A stationary weakly isotropic random field on a d-lattice (d-dimensional square lattice) is defined as a (weakly) stationary random field whose covariance function is invariant with respect to all the symmetries of the d-lattice. Under assumption of bounded range interactions the covariance function of such a field is expressed as a finite linear combination of product covariance functions. Analogous decomposition of the spectral density follows from that of the covariance function. A solution of an extrapolation problem of certain type is derived as a corollary of this result.

1. STATIONARY WEAKLY ISOTROPIC RANDOM FIELDS

Through $\mathbb{Z}^d$ we denote a d-dimensional square lattice, i.e. the set of all d-dimensional vectors whose coordinates are integer. The elements of $\mathbb{Z}^d$ will be called indeces.

Stationary random field on $\mathbb{Z}^d$ is a system of complex random variables $X = (X(K); K \in \mathbb{Z}^d)$ such that every variable from $X$ has zero mean and finite variance and there exists covariance function $B$ defined on $\mathbb{Z}^d$ such that, for every $K, L \in \mathbb{Z}^d$, $EX(K) \cdot X(L)^* = B(K - L)$ holds where asterisk denotes the complex conjugate.

Stationary random field on $\mathbb{Z}^d$ is weakly isotropic if, for every permutation $\sigma \in S_d$ (where $S_d$ denotes the symmetric group of degree $d$), every $d$-tuple $\alpha = (\alpha_1, \ldots, \alpha_d)$ where, for every $j = 1, \ldots, d$, $\alpha_j$ is either +1 or −1 and every $K = (K_1, \ldots, K_d) \in \mathbb{Z}^d$ the equality

$$B(K_1, \ldots, K_d) = B(\alpha_1K_{\alpha(1)}, \ldots, \alpha_dK_{\alpha(d)})$$

is true. Such a field may be viewed as some discrete analogy of the notion of a stationary isotropic random field on a d-dimensional Euclidean space $\mathbb{R}^d$ (cf. e.g. [3]).

A stationary random field on $\mathbb{Z}^d$ is said to have bounded range interactions if its covariance function is not zero just for finitely many indeces. It is obvious that the condition of bounded range interactions provides the existence of the spectral
density of the field given, that is, the existence of the function $f$ defined on $I = [-\pi, \pi]^d$ such that for every $K \in \mathbb{Z}^d$ it is

$$B(K) = \int f(\lambda) \, d\lambda.$$  

where $\langle \cdot, \cdot \rangle$ denotes the usual scalar product in $\mathbb{R}^d$.

Viz. in case of bounded range interactions the spectral density is given

$$f(\lambda) = (2\pi)^{-d} \sum_{K} e^{-\langle(K, \lambda)\rangle} B(K)$$

where the summation goes over all indices $K$ for which $B(K) \neq 0$.

1.1. Theorem. Let $X$ be a stationary random field on $\mathbb{Z}^d$ with the spectral density $f$. Then $X$ is weakly isotropic if and only if

$$f(\lambda) = f(\alpha X_1, \ldots, \alpha X_d)$$

holds for almost every $\lambda = (\lambda_1, \ldots, \lambda_d) \in I$, every $\alpha \in S_d$ and every $\alpha = (\alpha_1, \ldots, \alpha_d) \in \{\pm 1\}^d$.

Proof. The “if” is obvious and the “only if” follows immediately from the fact that the function

$$g(\lambda) = 2^{-d}(d!)^{-1} \sum_{\alpha} \sum_{x \in \{-1, 1\}^d} f(\alpha_1 X_1, \ldots, \alpha_d X_d)$$

has the property described in (3) and with respect to (1) both $f$ and $g$ have the same $d$-tuple Fourier coefficients (cf. [2]). 

Putting $\alpha_1 = \ldots = \alpha_d = -1$ in (3) we see that the spectral density of a weakly isotropic field is an even function. Hence the covariance function is a real one.

2. SPECTRUM DECOMPOSITION

In what follows we suppose that $X$ is a stationary weakly isotropic random field on $\mathbb{Z}^d$ with bounded range interactions. Its spectral density we denote through $f$. We say $f$ to be a product spectral density if there exists a function $\phi$ defined on $[\pi, \pi]$ such that, for every $\lambda \in I$, it is

$$f(\lambda) = \prod_{j=1}^{d} \phi(\lambda_j).$$

By aid of (2) it may be easily seen that the necessary and sufficient condition for a field $X$ to have a product spectral density is to have a product covariance function of the form

$$B(K) = \prod_{j=1}^{d} b(\lambda_j)$$

for every $K \in \mathbb{Z}^d$ where $b$ is some function defined on $\mathbb{Z}$, i.e. the set of all integers.
Our aim is to prove that under stated assumptions the spectral density is expressed as a finite linear combination of product ones.

We start with one auxiliary result.

2.1. Lemma. Let \( m, n \) be positive integers and \( A \in \mathbb{R}^{1, \ldots, m} \), that is, \( A = (A_{K(1), \ldots, K(n)}; K(1), \ldots, K(n) = 1, \ldots, m) \) and suppose that for every permutation \( \rho \in S_n \)

\[
A_{K(1), \ldots, K(n)} = A_{\rho(K(1)), \ldots, \rho(K(n))}
\]

holds. Further suppose that there exist \( t \in \mathbb{R}^n \) and \( \varepsilon > 0 \) such that for every \( t \in V(\varepsilon) = \{ t \in \mathbb{R}^n; |t_j - t_j^0| < \varepsilon \text{ for } j = 1, \ldots, m \} \) the relation

\[
\sum_{K(1)=1}^{m} \ldots \sum_{K(n)=1}^{m} A_{K(1), \ldots, K(n)} \prod_{j=1}^{n} t_{k(j)} = 0
\]

is true. Then \( A = 0 \), i.e. \( A_{K(1), \ldots, K(n)} = 0 \) for every choice \( K(1), \ldots, K(n) \in \{1, \ldots, m\} \).

Proof. Let us denote the left hand side of (5) through \( G(A, t) \). Obviously \( G(A, \cdot) \) is a smooth function on \( V(\varepsilon) \).

The proof will be carried out by induction.

1) Let \( n = 1 \). Then \( G(A, t) = \sum_{K(1)=1}^{m} A_{K(1)} t_{k(1)} \) and \( G(A, t) \equiv 0 \) on \( V(\varepsilon) \). Hence for every \( K = 1, \ldots, m \) there is \( A_{K} = (\partial G(A, t)) / (\partial t_{k(1)}) G(A, t) = 0 \).

2) Suppose the statement to be true for \( n - 1 \). For \( \{j(1), \ldots, j(k)\} \subset \{1, \ldots, n\} \)

we write \( \sum_{j(1), \ldots, j(k)} \) instead of \( \sum_{K(1)=1}^{m} \ldots \sum_{K(k)=1}^{m} \).

We calculate

\[
\frac{\partial}{\partial t_{k(1)}} G(A, t) = \sum_{(j(1), \ldots, j(k))} A_{K(1), \ldots, K(k)} \left( \frac{\partial}{\partial t_{k(1)}} (t_{k(1)}) \prod_{j=2}^{n} t_{k(j)} + \ldots + \prod_{j=1}^{n-1} t_{k(j)} \frac{\partial}{\partial t_{k(n)}} (t_{k(n)}) \right) =
\]

\[
= \sum_{(j(1), \ldots, j(k))} A_{K(2), \ldots, K(n)} \prod_{j=2}^{n} t_{k(j)} + \ldots + \sum_{(j(1), \ldots, j(n-1))} A_{K(1), \ldots, K(n-1), K(n)} \prod_{j=1}^{n-1} t_{k(j)}.
\]

According to (4) we obtain

\[
\frac{1}{n} \frac{\partial}{\partial t_{k(1)}} G(A, t) = \sum_{(j(1), \ldots, j(k))} A_{1, K(2), \ldots, K(n)} \prod_{j=2}^{n} t_{k(j)}.
\]

With respect to (5) it is \( G(A, t) \equiv 0 \) on \( V(\varepsilon) \) and therefore \( (\partial G(A, t)) / (\partial t_{k(1)}) G(A, t) \equiv 0 \) on \( V(\varepsilon) \). Hence the induction assumption and (6) yield \( A_{1, K(2), \ldots, K(n)} = 0 \) for every choice \( K(2), \ldots, K(n) \) from \( \{1, \ldots, m\} \).

Similar consideration of \( (\partial G(A, t)) / (\partial t_{k(j)}) \) for \( j = 1, \ldots, m \) completes the proof.

A simple one-dimensional spectral density is a strictly positive function \( \varphi \) defined on \( [-\pi, \pi] \) that is the spectral density of a stationary weakly isotropic random sequence with bounded range interactions, that is, the covariance function which
corresponds to \( \varphi \) is an even real function defined on \( \mathbb{Z} \) and is equal to zero outside some finite set of indices.

A simple product spectral density is a function \( g \) defined on \( I \) such that for every \( \lambda \in I \) it is \( g(\lambda) = \prod_{j=1}^{d} \varphi(\lambda_j) \) where \( \varphi \) is a simple one-dimensional spectral density.

2.2. Theorem. Let \( f \) be a spectral density of a stationary weakly isotropic random field \( X \) on \( \mathbb{Z}^d \) with bounded range interactions. Then there exist real numbers \( a_1, \ldots, a_m \) and simple product spectral densities \( f_1, \ldots, f_m \) such that \( f = \sum_{k=1}^{m} a_k f_k \).

Proof. With regard to (2) it is sufficient to express the covariance function \( B \) of \( X \) in the form

\[
B(K) = \sum_{k=1}^{m} a_k \prod_{j=1}^{d} b_j(K_j)
\]

for every \( K \in \mathbb{Z}^d \) where \( b_1, \ldots, b_m \) are covariance functions corresponding to simple one-dimensional spectral densities.

We denote \( S = \max \{ s; B(s, K_2, \ldots, K_d) = 0 \} \). Let \( \varepsilon \) be such that \( 0 < \varepsilon < 1 / (2S + 1) \). We put \( \tau_0 = 1, \tau_1 = \ldots = \tau_S = 0 \) and \( V(\varepsilon) = \{(t_0, \ldots, t_d); |t_i - \tau_i| < \varepsilon \} \) for \( s = 0, \ldots, S \). For every \( t \in V(\varepsilon) \) and every integer \( u \) we define

\[
b_t(u) = \begin{cases} t_u & \text{for } |u| \leq S \\ 0 & \text{otherwise} \end{cases}
\]

and for every \( t \in V(\varepsilon) \) and every \( x \in [-\pi, \pi] \) we define

\[
\varphi_t(x) = \frac{1}{2\pi} (t_0 + 2 \sum_{k=1}^{s} t_k \cos kx).
\]

According to the choice of \( \varepsilon \) the function \( \varphi_t \) is strictly positive and therefore \( \varphi_t \) is the simple one-dimensional spectral density corresponding to the covariance function \( b_t \).

Through \( B_t \) we denote the function defined on \( \mathbb{Z}^d \) as \( B_t(K) = \prod_{j=1}^{d} b_j(K_j) \).

For a function \( C \) defined on \( \mathbb{Z}^d \) we denote \( C_t \) the restriction of \( C \) onto the finite set of indices \( \{0, \ldots, S\}^d \).

We shall prove that there exist real numbers \( a_1, \ldots, a_m \) and vectors \( t(1), \ldots, t(m) \in V(\varepsilon) \) such that

\[
B' = \sum_{k=1}^{m} a_k B(t(k))
\]

holds. (8) obviously implies (7).

Let us denote through \( L \) the linear subspace of \( \mathbb{R}^{0 \ldots S} \) spanned by \( \{B_t; t \in V(\varepsilon)\} \).

If the expression (8) of \( B' \) were not possible, it would be \( B' = B^1 + B^2 \) where \( B^1 \in L \) and \( B^2 \neq 0 \) would be orthogonal to \( L \). But then \( B^2 \) would fulfill the assumptions for \( A \) in Lemma 2.1 and the contradiction \( B^2 = 0 \) would follow. \( \square \)
3. EXTRAPOLATION PROBLEM

Let $X$ be a stationary random field on $\mathbb{Z}^d$. We denote through $L(X)$ the linear space of all finite linear combinations (with complex coefficients) of random variables from $X$ and through $H(X)$ the Hilbert space that is the completion of $L(X)$, the scalar product being defined as covariance. $H_+(X)$ will denote the closed linear subspace of $H(X)$ that is spanned by the set $\{ X(K); K \in \mathbb{Z}^d_+ \}$ where $\mathbb{Z}^d_+=\{ K \in \mathbb{Z}^d; K_j \geq 0 \text{ for } j=1, \ldots, d \}$.

Given $L \in \mathbb{Z}^d \setminus \mathbb{Z}^d_+$, $\hat{X}(L)$ denotes that random variable from $H_+(X)$ for which $E[|X(L) - \hat{X}(L)|^2]$ is minimal, i.e. $\hat{X}(L)$ is the projection of $X(L)$ onto $H_+(X)$. Naturally, $\hat{X}(L)$ is viewed as the best linear extrapolation of $X(L)$ on the base of observed values $X(K), K \in \mathbb{Z}^d_+$.

It may occur that there exist coefficients $r_K, K \in \mathbb{Z}^d_+$ such that $X(L) \sim Y \sum_K X(K)$ where $K \in \mathbb{Z}^d_+$ is true, the convergence being understood in the mean, that is, with respect to the norm of $H(X)$. If it is the case and if, moreover, the series $\sum_K e^{i(K \cdot \lambda)}$ converges in the sense of $L_2(I) = L_2(I, \mathcal{B}, m)$ where $\mathcal{B}$ denotes the $\sigma$-algebra of all Borel subsets of $I$ and $m$ is the $d$-dimensional Lebesgue measure, we say that the extrapolation $\hat{X}(L)$ has the spectral characteristic

$$g_L(\lambda) = \sum_K r_K e^{i(K \cdot \lambda)} \quad \text{where } K \in \mathbb{Z}^d_+.$$

The spectral characteristic being known, the extrapolation coefficients may be determined by means of the relation

$$r_K = (2\pi)^{-d} \int_I g_L(\lambda) e^{-i(K \cdot \lambda)} \, d\lambda.$$

We shall therefore consider the spectral characteristic, if it exists, to be a solution of the extrapolation problem.

In case the random field $X$ has factorisable spectral density, i.e. there exists the function $h$ of $d$ complex variables such that both $h$ and $\partial h$ are holomorphic on the unit polydisc $U^d = \{(z_1, \ldots, z_d); |z_j| \leq 1 \text{ for } j=1, \ldots, d \}$ and the spectral density $f$ is expressed for every $\lambda \in I$ as $f(\lambda) = |h(e^{i\lambda_1}, \ldots, e^{i\lambda_d})|^2$, the spectral characteristic $g_L$ exists for every $L \in \mathbb{Z}^d \setminus \mathbb{Z}^d_+$ and its explicit form involves the factor function $h$. In fact,

$$g_L(\lambda) = \mathcal{H}(e^{i(L \cdot \lambda)} \cdot h(e^{i\lambda_1}, \ldots, e^{i\lambda_d})) \cdot (1/h(e^{i\lambda_1}, \ldots, e^{i\lambda_d}))$$

where $\mathcal{H}$ denotes the "holomorphic part of a function", more exactly, for every function $p$ on $I$ of the form

$$f(\cdot) = \sum_K p_K e^{i(K \cdot \lambda)} \quad \text{where } K \in \mathbb{Z}^d$$
the value of $\mathcal{H}(p(x))$ is, for every $x \in I$, defined as

$$\mathcal{H}(p(x)) = \sum_{K} e^{i(K,x)} \quad \text{where} \quad K \in \mathbb{Z}^d_+.$$ 

Theorem 3.5.3 in [1] implies that every (strictly) positive bounded lower semi-continuous function on $I$ is factorisable. Hence, if $X$ is a stationary random field on $\mathbb{Z}^d$ with bounded range interactions and the spectral density $f$ of $X$ is positive then the spectral density $f$ is factorisable because it is obviously bounded and continuous. So for every $L \in \mathbb{Z}^d \setminus \mathbb{Z}_+^d$ there exists the spectral characteristic $g_L$ of the extrapolation $\hat{X}(L)$. Unfortunately, the theorem referred to does not provide an effective way to obtain the factor function. In case $X$ is weakly isotropic, this difficulty may be overcome using the spectrum decomposition.

Let us, according to Theorem 2.2, write

$$f(x) = \sum_{j=1}^{m} a_j \prod_{k=1}^{d} \varphi_j(\lambda_k)$$

where $\varphi_1, \ldots, \varphi_m$ are simple one-dimensional spectral densities. For $j = 1, \ldots, m$ let $P_j$ denote the factor polynomial corresponding to $\varphi_j$. That is, $\varphi_j(x) = |P_j(e^{i\lambda})|^2$ for every $x \in [-\pi, \pi]$ and $1/P_j$ is holomorphic on the unit disc $U$. (Existence of such a factorization is an easy consequence of the definition of a simple one-dimensional spectral density.)

Denoting $h_j(x) = \prod_{k=1}^{d} P_j(e^{i\lambda_k})$ it is possible to express

$$f(x) = \sum_{j=1}^{m} a_j |h_j(x)|^2 .$$

3.1. Theorem. Let $X$ be a stationary weakly isotropic random field on $\mathbb{Z}^d$ with bounded range interactions and with strictly positive spectral density $f$. Then for every $L \in \mathbb{Z}^d \setminus \mathbb{Z}_+^d$, the spectral characteristic $g_L$ of the extrapolation $\hat{X}(L)$ is given

$$g_L(\lambda) = \frac{1}{f(\lambda)} \sum_{j=1}^{m} a_j \mathcal{H}(e^{i(L,j)} h_j(\lambda)) (h_j(\lambda))^*$$

for every $\lambda \in [-\pi, \pi]^d$.

Proof. The function $g_L$ is the spectral characteristic of $\hat{X}(L)$ if and only if

$$\int f(e^{i(L,j)} - g_L(\lambda)) e^{-i(K,j)} d\lambda = 0$$

is true for every $K \in \mathbb{Z}_+^d$. Viz. $\hat{X}(K)$ is the projection of $X(L)$ onto $H_+(X)$ if and only if $E(X(L) - \hat{X}(L)) \cdot (X(K))^* = 0$ for every $K \in \mathbb{Z}_+^d$, what implies (11) with regard to (2). Let us substitute (9) into (11). As $\int e^{i(M,j)} d\lambda + 0$ holds just in case $M = 0$, (11) is equivalent to

$$\sum_{j=1}^{m} a_j \mathcal{H}(e^{i(L,j)} - g_L(\lambda)) (h_j(\lambda))^* = 0 .$$

It is $\mathcal{H}(e^{i(L,j)} - g_L(\lambda)) h_j(\lambda) = \mathcal{H}(e^{i(L,j)} h_j(\lambda)) - g_L(\lambda) h_j(\lambda)$ because $\mathcal{H}(g_L) = g_L$ and $\mathcal{H}(h_j) = h_j$. Hence (12) is equivalent to (10). 

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