u}^{(1)}}(P_S\bar{F}^{-1}_{1,\bar{u}^{(1)}}(\{P_{R_1}\bar{y}\})) . \tag{21}$$

Notice that

$$\hat{\bar{y}} = \bar{F}_1(\bar{u}^{(1)}, \bar{F}_2(\bar{u}^{(2)}, \tilde{a}))$$

and

$$\bar{y}^* = \bar{F}_1(\bar{u}^{(1)}, \bar{F}_2(\bar{u}^{(2)}, a^*))$$

for the parameters $\tilde{a}$ and $a^*$ determined by 2SI and DI methods, respectively. The above – by the assumed uniqueness of the solutions of the 2SI and DI problems – yields the following equivalence condition

$$\hat{\bar{y}} = \bar{y}^*$$

or equivalently

$$\bar{F}_{1,\bar{u}^{(1)}}(P_S\bar{F}^{-1}_{1,\bar{u}^{(1)}}(\{P_{R_1}\bar{y}\})) = P_{R_2}\bar{y} \tag{22}$$

in terms of the results of input and output measurements. In order to examine the equivalence of DI and 2SI methods in the general form, i.e. using the condition (22) the forms of the operators $P_S$, $P_{R_1}$ and $P_{R_2}$ should be established for every case considered. Since, these operators depend of the form of $F_1$ and $F_2$, i.e. on the chosen classes of models and on the vectors of measurements $\bar{u}^{(1)}$, $\bar{u}^{(2)}$ obtained on the experiment, the general solution of this problem is not possible and each case should be considered individually. Further considerations regarding the equivalence of 2SI and DI methods will be confined to the case, when the models $F_1(u^{(1)}, x)$ and $F_2(u^{(2)}, a)$ are linear and continuous with respect to the parameters $x$ and $a$, respectively. Then the mappings $\bar{F}_1$ and $\bar{F}_2$ are linear and continuous with respect to $\bar{x}$ and $a$, i.e.

$$\bar{F}_1(\bar{u}^{(1)}, \bar{x}) = A_1\bar{x} \tag{23}$$

$$\bar{F}_2(\bar{u}^{(2)}, a) = A_2 a , \tag{24}$$

where the operators $A_1 = A_1(\bar{u}^{(1)})$ and $A_2 = A_2(\bar{u}^{(2)})$ depend on the measurement results of the subvectors $u^{(1)}$ and $u^{(2)}$, respectively, and $A_1 \in L(\bar{X}, \bar{Y})$, $A_2 \in L(\mathscr{A}, \bar{X})$. In the following the dependence of $u^{(1)}$ and $u^{(2)}$ measurements will remain in $A_1$ and $A_2$, respectively.

We shall assume that the mappings $A_1$ and $A_2$ are invertible (in the subspaces $R_1$ and $S$), respectively. Under the taken assumptions $R_1 \subset \bar{Y}$, $R_2 \subset \bar{Y}$ and $S \subset \bar{X}$ are closed subspaces ([23]) and the solutions of both identification problems are unique. In order to find, on the basis of the general condition (22), the equivalence conditions for the case considered, it is necessary to establish the form of the orthogonal projectors $P_{R_1}$, $P_{R_2}$ and $P_S$.

From the orthogonal projection theorem ([23]) it follows – under the assumption of invertibility of $A_1$ – that for every $\bar{y} \in \bar{Y}$:

$$\bar{y} = P_{R_1}\bar{y} + (\bar{y} - P_{R_1}\bar{y}),$$

holds, where $P_{R_1}\bar{y} \in R_1$ and $(\bar{y} - P_{R_1}\bar{y}) \in \ker(A_1^*)$. Hence

$$A_1^*\bar{y} - A_1^* P_{R_1}\bar{y} = 0 .$$

Simultaneously

$$P_{R_1}\bar{y} = A_1\bar{x} \tag{25}$$

for some $\bar{x} \in \bar{X}$, thus

$$A_1^*\bar{y} = A_1^*A_1\bar{x} .$$

Since, under the taken assumptions, the adjoint operators $A_1^*$ and $A_2^*$ are surjective, then by virtue of the well-known theorem of von Neumann ([24]) symmetric, self-adjoint operators $A_1^*A_1$ and $A_2^*A_2$ are invertible in spaces $\bar{X}$ and $\mathscr{A}$, respectively. Hence, for the case considered $\bar{x} = (A_1^*A_1)^{-1} A_1^*\bar{y}$, its substitution into (25) yields:

$$P_{R_1}\bar{y} = A_1(A_1^*A_1)^{-1} A_1^*\bar{y}$$

and finally

$$P_{R_1} = A_1(A_1^*A_1)^{-1} A_1^* . \tag{26}$$

Similarly it can be shown that

$$P_S = A_2(A_2^*A_2)^{-1} A_2^* . \tag{27}$$

It is easy to check that the operator $A_1A_2$ is invertible in the subspace $R_2$, thus

$$P_{R_2} = A_1A_2(A_2^*A_1^*A_1A_2)^{-1} A_2^*A_1^* . \tag{28}$$

Denoting the inverse image of the element $\bar{y}$ under the mapping $A_1$ by $A_1^{-1}\{\bar{y}\}$ the equality (22) in the case considered can be rewritten in form

$$A_1(A_2(A_2^*A_2)^{-1} A_2^*A_1^{-1}\{A_1(A_1^*A_1)^{-1} A_1^*\bar{y}\}) =$$
$$= A_1A_2(A_2^*A_1^*A_1A_2)^{-1} A_2^*A_1^*\bar{y} ,$$

or equivalently, after simple transformations:

$$A_2^*A_1^*\bar{y} = A_2^*A_1^*A_1A_2(A_2^*A_2)^{-1} A_2^*(A_1^*A_1)^{-1} A_1^*\bar{y} . \tag{29}$$

The above condition – being the equivalence condition (22) for the linear classes of models – may be called the *a posteriori* condition, since the equivalence of the two methods cannot be checked earlier than after the experiment is performed, i.e. after the inputs and outputs of the system are measured. It is rather difficult to use the condition in a direct fashion because of its complicated form. It may be expected that this condition will be simpler if, while examining the equivalence of both methods, not only the measurement results but also e.g. the results of two-stage identification are used. Suitable criterion will be presented in the sequel.

As far as identification practice is concerned it is more advisable to use so-called *a priori* conditions, i.e. such ones which enable the verification of the equivalence of direct and two-stage approaches before the experiment is realized (i.e. the system inputs and outputs are measured). Assuming arbitrareness of the vector $\bar{y}$, from (29) we obtain the following necessary and sufficient *a priori* equivalence condition of 2SI and DI methods:

$$A_2^* A_1^* A_1 A_2 (A_2^* A_2)^{-1} A_2^* (A_1^* A_1)^{-1} A_2 (A_2^* A_2)^{-1} = I \tag{30}$$

$I$ being the identity mapping in $\mathscr{A}$.

Since the operators $A_1$ and $A_2$ depend on $\bar{u}^{(1)}$ and $\bar{u}^{(2)}$, respectively, then on the basis of the above condition it is possible to design a suitable two-stage experiment ensuring the equivalence of both the methods for any identification system (see examples given in Section 4).

Using the condition (30) we shall examine when the equivalence of 2SI and DI methods holds

1. under arbitrary mapping $A_1$,
2. under arbitrary mapping $A_2$,

i.e. we shall consider $A_1$-free and $A_2$-free *a priori* equivalence conditions in the class of linear continuous mappings $A_1$ and $A_2$ invertible in $R_1$ and $S$, respectively.

For the first case the following equivalence condition holds

**Condition 1.** ($A_1$-free.) If $A_2$ is surjective, then 2SI and DI methods are equivalent for any $A_1$.

The above condition follows from the fact that for the surjective $A_2$ we have $(A_2^* A_2)^{-1} = A_2^{-1} A_2^{*-1}$.

In turn it is easy to check that

**Condition 2.** ($A_2$-free.) If $A_1$ is such that

$$A_1^* A_1 = \gamma I \,, \quad \gamma > 0 \,, \tag{31}$$

then 2SI and DI methods are equivalent for any $A_2$.

The above condition can be reformulated using the following lemma

**Lemma.** Let $X$, $Y$ be the real Hilbert spaces. If $A \in L(X, Y)$, then

$$A^* A = c^2 I \,, \quad c \neq 0$$

if and only if

$$\|Ax\| = \|A\| \, \|x\| \,, \quad x \in X \,. \tag{32}$$

Proof. If $A^*A = c^2 I$, then $(Ax, Ax) = c^2(x, x)$. At the same time $(Ax, Ax) \leqq \leqq \|A\|^2 (x, x)$, whence $\|A\| = |c|$, which completes the proof of necessity.

The condition (32), or equivalently,

$$(Ax, Ax) = \|A\|^2 (x, x), \quad x \in X,$$

denoting $B \triangleq A^*A - \|A\|^2 I$, can be rewritten in the following form

$$(33) \qquad (Bx, x) = 0, \quad x \in X.$$

Since $B$ is a Hermitian operator for an arbitrary pair $x_1, x_2 \in X$

$$(Bx_1, x_2) = \tfrac{1}{4}[(B(x_1 + x_2), x_1 + x_2) - (B(x_1 - x_2), x_1 - x_2)]$$

holds on the basis of the polarization formula. In view of (33) we have

$$(Bx_1, x_2) = 0, \quad x_1 \in X, \quad x_2 \in X,$$

hence $B = 0$. Thus for $|c| = \|A\|$ we have $A^*A = c^2 I$ and the sufficiency is proved. □

By virtue of the above lemma the condition (31) may be presented in the form

$$(34) \qquad \|A_1 \bar{x}\| = \|A_1\| \, \|\bar{x}\|, \quad \bar{x} \in \bar{X}.$$

Since the adjoint operator $A_1^*$ does not appear in (34) such a representation of the condition (31) seems to be more convenient.

For the memoryless (static) systems the necessity of the requirements for $A_1$-free and $A_2$-free equivalence of the 2SI and DI methods stated in the Conditions 1 and 2 (after slight modification) can be also proved.

We formulate the following theorems

**Theorem 1.** If the identification system is memoryless and the models in consecutive stages are assumed in the form $\bar{y}_M = \bar{F}_1(\bar{u}^{(1)}, \bar{x}) = A_1 \bar{x}$, $\bar{x}_M = \bar{F}_2(\bar{u}^{(2)}, a) = A_2 a$ (see (23), (24)), the matrices $A_1$ and $A_2$ being dependent on $\bar{u}^{(1)}$ and $\bar{u}^{(2)}$, respectively, then 2SI and DI methods are equivalent for any $A_1$ iff $A_2$, after reordering of the rows can be reduced to the form

$$(35) \qquad \tilde{A}_2 = \begin{bmatrix} N \\ 0 \end{bmatrix},$$

where $N$ is nonsingular matrix formed by nonzero rows of the matrix $A_2$.

Proof. Notice, taking account of (10), that the matrix $A_1$ has the block-diagonal form (see also Example 1)

$$A_1 = \text{block diag}\,(A_{11}, A_{22}, \ldots, A_{n_2 n_2}).$$

*Sufficiency*. Assume that the matrix $A_2$ can be reduced to the form (35). The respective reordering of the rows of the matrix $A_1$ yields

$$\tilde{A}_1 = \text{block diag}\,(R, Q).$$

Since

$$(\tilde{A}_2^{\mathrm{T}}\tilde{A}_2)^{-1} = N^{-1}(N^{-1})^{\mathrm{T}};$$

taking account of the form of $\tilde{A}_1$ it is easy to verified that the condition (30) holds for every $R$ and $Q$.

*Necessity*. Taking account of invertibility of $A_2$ without any loss of generality we can consider only the matrices $A_2$ of the form

$$A_2 = \begin{bmatrix} N \\ B \end{bmatrix}, \tag{36}$$

where $N$ is nonsingular. Such a form can be generally obtained by the appropriate reordering of the rows of $A_2$. The structure of the matrix $A_1$ corresponding to the assumed partition of $A_2$ is the following

$$A_1 = \text{block diag}(R, Q).$$

Let (30) hold for every $A_1$. In particular for $A_1$ such that $R^{\mathrm{T}}R = I$ and $Q^{\mathrm{T}}Q = \gamma I$, $\gamma > 0$, it takes the form

$$(N^{\mathrm{T}}N + \gamma B^{\mathrm{T}}B)\, D^{-1}(N^{\mathrm{T}}N + 1/\gamma(B^{\mathrm{T}}B))\, D^{-1} = I, \tag{37}$$

where $D = (N^{\mathrm{T}}N + B^{\mathrm{T}}B)$. Denote the left hand side of (37) by $L(\gamma)$. Obviously $\mathrm{d}L(\gamma)/\mathrm{d}\gamma = 0$, $\gamma > 0$, whence

$$B^{\mathrm{T}}BD^{-1}N^{\mathrm{T}}N = 1/\gamma^2(N^{\mathrm{T}}N)\, D^{-1}B^{\mathrm{T}}B$$

for arbitrary $\gamma > 0$. Hence

$$N^{\mathrm{T}}ND^{-1}B^{\mathrm{T}}B = 0$$

and because of the nonsingularity of $N$ we obtain $B^{\mathrm{T}}B = 0$, i.e. $B = 0$. □

**Theorem 2.** If the identification system is memoryless and the models in both stages are assumed in the form $\bar{y}_M = \bar{F}_1(\bar{u}^{(1)}, \bar{x}) = A_1\bar{x}$, $\bar{x}_M = \bar{F}_2(\bar{u}^{(2)}, a) = A_2 a$ ($A_1$, $A_2$ depend on $\bar{u}^{(1)}$, $\bar{u}^{(2)}$, respectively), then 2SI and DI methods are equivalent for any $A_2$ iff $A_1$ is such that

$$A_1^{\mathrm{T}}A_1 = \gamma I, \quad \gamma > 0. \tag{38}$$

Proof. *Sufficiency* follows immediately from the Condition 2.

*Necessity*. Assume that the equation (30) hold for arbitrary invertible $A_2$, thus in particular for $A_2$ of the block form

$$A_2 = \begin{bmatrix} N \\ B \end{bmatrix},$$

with nonsingular $N$. The corresponding structuralization of $A_1$ is the following

$$A_1 = \text{block diag}(R, Q).$$

Assuming that $N = wI$, $w \neq 0$, and taking into account two possible relations between the number of rows and columns of submatrix $B$ in matrix $A_2$ we shall consider the two following cases

(i) if the matrix $B$ satisfies the condition

$$BB^{\mathrm{T}} = I \quad (B^{\mathrm{T}} - \text{isometric matrix}) \tag{39}$$

(ii) if the matrix $B$ satisfies the condition

$$B^{\mathrm{T}}B = I \quad (B - \text{isometric matrix}) \tag{40}$$

*Case* (i). Reformulating (30) we have

$$A_2^{\mathrm{T}}A_1^{\mathrm{T}}A_1 = A_2^{\mathrm{T}}A_1^{\mathrm{T}}A_1A_2(A_2^{\mathrm{T}}A_2)^{-1}\,A_2^{\mathrm{T}}\,.$$

Taking advantage of the forms of $A_1$ and $A_2$ and using (39) we obtain

$$\begin{aligned}&[wR^{\mathrm{T}}R \mid B^{\mathrm{T}}Q^{\mathrm{T}}Q] = \\ &= (w^2R^{\mathrm{T}}R + B^{\mathrm{T}}Q^{\mathrm{T}}QB)(w^2I + B^{\mathrm{T}}B)^{-1}\,[wI \mid B^{\mathrm{T}}]\,.\end{aligned} \tag{41}$$

Since for isometric $B^{\mathrm{T}}$

$$(w^2I + B^{\mathrm{T}}B)^{-1} = (1/z)\,I - 1/(z(z+1))\,(B^{\mathrm{T}}B)\,, \quad z = w^2\,,$$

from (41) we obtain

$$R^{\mathrm{T}}RB^{\mathrm{T}}B = B^{\mathrm{T}}Q^{\mathrm{T}}QB$$

and hence

$$BR^{\mathrm{T}}RB^{\mathrm{T}} = Q^{\mathrm{T}}Q \tag{42}$$

for arbitrary $B$ such that $BB^{\mathrm{T}} = I$.

In particular for the matrix $B_i$ of the form

$$B_i^{\mathrm{T}} = [P_{i_1}, P_{i_2}, \ldots, P_{i_m}]\,,$$

with $\{P_{i_1}, P_{i_2}, \ldots, P_{i_m}\} \subset \{P_1, P_2, \ldots, P_n\}$, $P_i$, $i = 1, \ldots, n$, being the eigenvectors of the matrix $R^{\mathrm{T}}R$, where $n$ and $m \leqq n$ are dimensions of the matrices $R^{\mathrm{T}}R$ and $Q^{\mathrm{T}}Q$, respectively, the equation (42) takes the form

$$B_iR^{\mathrm{T}}RB_i^{\mathrm{T}} = Q^{\mathrm{T}}Q = \Lambda_i\,, \tag{43}$$

with $\Lambda_i = \operatorname{diag}(\lambda_{i_1}, \ldots, \lambda_{i_m})$, $\lambda_{i_j}$ being the eigenvalues of $R^{\mathrm{T}}R$ corresponding to eigenvectors $P_{i_j}$. The equation (43) holds for an arbitrary subset $\{P_{i_1}, \ldots, P_{i_m}\}$ of the set $\{P_1, \ldots, P_n\}$, therefore $\lambda_1 = \lambda_2 = \ldots = \lambda_n \triangleq \gamma$, $Q^{\mathrm{T}}Q = \gamma I$ and including the positive definiteness of $R^{\mathrm{T}}R$ we have $R^{\mathrm{T}}R = \gamma I$, $\gamma > 0$.

*Case* (ii). For $B$ satisfying (40) and $N = wI$ ($w \neq 0$) the equation (30) takes the form

$$1/(z+1)^2\,[zP^{\mathrm{T}}P + B^{\mathrm{T}}Q^{\mathrm{T}}QB]\,[z(P^{\mathrm{T}}P)^{-1} + B^{\mathrm{T}}(Q^{\mathrm{T}}Q)^{-1}B] = I\,,$$

$z \triangleq w^2$. Denoting the left hand side of the above by $L(z)$ and using the fact that $\mathrm{d}L(z)/\mathrm{d}z = 0$ we obtain

$$P^{\mathrm{T}}PB^{\mathrm{T}}(Q^{\mathrm{T}}Q)^{-1}\,B + B^{\mathrm{T}}(Q^{\mathrm{T}}Q)\,B(P^{\mathrm{T}}P)^{-1} = 2I\,. \tag{44}$$

Since

$$B^{\mathrm{T}}(Q^{\mathrm{T}}Q)\,BB^{\mathrm{T}}(Q^{\mathrm{T}}Q)^{-1}\,B = I\,, \tag{45}$$

the equation (44) can be rewritten as follows

$$P^{\mathrm{T}}PB^{\mathrm{T}}(Q^{\mathrm{T}}Q)^{-1}B + (P^{\mathrm{T}}PB^{\mathrm{T}}(Q^{\mathrm{T}}Q)^{-1}B)^{-1} = 2I .$$

Hence $P^{\mathrm{T}}PB^{\mathrm{T}}(Q^{\mathrm{T}}Q)^{-1} B = I$ and taking account of (45) we obtain

$$B^{\mathrm{T}}Q^{\mathrm{T}}QB = P^{\mathrm{T}}P$$

for arbitrary $B$ fulfilling (40). The above (similarly as in case (i)) leads to the conclusion that

$$P^{\mathrm{T}}P = \gamma I , \quad Q^{\mathrm{T}}Q = \gamma I , \quad \gamma > 0 . \qquad \square$$

By the above $A_1$-free and $A_2$-free results (the Conditions 1 and 2 and Theorems 1 and 2) the choice of the two-stage experiment design ensuring the equivalence of DI and 2SI methods can be substantially simplified, if compared with the case when the condition (30) is used.

This simplification consists in the fact the choice of appropriate experiment design is reduced to the choice of either the design points $u_i^{(2)}$ in the second stage or the design points $u_{ij}^{(1)}$ in the first stage of the two-stage experiment. The choice of the design points in the remaining stage is then arbitrary (provided that the uniqueness of the choice of the best model is guaranteed).

In the case when the result of two-stage identification (the parameter $\tilde{a}$) is already known then, using the condition (29) it is possible to formulate a simple criterion enabling the examination of the equivalence of both identification methods – regardless whether or not the best model according to the direct method was computed

To this end let us notice that (26) yields

$$\tilde{\tilde{x}} = (A_1^*A_1)^{-1} A_1^*\bar{y} \tag{46}$$

and, on the basis of (27),

$$\tilde{a} = (A_2^*A_2)^{-1} A_2^*\tilde{\tilde{x}} . \tag{47}$$

Hence and from the condition (29) the following *a posteriori* equivalence criterion of the two methods can be obtained

$$A_2^*A_1^*\bar{y} = A_2^*A_1^*A_1A_2\tilde{a} . \tag{48}$$

If for the measurements results obtained in the experiment and the parameter $\tilde{a}$ computed in two stages the condition (48) holds, then 2SI and DI methods are equivalent. In the opposite case the equivalence does not hold. Notice that (46) and (47) determine jointly the identification algorithm for the two-stage approach.

Because of the fundamental role of the assumption of uniqueness of the solutions for the DI and 2SI methods in the presented paper, the examinations of the relations between uniqueness of the solutions of the two methods is significant. This problem for linear classes of models (with preassumed properties) is stated in the following

**Theorem 3.**

A. If for the given MIMO system and given input series $[\bar{u}^{(1)\mathrm{T}}, \bar{u}^{(2)\mathrm{T}}]^{\mathrm{T}}$ the solution of the 2SI problem is unique, then the solution of the DI problem is also unique.

B. If for the given MIMO system and given input series $[\bar{u}^{(1)\mathrm{T}}, \bar{u}^{(2)\mathrm{T}}]^{\mathrm{T}}$ the solution of the DI problem is unique, and for 2SI problem:

(i) the solution of the optimization problem in the 1S is unique, or

(ii) $S = \bar{X}$ (see Section 3),

then the solution of the 2SI problem is also unique.

Proof. The proof will be based on two elementary properties ([24]):

1. A linear operator $A \in L(X, Y)$ ($X$, $Y$ – Hilbert spaces) is a surjective iff there exists a constant $m > 0$ such that

$$\|A^* y\| \geqq m\|y\| , \quad y \in Y .$$

2. A linear operator $A \in L(X, Y)$ ($X$, $Y$ – Hilbert spaces) has the continuous inverse (defined on the range of $A$) iff there exists a constant $m > 0$ such that

$$\|Ax\| \geqq m\|x\| , \quad x \in X ,$$

and on the von Neumann invertibility theorem ([24]).

In order to prove the part A of the theorem let us observe that in case when the solution of 2SI problem is unique we have (by property 2).

$$\|A_1 A_2 a\| \geqq m_1 m_2 \|a\| , \quad a \in \mathscr{A} ,$$

whence, by virtue of the von Neumann theorem, the operator $A_2^* A_1^* A_1 A_2$ is invertible, i.e. – taking account of (28) – the solution of the DI problem is unique. To prove the part B let us note that – by the von Neumann invertibility theorem – the uniqueness of the solution of DI problem implies

$$\|A_1 A_2 a\| \geqq m\|a\| , \quad a \in \mathscr{A} . \tag{49}$$

Hence by property 1 $A_2^* A_1^*$ is a surjective operator, so $A_2^*$ is surjective too, whence – by the properties 1 and 2 and the von Neumann invertibility theorem – $A_2^* A_2$ is invertible in the parameter space $\mathscr{A}$. The last together with (i) completes the proof of uniqueness for two-stage approach.

If, in turn, $S = \bar{X}$ (ii)) then for every $\bar{x} \in \bar{X}$ there exists $a \in \mathscr{A}$ such that $\bar{x} = A_2 a$, and by virtue of (49) it is easy to show that

$$\|A_1 \bar{x}\| \geqq \frac{m}{\|A_2\|} \|\bar{x}\| , \quad \bar{x} \in \bar{X} .$$

Whence by property 2 and von Neumann invertibility theorem the operator $A_1^* A_1$ is invertible which, together with the invertibility of $A_2^* A_2$ proved before, completes the proof of uniqueness two-stage approach for the case (ii). □

In the next section we shall give some examples of the derived equivalence conditions of 2SI and DI methods and discuss their implementation to the problem of experiment design.

## 4. EXAMPLES

The equivalence conditions for two-stage and direct choice of the best model will be illustrated for some static and some dynamic systems. For the sake of simplicity the examples are restricted to the case of two-inputs, one-output systems.

**Example 1.** For the static system with scalar inputs $u^{(1)}$ and $u^{(2)}$ and scalar output $y$ we assume a typical model of the form ([12]):

$$y_M = a u^{(1)} u^{(2)} .$$

Denote

$$F_1(u^{(1)}, x) \triangleq x u^{(1)} ,$$

$$F_2(u^{(2)}, a) \triangleq a u^{(2)} .$$

Denoting additionally:

$$\bar{x} \triangleq [x_1, \ldots, x_{n_2}], \quad \bar{u}^{(2)} \triangleq [u_1^{(2)}, \ldots, u_{n_2}^{(2)}]$$

$$\bar{y} \triangleq [y_{11}, \ldots, y_{1n_1}, y_{21}, \ldots, y_{n_2 n_1}]$$

$$\bar{U}^{(1)} \triangleq \left[\begin{array}{ccc|ccc|ccc} u_{11}^{(1)} & \ldots & u_{1n_1}^{(1)} & 0 & \ldots & 0 & & \ldots\ldots\ldots\ldots & 0 \\ 0 & \ldots & 0 & u_{21}^{(1)} & \ldots & u_{2n_1}^{(1)} & & \ldots\ldots\ldots\ldots & 0 \\ \hline 0 & \ldots & 0 & 0 & \ldots\ldots\ldots\ldots & & u_{n_2 1}^{(1)} & \ldots & u_{n_2 n_1}^{(1)} \end{array}\right]$$

the linear operator $A_1$ from the formula (23) is the matrix operator of the form

$$\bar{y} = \bar{U}^{(1)} \bar{x}$$

and the operator $A_2$ (formula (24)) has the form:

$$\bar{x} = \bar{u}^{(2)} a .$$

In the case considered – for square identification quality criterions $Q_1$, $Q_2$ and $Q$ – the *a priori* equivalence condition (30) takes the form:

$$\text{(50)} \quad \bar{u}^{(2)T} \bar{U}^{(1)T} \bar{U}^{(1)} \bar{u}^{(2)} (\bar{u}^{(2)T} \bar{u}^{(2)})^{-1} \, \bar{u}^{(2)T} (\bar{U}^{(1)T} \bar{U}^{(1)})^{-1} \, \bar{u}^{(2)} (\bar{u}^{(2)T} \bar{u}^{(2)})^{-1} = 1 .$$

Since $\bar{u}^{(2)T} \bar{u}^{(2)} = \sum_{i=1}^{n_2} u_i^{(2)^2} \triangleq w$ is a scalar value and

$$\text{(51)} \qquad \bar{U}^{1T} \bar{U}^{1} = \operatorname{diag}\left(\sum_{j=1}^{n_1} u_{1j}^{(1)^2}, \ldots, \sum_{j=1}^{n_1} u_{n_2 j}^{(1)^2}\right)$$

the condition (50) can be rewritten in the following form:

$$\text{(52)} \qquad \left[\sum_{i=1}^{n_2} u_i^{(2)^2} \left(\sum_{j=1}^{n_1} u_{ij}^{(1)^2}\right)\right] \left[\sum_{i=1}^{n_2} u_i^{(2)^2} / \left(\sum_{j=1}^{n_1} u_{ij}^{(1)^2}\right)\right] = \left[\sum_{i=1}^{n_2} u_i^{(2)^2}\right]^2 .$$

So, in order to obtain the two-stage experiment design ensuring the equivalence of 2SI and DI methods the inputs $u_{ij}^{(1)}$, $u_i^{(2)}$ should be chosen in such a way that

$w > 0, \sum_{j=1}^{n_1} u_{ij}^{(1)2} > 0,\ i = 1, \ldots, n_2,$ (the condition of the uniqueness of the choice of the best model) and (52) holds. Notice that such an experiment design does exist and, in general, is not determined uniquely.

We shall now examine the form of the experiment designs for which $A_2$-free and $A_1$-free equivalence conditions (Theorems 2 and 1) hold. Notice, that the condition (38) ($A_2$-free) takes now the form:

$$\bar{U}^{(1)\mathrm{T}} \bar{U}^{(1)} = \gamma I\,, \quad \gamma > 0\,,$$

where $I$ denotes the unit matrix of the rank $n_2$. Taking account of (51) it can be presented as follows:

$$\sum_{j=1}^{n_1} u_{ij}^{(1)2} = \gamma\,, \quad i = 1, \ldots, n_2\,, \quad \gamma > 0\,. \tag{53}$$

If the above condition is satisfied (the first stage experiment design is proper) the equivalence of 2SI and DI methods is guaranteed regardless of the design chosen for the second stage. Notice that the condition (53) is, in particular, satisfied if:

$$u_{1j}^{(1)} = u_{2j}^{(1)} = \ldots u_{n_2 j}^{(1)}\,, \quad j = 1, \ldots, n_1\,,$$

i.e. if for the consecutively fixed values $u_i^{(2)}$ the corresponding input series $u_{i1}^{(1)}, \ldots$ $\ldots, u_{in_1}^{(1)}$ are identical. Such a case was studied in [12].

For the case considered $A_1$-free necessary and sufficient equivalence condition (Theorem 1) has the form

$$\exists!\, i_0 \quad u_{i_0}^{(2)} \neq 0 \quad \text{and} \quad u_i^{(2)} = 0 \quad \text{for} \quad i \neq i_0\,, \quad i = 1, \ldots, n_2\,.$$

**Example 2.** For the dynamic system with scalar inputs $u^{(1)}, u^{(2)}$ and scalar output $y$ we assume the model of the form

$$y_{\mathrm{M}}(t) = \int_0^t K(s)\, u^{(1)}(t - \tau)\, u^{(2)}(\tau - s)\, \mathrm{d}s\, \mathrm{d}\tau\,, \quad t \in [0, T]\,, \tag{54}$$

where $T$ is a finite time horizon for the continuous-time experiment. Denote $F_1(u^{(1)}, x)(t) = \int_0^t u^{(1)}(t - s)\, x(s)\, \mathrm{d}s$ and $F_2(u^{(2)}, K)(t) = \int_0^t K(s)\, u^{(2)}(t - s)\, \mathrm{d}s$. For this model we shall examine the form of the $A_2$-free equivalence condition (31). Let additionally

$$\bar{x}(t) = [x_1(t), \ldots, x_{n_2}(t)]$$

$$\bar{y}(t) = [y_{11}(t), \ldots, y_{1n_1}(t), y_{21}(t), \ldots, y_{n_2 n_1}(t)]^{\mathrm{T}}$$

$$\bar{U}^{(1)}(t) = \begin{bmatrix} \begin{matrix} u_{11}^{(1)}(t) \\ \ldots \\ u_{1n_1}^{(1)}(t) \end{matrix} & 0 \ldots & \ldots 0 \\ 0 & \begin{matrix} u_{21}^{(1)}(t) \\ \ldots \\ u_{2n_1}^{(1)}(t) \end{matrix} & \ldots 0 \\ 0 \ldots & & \begin{matrix} u_{n_2 1}^{(1)}(t) \\ \ldots \\ u_{n_2 n_1}^{(1)}(t) \end{matrix} \end{bmatrix}$$

For the case considered the operator $A_2$ from the formula (24) is

$$(A_1\bar{x})(t) = \int_0^t \bar{U}^{(1)}(t - \tau)\,\bar{y}(\tau)\,\mathrm{d}\tau\,, \quad t \in [0, T]\,.$$

It is an integral Fredholm operator with the difference matrix kernel $\bar{U}^{(1)}(\cdot)$. The adjoint operator $A_1^*$ is defined by

$$(A_1^*\bar{y})(t) = \int_0^T \bar{U}^{(1)\mathrm{T}}(\tau - t)\,\bar{y}(\tau)\,\mathrm{d}\tau\,, \quad t \in [0, T]\,.$$

Since

$$(A_1^* A_1 \bar{x})(t) = \int_0^T N_1(t, w)\,\bar{x}(w)\,\mathrm{d}w\,, \quad t \in [0, T]\,,$$

where $N_1(t, w) = \int_0^T \bar{U}^{(1)\mathrm{T}}(\tau - t)\,\bar{U}^{(1)}(\tau - w)\,\mathrm{d}\tau$, the condition (31) takes the form

$$\int_0^T \sum_{j=1}^{n_1} u_{ij}^{(1)}(\tau - t)\,u_{ij}^{(1)}(\tau - w)\,\mathrm{d}\tau = \gamma\,\delta(t - w)\,, \tag{55}$$

$i = 1, \ldots, n_2$, $\delta(\cdot)$ is the Dirac impulse signal.

If the condition (55) holds for experiment design taken in the 1st stage, then – by virtue of the Condition 1 – the models of the form (54) determined by 2SI and DI methods are identical independently of the choice of the input series in the 2nd stage. In particular, the condition (55) is fulfilled for the series of the form:

$$u_{ij}^{(1)}(t) = c_{ij}\,\delta(t)\,, \quad i = 1, \ldots, n_2\,, \quad j = 1, \ldots, n_1\,,$$

where $\sum_{j=1}^{n_2} c_{ij}^2 = c$, $c > 0$, $i = 1, \ldots, n_2$.

From the above examples it follows that the general condition (30) can be considered as a basis enabling us to derive constructive instructions concerning the choice of suitable design of the experiment and to obtain the useful equivalence conditions of the both methods in particular cases.

## 5. FINAL REMARKS

The equivalence conditions of two-stage and direct approaches to the problem of the choice of the best model from the given parametric class have been studied.

The general equivalence conditions of 2SI and DI methods have been given for linear classes of models. In particular, sufficient (necessary and sufficient for a memoryless case) equivalence conditions of $A_1$-free and $A_2$-free kinds (Section 3), i.e. the conditions ensuring the equivalence of both methods regardless of the choice of the form of the model (linear with respect to the parameters) in the 1st and 2nd stages, respectively, have been established.

The presented equivalence conditions refer generally to the static as well as to the dynamic systems for continuous and discrete-time experiments and enable us to formulate some instructions regarding the choice of such an experiment design for which the best, for the direct approach, model can be obtained by more convenient two-stage method.

The extension of the two-stage experiment design problems and further investigations of the two-stage identification method seem to be important.

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