SPECTRUM DECOMPOSITION FOR STATIONARY WEAKLY ISOTROPIC RANDOM FIELDS WITH BOUNDED RANGE INTERACTIONS

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A stationary weakly isotropic random field on a d-lattice (d-dimensional square lattice) is defined as a (weakly) stationary random field whose covariance function is invariant with respect to all the symmetries of the d-lattice. Under assumption of bounded range interactions the covariance function of such a field is expressed as a finite linear combination of product covariance functions. Analogous decomposition of the spectral density follows from that of the covariance function. A solution of an extrapolation problem of certain type is derived as a corollary of this result.

1. STATIONARY WEAKLY ISOTROPIC RANDOM FIELDS

Through $\mathbb{Z}^d$ we denote a *d-dimensional square lattice*, i.e. the set of all d-dimensional vectors whose coordinates are integer. The elements of $\mathbb{Z}^d$ will be called *indeces.*

Stationary random field on $\mathbb{Z}^d$ is a system of complex random variables $X = (X(K): K \in \mathbb{Z}^d)$ such that every variable from X has zero mean and finite variance and there exists *covariance function B* defined on $\mathbb{Z}^d$ such that, for every $K, L \in \mathbb{Z}^d$, $\mathsf{E}X(K) . X(L)^* = B(K - L)$ holds where asterisk denotes the complex conjugate.

Stationary random field on $\mathbb{Z}^d$ is *weakly isotropic* if, for every permutation $\varrho \in S_d$ (where S_d denotes the symmetric group of degree d), every d-tuple $\alpha = (\alpha_1, \ldots, \alpha_d)$ where, for every $j = 1, \ldots, d$, α_j is either $+1$ or -1 and every $K = (K_1, \ldots, K_d) \in \mathbb{Z}^d$ the equality

$$(1) \qquad B(K_1, \ldots, K_d) = B(\alpha_1 K_{\varrho(1)}, \ldots, \alpha_d K_{\varrho(d)})$$

is true. Such a field may be viewed on as some discrete analogy of the notion of a stationary isotropic random field on a d-dimensional Euclidean space $\mathbb{R}^d$ (cf. e.g. [3]).

A stationary random field on $\mathbb{Z}^d$ is said to have *bounded range interactions* if its covariance function is not zero just for finitely many indeces. It is obvious that the condition of bounded range interactions provides the existence of the *spectral* *density* of the field given, that is, the existence of the function f defined on $I = [-\pi, \pi]^d$ such that for every $K \in \mathbb{Z}^d$ it is

$$B(K) = \int_I e^{i\langle K,\lambda\rangle} f(\lambda)\, d\lambda \tag{2}$$

where $\langle \cdot, \cdot \rangle$ denotes the usual scalar product in $\mathbb{R}^d$.

Viz. in case of bounded range interactions the spectral density is given

$$f(\lambda) = (2\pi)^{-d} \sum_K e^{-i\langle K,\lambda\rangle} B(K)$$

where the summation goes over all indeces K for which $B(K) \neq 0$.

1.1. Theorem. Let X be a stationary random field on $\mathbb{Z}^d$ with the spectral density f. Then X is weakly isotropic if and only if

$$f(\lambda) = f(\alpha_1 \lambda_{\varrho(1)}, \ldots, \alpha_d \lambda_{\varrho(d)}) \tag{3}$$

holds for almost every $\lambda = (\lambda_1, \ldots, \lambda_d) \in I$, every $\varrho \in S_d$ and every $\alpha = (\alpha_1, \ldots, \alpha_d) \in \{-1, 1\}^d$.

Proof. The "if" is obvious and the "only if" follows immediately from that the function

$$g(\lambda) = 2^{-d}(d!)^{-1} \sum_{\varrho \in S_d} \sum_{\alpha \in \{-1,1\}^d} f(\alpha_1 \lambda_{\varrho(1)}, \ldots, \alpha_d \lambda_{\varrho(d)})$$

has the property described in (3) and with respect to (1) both f and g have the same d-tuple Fourier coefficients (cf. [2]). □

Putting $\alpha_1 = \ldots = \alpha_d = -1$ in (3) we see that the spectral density of a weakly isotropic field is an even function. Hence the covariance function is a real one.

2. SPECTRUM DECOMPOSITION

In what follows we suppose that X is a stationary weakly isotropic random field on $\mathbb{Z}^d$ with bounded range interactions. Its spectral density we denote through f. We say f to be a *product spectral density* if there exists a function φ defined on $[-\pi, \pi]$ such that, for every $\lambda \in I$, it is

$$f(\lambda) = \prod_{j=1}^{d} \varphi(\lambda_j) .$$

By aid of (2) it may be easily seen that the necessary and sufficient condition for a field X to have a product spectral density is to have a product covariance function of the form

$$B(K) = \prod_{j=1}^{d} b(K_j)$$

for every $K \in \mathbb{Z}^d$ where b is some function defined on $\mathbb{Z}$, i.e. the set of all integers.

Our aim is to prove that under stated assumptions the spectral density is expressed as a finite linear combination of product ones.

We start with one auxiliary result.

2.1. Lemma. Let m, n be positive integers and $A \in \mathbb{R}^{\{1,\ldots,m\}^n}$, that is, $A = (A_{K(1),\ldots,K(n)}: K(1), \ldots, K(n) = 1, \ldots, m)$ and suppose that for every permutation $\varrho \in S_n$

$$A_{K(1),\ldots,K(n)} = A_{K(\varrho(1)),\ldots,K(\varrho(n))} \tag{4}$$

holds. Further suppose that there exist $\tau \in \mathbb{R}^m$ and $\varepsilon > 0$ such that for every $t \in V_\varepsilon(\tau) = \{t \in \mathbb{R}^m: |t_j - \tau_j| < \varepsilon \text{ for } j = 1, \ldots, m\}$ the relation

$$\sum_{K(1)=1}^{m} \cdots \sum_{K(n)=1}^{m} A_{K(1),\cdots,K(n)} \prod_{j=1}^{n} t_{K(j)} = 0 \tag{5}$$

is true. Then $A = 0$, i.e. $A_{K(1),\cdots,K(n)} = 0$ for every choice $K(1), \ldots, K(n) \in \{1, \ldots, m\}$.

Proof. Let us denote the left hand side of (5) through $G(A, t)$. Obviously $G(A, \cdot)$ is a smooth function on $V_\varepsilon(\tau)$.

The proof will be carried out by induction.

1) Let $n = 1$. Then $G(A, t) = \sum_{K=1}^{m} A_K t_K$ and $G(A, t) \equiv 0$ on $V_\varepsilon(\tau)$. Hence for every $K = 1, \ldots, m$ there is $A_K = (\partial/\partial t_K)\, G(A, t) = 0$.

2) Suppose the statement to be true for $n - 1$. For $\{j(1), \ldots, j(k)\} \subset \{1, \ldots, n\}$ we write $\sum_{(j(1),\ldots,j(k))}$ instead of $\sum_{K(j(1))=1}^{m} \cdots \sum_{K(j(k))=1}^{m}$.

We calculate

$$\frac{\partial}{\partial t_1} G(A, t) = \sum_{(1,\ldots,n)} A_{K(1),\cdots,K(n)} \left(\frac{\partial}{\partial t_1} (t_{K(1)}) \prod_{j=2}^{n} t_{K(j)} + \ldots + \prod_{j=1}^{n-1} t_{K(j)} \frac{\partial}{\partial t_1} (t_{K(n)}) \right) =$$

$$= \sum_{(2,\ldots,n)} A_{1,K(2),\cdots,K(n)} \prod_{j=2}^{n} t_{K(j)} + \ldots + \sum_{(1,\ldots,n-1)} A_{K(1),\cdots,K(n-1),1} \prod_{j=1}^{n-1} t_{K(j)} .$$

According to (4) we obtain

$$\frac{1}{n} \frac{\partial}{\partial t_1} G(A, t) = \sum_{(2,\ldots,n)} A_{1,K(2),\cdots,K(n)} \prod_{j=2}^{n} t_{K(j)} . \tag{6}$$

With respect to (5) it is $G(A, t) \equiv 0$ on $V_\varepsilon(\tau)$ and therefore $(\partial/\partial t_1)\, G(A, t) \equiv 0$ on $V_\varepsilon(\tau)$. Hence the induction assumption and (6) yield $A_{1,K(2),\cdots,K(n)} = 0$ for every choice $K(2), \ldots, K(n)$ from $\{1, \ldots, m\}$.

Similar consideration of $(\partial/\partial t_j)\, G(A, t)$ for $j = 1, \ldots, m$ completes the proof. □

A *simple one-dimensional spectral density* is a strictly positive function φ defined on $[-\pi, \pi]$ that is the spectral density of a stationary weakly isotropic random sequence with bounded range interactions, that is, the covariance function which corresponds to φ is an even real function defined on $\mathbb{Z}$ and is equal to zero outside some finite set of indices.

A *simple product spectral density* is a function g defined on I such that for every $\lambda \in I$ it is $g(\lambda) = \prod_{j=1}^{d} \varphi(\lambda_j)$ where φ is a simple one-dimensional spectral density.

2.2. Theorem. Let f be a spectral density of a stationary weakly isotropic random field X on $\mathbb{Z}^d$ with bounded range interactions. Then there exist real numbers $a_1, \ldots \ldots, a_m$ and simple product spectral densities $f_1, \ldots, f_m$ such that $f = \sum_{k=1}^{m} a_k f_k$.

Proof. With regard to (2) it is sufficient to express the covariance function B of X in the form

$$B(K) = \sum_{k=1}^{m} a_k \prod_{j=1}^{d} b_k(K_j) \tag{7}$$

for every $K \in \mathbb{Z}^d$ where $b_1, \ldots, b_m$ are covariance functions corresponding to simple one-dimensional spectral densities.

We denote $S = \max\{s: B(s, K_2, \ldots, K_d) \neq 0\}$. Let ε be such that $0 < \varepsilon < 1 / (2S + 1)$. We put $\tau_0 = 1, \tau_1 = \ldots = \tau_S = 0$ and $V_\varepsilon(\tau) = \{(t_0, \ldots, t_S): |t_s - \tau_s| < \varepsilon$ for $s = 0, \ldots, S\}$. For every $t \in V_\varepsilon(\tau)$ and every integer u we define

$$b_t(u) = \begin{cases} t_{|u|} & \text{for } |u| \leqq S \\ 0 & \text{otherwise} \end{cases}$$

and for every $t \in V_\varepsilon(\tau)$ and every $x \in [-\pi, \pi]$ we define

$$\varphi_t(x) = (1/2\pi)\left(t_0 + 2 \sum_{k=1}^{S} t_k \cos kx\right).$$

According to the choice of ε the function φ_t is strictly positive and therefore φ_t is the simple one-dimensional spectral density corresponding to the covariance function b_t.

Through B_t we denote the function defined on $\mathbb{Z}^d$ as $B_t(K) = \prod_{j=1}^{d} b_t(K_j)$.

For a function C defined on $\mathbb{Z}^d$ we denote C' the restriction of C onto the finite set of indeces $\{0, \ldots, S\}^d$.

We shall prove that there exist real numbers $a_1, \ldots, a_m$ and vectors $t(1), \ldots, t(m) \in V_\varepsilon(\tau)$ such that

$$B' = \sum_{k=1}^{m} a_k B'_{t(k)} \tag{8}$$

holds. (8) obviously implies (7).

Let us denote through $\mathbf{L}$ the linear subspace of $\mathbb{R}^{\{0,\ldots,S\}^d}$ spanned by $\{B'_t: t \in V_\varepsilon(\tau)\}$. If the expression (8) of B' were not possible, it would be $B' = B^1 + B^2$ where $B^1 \in \mathbf{L}$ and $B^2 \neq 0$ would be orthogonal to $\mathbf{L}$. But then B^2 would fulfil the assumptions for A in Lemma 2.1 and the contradiction $B^2 = 0$ would follow. □

3. EXTRAPOLATION PROBLEM

Let X be a stationary random field on $\mathbb{Z}^d$. We denote through $\mathbf{L}(X)$ the linear space of all finite linear combinations (with complex coefficients) of random variables from X and through $\mathbf{H}(X)$ the Hilbert space that is the completion of $\mathbf{L}(X)$, the scalar product being defined as covariance. $\mathbf{H}_+(X)$ will denote the closed linear subspace of $\mathbf{H}(X)$ that is spanned by the set $\{X(K): K \in \mathbb{Z}^d_+\}$ where $\mathbb{Z}^d_+ = \{K \in \mathbb{Z}^d: K_j \geqq 0$ for $j = 1, \ldots, d\}$.

Given $L \in \mathbb{Z}^d \setminus \mathbb{Z}^d_+$, $\hat{X}(L)$ denotes that random variable from $\mathbf{H}_+(X)$ for which $\mathsf{E}|X(L) - \hat{X}(L)|^2$ is minimal, i.e. $\hat{X}(L)$ is the projection of $X(L)$ onto $\mathbf{H}_+(X)$. Naturally, $\hat{X}(L)$ is viewed on as the best linear extrapolation of $X(L)$ on the base of observed values $X(K)$, $K \in \mathbb{Z}^d_+$.

It may occur that there exist coefficients r_K, $K \in \mathbb{Z}^d_+$ such that

$$\hat{X}(L) = \sum_K r_K X(K) \quad \text{where} \quad K \in \mathbb{Z}^d_+$$

is true, the convergence being understood in the mean, that is, with respect to the norm of $\mathbf{H}(X)$. If it is the case and if, moreover, the series $\sum_K r_K \mathrm{e}^{\mathrm{i}\langle K,\cdot\rangle}$ converges in the sense of $L_2(I) = L_2(I, \mathscr{B}, \mathsf{m}_d)$ where $\mathscr{B}$ denotes the σ-algebra of all Borel subsets of I and m_d is the d-dimensional Lebesgue measure, we say that the extrapolation $\hat{X}(L)$ has the spectral characteristic

$$g_L(\cdot) = \sum_K r_K \mathrm{e}^{\mathrm{i}\langle K,\cdot\rangle} \quad \text{where} \quad K \in \mathbb{Z}^d_+ .$$

The spectral characteristic being known, the extrapolation coefficients may be determined by means of the relation

$$r_K = (2\pi)^{-d} \int_I g_L(\lambda)\, \mathrm{e}^{-\mathrm{i}\langle K,\lambda\rangle}\, \mathrm{d}\lambda .$$

We shall therefore consider the spectral characteristic, if it exists, to be a solution of the extrapolation problem.

In case the random field X has factorisable spectral density, i.e. there exists the function h of d complex variables such that both h and $1/h$ are holomorphic on the unit polydisc $U^d = \{(z_1, \ldots, z_d): |z_j| \leqq 1 \text{ for } j = 1, \ldots, d\}$ and the spectral density f is expressed for every $\lambda \in I$ as $f(\lambda) = |h(\mathrm{e}^{\mathrm{i}\lambda_1}, \ldots, \mathrm{e}^{\mathrm{i}\lambda_d})|^2$, the spectral characteristic g_L exists for every $L \in \mathbb{Z}^d \setminus \mathbb{Z}^d_+$ and its explicit form involves the factor function h. In fact,

$$g_L(\lambda) = \mathscr{H}(\mathrm{e}^{\mathrm{i}\langle L,\lambda\rangle} \,.\, h(\mathrm{e}^{\mathrm{i}\lambda_1}, \ldots, \mathrm{e}^{\mathrm{i}\lambda_d}))\,.\,(1/h(\mathrm{e}^{\mathrm{i}\lambda_1}, \ldots, \mathrm{e}^{\mathrm{i}\lambda_d}))$$

where $\mathscr{H}$ denotes the "holomorphic part of a function", more exactly, for every function p on I of the form

$$p(\cdot) = \sum_K p_K \mathrm{e}^{\mathrm{i}\langle K,\cdot\rangle} \quad \text{where} \quad K \in \mathbb{Z}^d$$

the value of $\mathscr{H}(p(\lambda))$ is, for every $\lambda \in I$, defined as

$$\mathscr{H}(p(\lambda)) = \sum_K p_K \, e^{i\langle K, \lambda \rangle} \quad \text{where} \quad K \in \mathbb{Z}_d^+ .$$

Theorem 3.5.3 in [1] implies that every (strictly) positive bounded lower semi-continous function on I is factorisable. Hence, if X is a stationary random field on $\mathbb{Z}^d$ with bounded range interactions and the spectral density f of X is positive then the spectral density f is factorisable because it is obviously bounded and continuous. So for every $L \in \mathbb{Z}^d \setminus \mathbb{Z}_+^d$ there exists the spectral characteristic g_L of the extrapolation $\hat{X}(L)$. Unfortunately, the theorem referred to does not provide an effective way to obtain the factor function. In case X is weakly isotropic, this difficulty may be overcome using the spectrum decomposition.

Let us, according to Theorem 2.2, write

$$f(\lambda) = \sum_{j=1}^{m} a_j \prod_{k=1}^{d} \varphi_j(\lambda_k)$$

where $\varphi_1, \ldots, \varphi_m$ are simple one-dimensional spectral densities. For $j = 1, \ldots, m$ let P_j denote the factor polynomial corresponding to φ_j. That is, $\varphi_j(x) = |P_j(e^{ix})|^2$ for every $x \in [-\pi, \pi]$ and $1/P_j$ is holomorphic on the unit disc U. (Existence of such a factorization is an easy consequence of the definition of a simple one-dimensional spectral density.)

Denoting $h_j(\lambda) = \prod_{k=1}^{d} P_j(e^{i\lambda_k})$ it is possible to express

$$f(\lambda) = \sum_{j=1}^{m} a_j |h_j(\lambda)|^2 . \tag{9}$$

3.1. Theorem. Let X be a stationary weakly isotropic random field on $\mathbb{Z}^d$ with bounded range interactions and with strictly positive spectral density f. Then for every $L \in \mathbb{Z}^d \setminus \mathbb{Z}_+^d$ the spectral characteristic g_L of the extrapolation $\hat{X}(L)$ is given

$$g_L(\lambda) = (1/f(\lambda)) \, . \sum_{j=1}^{m} a_j \, \mathscr{H}(e^{i\langle L, \lambda \rangle} \, h_j(\lambda)) \, (h_j(\lambda))^* \tag{10}$$

for every $\lambda \in I = [-\pi, \pi]^d$.

Proof. The function g_L is the spectral characteristic of $\hat{X}(L)$ if and only if

$$\int_I (e^{i\langle L, \lambda \rangle} - g_L(\lambda)) f(\lambda) \, e^{-i\langle K, \lambda \rangle} \, d\lambda = 0 \tag{11}$$

is true for every $K \in \mathbb{Z}_+^d$. Viz. $\hat{X}(K)$ is the projection of $X(L)$ onto $\mathbf{H}_+(X)$ if and only if $\mathsf{E}(X(L) - \hat{X}(L)) \, . \, (X(K))^* = 0$ for every $K \in \mathbb{Z}_+^d$ what implies (11) with regard to (2). Let us substitute (9) into (11). As $\int_I e^{i\langle M, \lambda \rangle} \, d\lambda \neq 0$ holds just in case $M = 0$, (11) is equivalent to

$$\sum_{j=1}^{m} a_j \, \mathscr{H}((e^{i\langle L, \lambda \rangle} - g_L(\lambda)) \, h_j(\lambda)) \, (h_j(\lambda))^* = 0 . \tag{12}$$

It is $\mathscr{H}((e^{i\langle L, \lambda \rangle} - g_L(\lambda)) \, h_j(\lambda)) = \mathscr{H}(e^{i\langle L, \lambda \rangle} \, h_j(\lambda)) - g_L(\lambda) \, h_j(\lambda)$ because $\mathscr{H}(g_L) = g_L$ and $\mathscr{H}(h_j) = h_j$. Hence (12) is equivalent to (10).

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