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TIME-OPTIMAL CONTROL OF NONLINEAR PARABOLIC SYSTEMS WITH CONSTRAINED DERIVATIVE OF CONTROL, EXISTENCE THEOREM

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Nonlinear autonomous systems in abstract Banach spaces are considered. Supposing controllability and certain correctness of the controlled system, existence of a time-optimal control is shown. The proof essentially employs the constraint imposed on the derivative of the control. Finally, systems with a monotone operator are investigated.

1. INTRODUCTION AND NOTATION

The studies of the control problems for an evolution system with a bounded derivative (= the derivative with respect to time) of the control were stimulated by solving practical problems. Our problem is to find an optimal control of a thermal process where the control acts by means of boundary conditions. In more details, an iron body to be heated up is placed into a furnace which temperature is considered to be everywhere constant at each fixed time. The furnace temperature may change within time according to a plan determined in advance, but the speed of the temperature changes, i.e. the derivative of the control, is bounded due to construction parameters of the furnace. Our aim is to heat the body up in a minimum time and, at the same time, the furnace-temperature changes must not exceed the maximal possible speed, and also certain constraints on the temperature inside the body (= the state-space constraints) must be fulfilled, namely an effective thermoelastic stress must not exceed the critical level prescribed in advance. However, in this paper the state-space constraints will be considered only in a general manner and the problem is thus reduced to a nonlinear parabolic system without any consideration of the quasistationary elliptic problem arising from the elastic-stress equation. Moreover, also the heat transfer operator is considered only in an abstract manner.

Generally speaking, in practical situations the derivative of the control is, in fact, constrained very often in consequence of various construction reasons, but mostly the changes of the control may be far quicker than the changes of the state in the controlled system. In such a case it is highly apposite to admit the controls of the bang-

bang type. On the other hand, if the maximal speed of the control changes is, roughly speaking, comparable with or slower than the speed of the state changes, then the constraint on the derivative of the control must be taken into consideration.

The time-optimal control problem for a parabolic equation without the constraint on the derivative of the control was investigated by Lions [4] (a linear case) and Ahmed [1] (a nonlinear case). In those works the weak topology in relevant Banach spaces was used in an essential manner to prove existence of an optimal control. The constraint on the derivative of the control, however, enables by means of the well-known Arzelà-Ascoli theorem to use only the strong topologies. Thus we may admit more general situation than we might without the constrained derivative. Convexity will be required only for the constraint set of the derivative of the control itself. The existence theorem for the general situation thus obtained will be stated in Section 2. Furthermore, a more detailed situation of the controlled system with a monotone operator will be investigated in Section 3. However, in contrast to [1] we shall have to suppose that this operator is even "uniformly" monotone to ensure the continuity of the mapping from the controls to the states.

Now we begin with some notations. Let real Banach spaces U, X; operators $A : X \to X$, $B : U \to X$; an element $x_0 \in X$; and sets $D \subset U$; $M_u(\tau) \subset U$, $M_x(\tau) \subset X$ for $\tau \in [0, 1]$ be given. The element x_0 represents an initial state of the controlled system, the set D represents the constraint on the derivative of the control, and the collections $M_u(\tau)$ and $M_x(\tau)$ represent the control and the state-space constraints, respectively. Furthermore, $\mathbb{R}_0^+$ denotes the set of all non-negative reals and $C(0, \vartheta; U)$ denotes the Banach space of all U-valued continuous functions on the interval $[0, \vartheta]$, where $\vartheta \in \mathbb{R}_0^+$ is the time of the whole control process. We will engage in the following time-optimal control problem:

$$\vartheta \to \inf \quad \text{(the minimal-time criterion)}$$

subject to

$$(1) \qquad \begin{cases} (\vartheta, u) \in \mathbb{R}_0^+ \times C(0, \vartheta; U) \quad \text{and} \quad u \quad \text{is Lipschitz}, \\ \mathrm{d}x/\mathrm{d}t + Ax = Bu \quad \text{(the state equation)}, \\ x(0) = x_0 , \\ \mathrm{d}u/\mathrm{d}t \in D \quad \text{for a.a.} \quad t \in [0, \vartheta] , \\ u(t) \in M_u(t/\vartheta) \quad \text{for all} \quad t \in [0, \vartheta] , \\ x(t) \in M_x(t/\vartheta) \quad \text{for all} \quad t \in [0, \vartheta] . \end{cases}$$

The exact meaning of the fact that $x : [0, \vartheta] \to X$ solves the state equation is not needed for the purpose of Section 2. But in Section 3 the solution of the state equation will be considered, as usual, in the distributive sense; for details see e.g. [3, 4].

Note that the above framework enables to consider the problems with only terminal-state constraint (taking $M_x(\tau) = X$ for $\tau < 1$), or other special situations.

For further investigation it is useful to rewrite problem (1) onto a fixed time-interval, say $[0, 1]$. The transformed time will be denoted also by t without causing any misunderstanding. The control is thus a couple of the real parameter ϑ and the function $u : [0, 1] \to U$. The set of the controls admissible with respect to the control constraints only, denoted by U_{ad}, is given by

$$U_{ad} = \{(\vartheta, u) \in \mathbb{R}_0^+ \times C(0, 1; U) ; \quad u \quad \text{is Lipschitz}, \quad du/dt \in \vartheta . D \quad \text{for} \\ \text{a.a.} \quad t \in [0, 1] \quad \text{and} \quad u(t) \in M_u(t) \quad \text{for all} \quad t \in [0, 1]\} .$$

The cost function augmented by the indicator function corresponding to the state-space constraints, denoted by J, is given by

$$J(\vartheta, u) = \begin{cases} \vartheta & \text{for} \quad (\vartheta, Bu) \in F_{ad} , \\ +\infty & \text{elsewhere,} \end{cases}$$

where F_{ad} is the set of all admissible (with respect to the state-space constraints) right-hand sides of the state equation, i.e.

$$F_{ad} = \{(\vartheta, f) \in \mathbb{R}_0^+ \times \mathscr{X} ; \quad \exists x , \quad dx/dt + \vartheta Ax = \vartheta f , \quad x(0) = x_0 \quad \text{and} \\ \forall t \in [0, 1], \; x(t) \in M_x(t)\} ,$$

where $\mathscr{X}$ is a sufficiently large Banach space of the functions $[0, 1] \to X$, e.g. $B \circ C(0, 1; U) \subset \mathscr{X}$. Supposing, as usual, the controllability condition

$$\inf \{J(\vartheta, u); \; (\vartheta, u) \in U_{ad}\} < +\infty , \tag{2}$$

the problem of the time-optimal control of (1) is thus equivalent to the problem to find

$$(\vartheta_{opt}, u_{opt}) \in \operatorname{Arg\,inf} \{J(\vartheta, u); (\vartheta, u) \in U_{ad}\} . \tag{3}$$

In the following we shall suppose that the control constraints fulfil the conditions

$$\left\{ \begin{array}{ll} M_u(t) & \text{are closed in } U \text{ for all } t \in [0, 1] , \\ M_u(t_0) & \text{is compact in } U \text{ for some } t_0 \in [0, 1], \text{ and} \\ D & \text{is compact and convex in } U . \end{array} \right. \tag{4}$$

We remark that every Banach space will be considered only with its strong topology and the norms in the function spaces used below will be taken in a usual manner. Finally, we recall the notion of the Gelfand-Banach space frequently used in what follows. U is called a Gelfand-Banach space iff it is a Banach space and every Lipschitz function $[0, 1] \to U$ is differentiable a.e. in $[0, 1]$. For a definition of a more general space, namely the Gelfand-Fréchet space, see Mankiewicz [5]. The class of such spaces is fairly broad, e.g. every separable conjugate Banach space is the Gelfand-Banach space as well (this assertion was established by Gelfand).

2. GENERAL EXISTENCE THEOREM

First, we introduce some notions used in this section.

Definition 1. Problem (1) is called to be correct on the space $\mathscr{X}$ with respect to the collection M_x iff the set F_{ad} defined above is closed in $\mathbb{R} \times \mathscr{X}$.

Definition 2. Let M be a bounded subset in U. The function $u : [0, 1] \to U$ is called to be M-Lipschitz iff

$$\forall t_1, t_2 \in [0, 1] : u(t_1) - u(t_2) \in (t_1 - t_2) M .$$

The following assertions are direct consequence of Definition 2. An M-Lipschitz function is Lipschitz in the usual sense as well, because M is bounded. If there exists the derivative of an M-Lipschitz function, then this derivative belongs to the closure of M. If a Lipschitz function u is differentiable a.e. in $[0, 1]$ and $\mathrm{d}u/\mathrm{d}t \in M$, then u is $\overline{\mathrm{co}}\, M$-Lipschitz ($\overline{\mathrm{co}}\, M$ is the closed convex hull of M), but generally u is not M-Lipschitz.

For $\vartheta_0 \in \mathbb{R}$ we denote $U_{ad}(\vartheta_0) = \{(\vartheta, u) \in U_{ad};\ \vartheta \leqq \vartheta_0\}$.

Lemma 1. Let (4) be fulfilled and let U be a Gelfand-Banach space. Then $U_{ad}(\vartheta_0)$ is compact in $\mathbb{R} \times C(0, 1; U)$.

Proof. Obviously, we have the estimate: $U_{ad}(\vartheta_0) \subset [0, \vartheta_0] \times G(\vartheta_0)$, where $G(\vartheta_0) = \{u \in C(0, 1; U);\ \forall t \in [0, 1] : u(t) \in G_0$ and $\mathrm{d}u/\mathrm{d}t \in G_1$ a.e.$\}$, with $G_0 = M_u(t_0) + G_1$ (t_0 is taken from (4)) and $G_1 = \vartheta_0 \,.\, \mathrm{co}(\{0\} \cap D)$. Since G_1 is bounded and G_0 is precompact, $G(\vartheta_0)$ is precompact in $C(0, 1; U)$ as a consequence of the well-known Arzelà-Ascoli theorem. Thus $U_{ad}(\vartheta_0)$ is precompact in $\mathbb{R} \times C(0, 1; U)$.

Now we have to prove that $U_{ad}(\vartheta_0)$ is closed. Let a convergent sequence $(\vartheta_i, u_i) \in U_{ad}(\vartheta_0)$ and $\varepsilon > 0$ be given. Denote $(\vartheta, u) = \lim_{i \to \infty} (\vartheta_i, u_i)$. The set $M_{\vartheta,\varepsilon} = I_{\vartheta,\varepsilon} \,.\, D$ is convex and compact in U, where $I_{\vartheta,\varepsilon}$ is the interval $[\max(0, \vartheta - \varepsilon), \vartheta + \varepsilon]$. For a sufficiently large i we have $\vartheta_i \in I_{\vartheta,\varepsilon}$ and, owing to the convexity of $M_{\vartheta,\varepsilon}$, the functions u_i are $M_{\vartheta,\varepsilon}$-Lipschitz. Therefore, the limit function u is $M_{\vartheta,\varepsilon}$-Lipschitz, too. Then $\mathrm{d}u/\mathrm{d}t \in M_{\vartheta,\varepsilon}$ a.e. in $[0, 1]$ provided that U is a Gelfand-Banach space. As ε has been arbitrary positive, $\mathrm{d}u/\mathrm{d}t \in \bigcap_{\varepsilon > 0} M_{\vartheta,\varepsilon}$. Since D is closed, $\bigcap_{\varepsilon > 0} M_{\vartheta,\varepsilon} = \vartheta \,.\, D$. The other required properties of u are obvious, hence $(\vartheta, u) \in U_{ad}(\vartheta_0)$ and $U_{ad}(\vartheta_0)$ is closed. □

Corollary. J is coercive on U_{ad} provided (4) is fulfilled.

Proof. As shown in the proof of Lemma 1, the condition (4) implies that $U_{ad}(\vartheta_0)$ is precompact, and thus bounded as well. Therefore, the coercivity of J follows clearly from the inequality $J(\vartheta, u) \geqq \vartheta$. □

Remark 1. Lemma 1 need not hold if D is not convex. To show it, we outline the following simple example. Consider $U = \mathbb{R}$, $D = \{1, -1\}$ and a sequence of Lipschitz functions $u_i \in C(0, 1; \mathbb{R})$ such that $\mathrm{d}u_i/\mathrm{d}t \in D$ a.e. in $[0, 1]$ and $u_i \to 0$ in $C(0, 1; \mathbb{R})$. Such a sequence clearly exists. Obviously, $(1, u_i) \in U_{\mathrm{ad}}(1)$ and $(1, u_i) \to$ $\to (1, u)$ with $u = 0$; however, $\mathrm{d}u/\mathrm{d}t \notin D$, hence $U_{\mathrm{ad}}(1)$ is not closed.

Further, we define the mapping $\mathscr{B}$ by the formula $u \mapsto Bu$. If $B : U \to X$ is uniformly continuous, then $\mathscr{B}$ is continuous (and everywhere defined) as a mapping $C(0, 1; U) \to C(0, 1; X)$. If, in addition, U is locally compact (i.e. finite-dimensional), then for the continuity of $\mathscr{B}$ it is sufficient that B is only continuous (see [2], Chap. X, § 3).

Now we can formulate the general existence theorem.

Theorem 1. Let the control constraints fulfil (4), U be a Gelfand-Banach space, B be uniformly continuous and the problem (1) be controllable and correct on $C(0, 1; X)$ with respect to the collection M_x. Then there exists a time-optimal control of the problem (1). The mere continuity of B can be supposed in case U is finite-dimensional.

Proof. As (1) is correct and $\mathscr{B}$ is continuous, J is lower semi-continuous on $\mathbb{R} \times$ $\times\, C(0, 1; U)$. Owing to the controllability (2), we can choose such ϑ_0 that inf . $. \{J(\vartheta, u); (\vartheta, u) \in U_{\mathrm{ad}}\} < \vartheta_0 < +\infty$. Then $\operatorname{Arg\,inf} \{J(\vartheta, u); (\vartheta, u) \in U_{\mathrm{ad}}\} =$ $= \operatorname{Arg\,inf} \{J(\vartheta, u); (\vartheta, u) \in U_{\mathrm{ad}}(\vartheta_0)\}$. Since $U_{\mathrm{ad}}(\vartheta_0)$ is compact, there exists a solution $(\vartheta_{\mathrm{opt}}, u_{\mathrm{opt}})$ of the problem (3). Transforming the function u_{opt} onto the interval $[0, \vartheta_{\mathrm{opt}}]$, we obtain a time-optimal control of (1). □

Remark 2. Theorem 1 remains valid if the space $C(0, 1; X)$ is replaced by any space $\mathscr{X}$ into which $C(0, 1; X)$ is continuously imbedded, because in such a case the problem (1) is correct on the space $C(0, 1; X)$, too.

3. STATE EQUATION WITH MONOTONE OPERATOR

In this section we shall study a more detailed structure corresponding to the cases of nonlinear parabolic equations, where the notion of correctness (from Definition 1) will be specified.

Let V be a reflexive Banach space continuously and densely imbedded into a Hilbert space H. Denoting V^* the dual of V and identifying H with its own dual, we have $V \subset H \subset V^*$. The duality pairing of V^* and V is denoted by $\langle .\,, .\rangle$. We suppose that $A : V \to V^*$ satisfies:

$$\forall u, v \in V \quad w\text{-}\lim_{t \to 0} A(u + tv) = Au \quad \text{(hemicontinuity)}\,;$$

$$\forall u, v \in V \quad \langle Au - Av, u - v\rangle \geqq 0 \quad \text{(monotonicity)}\,;$$

$$\exists \alpha > 0,\ C < +\infty,\ 1 < p < +\infty\ \forall v \in V \quad \|Av\|_{V^*} \leqq C\|v\|_V^{p-1}\,,$$

$$\langle Av, v\rangle \geqq \alpha\|v\|_V^p\,,$$

where $\|.\|$ denotes the norm in the corresponding space.

Denote $\mathscr{V} = L^p(0, 1; V)$, $\mathscr{V}^* = L^{p'}(0, 1; V^*)$, where $p' = p/(p - 1)$. From these assumptions we can deduce (see Lions [3], Chap. II, § 1.4) that for every $f \in \mathscr{V}^*$ there is a unique solution of the initial-value problem $\mathrm{d}x/\mathrm{d}t + Ax = f$, $x(0) = x_0 \in H$ such that $x \in \mathscr{V} \cap C(0, 1; H)$. This implies that for every $\vartheta \geqq 0$, $f \in \mathscr{V}^*$ there is a unique solution of the problem

$$\frac{\mathrm{d}x}{\mathrm{d}t} + \vartheta Ax = \vartheta f\,, \quad x(0) = x_0 \in H \tag{5}$$

such that $x \in C(0, 1; H)$. If in addition $\vartheta > 0$, $x \in \mathscr{V}$.

Lemma 2. Let the assumptions stated above in this section be fulfilled. Let, in addition, $\forall u, v \in V: \langle Au - Av, u - v\rangle \geqq \alpha\|u - v\|_V^p$ with $\alpha > 0$. Then the mapping $\mathbb{R}_0^+ \times \mathscr{V}^* \to C(0, 1; H)$ defined by $(\vartheta, f) \mapsto x$, where x is the solution of (5), is continuous.

Proof. Let fixed $\vartheta > 0$, $f \in \mathscr{V}^*$ be given. The case $\vartheta = 0$ will be investigated later. Moreover, let $\varepsilon \in (0, \vartheta/2)$ and $\tilde{\vartheta} \in \mathbb{R}$, $\tilde{f} \in \mathscr{V}^*$ fulfil $|\vartheta - \tilde{\vartheta}| < \varepsilon$, $\|f - \tilde{f}\|_{\mathscr{V}^*} < \varepsilon$. Denote $\tilde{x}$ the solution of the perturbed problem

$$\frac{\mathrm{d}\tilde{x}}{\mathrm{d}t} + \tilde{\vartheta} A\tilde{x} = \tilde{\vartheta}\tilde{f}\,, \quad \tilde{x} = x_0\,. \tag{$\tilde{5}$}$$

We will deduce an estimate for $\|x - \tilde{x}\|_H$, where $\|x - \tilde{x}\|_H$ means a function $[0, 1] \to \mathbb{R}$; and similarly $\langle f, x\rangle$ etc.

By the usual manner, we obtain

(6)
$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|x - \tilde{x}\|_H^2 + \langle \vartheta Ax - \tilde{\vartheta}A\tilde{x}, x - \tilde{x}\rangle = \langle \vartheta f - \tilde{\vartheta}\tilde{f}, x - \tilde{x}\rangle\,, (x - \tilde{x})(0) = 0\,.$$

For the expression in the brackets on the left-hand side we have

$$\langle \vartheta Ax - \tilde{\vartheta}A\tilde{x}, x - \tilde{x}\rangle = \tilde{\vartheta}\langle Ax - A\tilde{x}, x - \tilde{x}\rangle + (\vartheta - \tilde{\vartheta})\langle Ax, x - \tilde{x}\rangle \geqq$$
$$\geqq \tilde{\vartheta}\alpha\|x - \tilde{x}\|_V^p - \varepsilon C\|x\|_V^{p-1}\,\|x - \tilde{x}\|_V$$

and for the right-hand side we have

$$\langle \vartheta f - \tilde{\vartheta}\tilde{f}, x - \tilde{x}\rangle \leqq \tfrac{1}{2}\tilde{\vartheta}\alpha\|x - \tilde{x}\|_V^p + C_1\|\vartheta f - \tilde{\vartheta}\tilde{f}\|_{V^*}^{p'}$$

provided C_1 is sufficiently large. Because of $\tilde{\vartheta} > \vartheta/2$, we can suppose that the constant C_1 depends only on ϑ, α and p. Therefore from (6) we obtain for every $t \in [0, 1]$

$$\|(x - \tilde{x})(t)\|_H^2 \leqq 2\int_0^1 (\varepsilon C\|x\|_V^{p-1}\,\|x - \tilde{x}\|_V + C_1\|\vartheta f - \tilde{\vartheta}\tilde{f}\|_{V^*}^{p'})\,\mathrm{d}t\,.$$

There is C_2 (depending on C and p) such that $Ca^{p-1}b \leqq C_2(a^p + b^p)$ for every positive a, b. Hence

$$\|(x - \tilde{x})(t)\|_H^2 \leqq 2\varepsilon C_2(\|x\|_{\mathscr{V}}^p + (\|x\|_{\mathscr{V}} + \|\tilde{x}\|_{\mathscr{V}})^p) + 2\varepsilon^{p'}C_1(\tfrac{3}{2}\vartheta + \|f\|_{\mathscr{V}^*})^{p'}\,. \tag{7}$$

We used the inequality

$$\|\vartheta f - \tilde{\vartheta}\tilde{f}\|_{\mathscr{V}^*}^{p'} = \|\tilde{\vartheta}(f - \tilde{f}) + (\vartheta - \tilde{\vartheta}) f\|_{\mathscr{V}^*}^{p'} \leqq (\tfrac{3}{2}\vartheta\|f - \tilde{f}\|_{\mathscr{V}^*} + \varepsilon\|f\|_{\mathscr{V}^*})^{p'} .$$

Now we need to estimate $\|\tilde{x}\|_{\mathscr{V}}$. Similarly as in the foregoing we can obtain the inequality

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\tilde{x}\|_H^2 + \tilde{\vartheta}\alpha\|\tilde{x}\|_V^p \leqq \frac{\tilde{\vartheta}\alpha}{2}\|\tilde{x}\|_V^p + \tilde{\vartheta}C_3\|\tilde{f}\|_{V^*}^{p'} \tag{8}$$

with the constant C_3 depending on α and p only. Via integration of (8) we obtain

$$\|x(1)\|_H^2 + \tilde{\vartheta}\alpha\|\tilde{x}\|_{\mathscr{V}}^p \leqq 2\tilde{\vartheta}C_3\|\tilde{f}\|_{\mathscr{V}^*}^{p'} + \|x_0\|_H^2 ,$$

hence

$$\|\tilde{x}\|_{\mathscr{V}}^p \leqq \frac{2C_3}{\alpha}(\|f\|_{\mathscr{V}^*} + \varepsilon)^{p'} + \frac{2}{\vartheta\alpha}\|x_0\|_H^2 \leqq C_4^p ,$$

where C_4 depends on x_0, α, p, ϑ and f only (because $\varepsilon < \vartheta/2$). Using (7) we obtain

$$\|x - \tilde{x}\|_{C(0,1;H)} \leqq (2\varepsilon C_2(\|x\|_{\mathscr{V}}^p + (\|x\|_{\mathscr{V}} + C_4)^p) + 2\varepsilon^{p'}C_1(\tfrac{3}{2}\vartheta + \|f\|_{\mathscr{V}^*})^{p'})^{1/2}$$

and the right-hand side tends to zero together with $\varepsilon \to 0$.

Now the degenerate case $\vartheta = 0$ remains to be proved. The solution of (5) is trivial $x = \text{const.} = x_0$. Let for $\tilde{\vartheta}$, $\tilde{f}$ the inequalities $0 \leqq \tilde{\vartheta} < \varepsilon$, $\|f - \tilde{f}\|_{\mathscr{V}^*} < \varepsilon$ hold and $\tilde{x}_0 \in V$ be given. We may consider $\tilde{\vartheta} > 0$. Thus $\tilde{x}(t) - \tilde{x}_0 \in V$ a.e. in $[0,1]$ and from $(\tilde{5})$ we obtain

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\tilde{x} - \tilde{x}_0\|_H^2 + \tilde{\vartheta}\langle A\tilde{x}, \tilde{x}\rangle = \tilde{\vartheta}\langle\tilde{f}, \tilde{x}\rangle + \tilde{\vartheta}\langle A\tilde{x} - \tilde{f}, \tilde{x}_0\rangle , \quad \tilde{x}(0) = x_0 .$$

There are C_5 (depending on C, α, p) and C_6 (depending on p) such that $Ca^{p-1}b \leqq \leqq \frac{1}{4}\alpha a^p + C_5 b^p$ and $ab \leqq C_6(a^p + b^{p'})$ for every positive a, b. Hence

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\tilde{x} - \tilde{x}_0\|_H^2 + \tilde{\vartheta}\alpha\|\tilde{x}\|_V^p \leqq \frac{\tilde{\vartheta}\alpha}{2}\|\tilde{x}\|_V^p + \tilde{\vartheta}C_3\|\tilde{f}\|_{V^*}^{p'} + \tilde{\vartheta}(C\|\tilde{x}\|_V^{p-1} + \|\tilde{f}\|_{V^*})\|\tilde{x}_0\|_V \leqq$$

$$\leqq \frac{3\tilde{\vartheta}\alpha}{4}\|\tilde{x}\|_V^p + \tilde{\vartheta}(C_3 + C_6)\|\tilde{f}\|_{V^*}^{p'} + \tilde{\vartheta}(C_5 + C_6)\|\tilde{x}_0\|_V^p$$

and via integration we obtain

$$\|\tilde{x} - x_0\|_{C(0,1;H)} \leqq (2\varepsilon(C_3 + C_6)(\|f\|_{\mathscr{V}^*} + \varepsilon)^{p'} + 2\varepsilon(C_5 + C_6)\|\tilde{x}_0\|_V^p + \|x_0 - \tilde{x}_0\|_H^2)^{1/2} + \|x_0 - \tilde{x}_0\|_H . \tag{9}$$

As V is dense in H, we can choose such $\tilde{x}_0$ and ε that the right-hand side of (9) is an arbitrary small positive number. Finally, we remark that it was not possible to set $\tilde{x}_0 = x_0$, because $\tilde{x} - x_0$ would not be a "good" test function, namely $\tilde{x} - x_0 \notin \mathscr{V}$. □

If the sets $M_x(t)$ are closed in H for every $t \in [0, 1]$, then from Lemma 2 it follows that the set F_{ad} from Section 1 is closed and our problem is thus correct on $\mathscr{V}^*$ with respect to M_x. Then, using Theorem 1, we obtain the existence of the time-optimal control of the problem in question. In the following theorem we summarize all the preceding results.

Theorem 2. Let U be a Gelfand-Banach space; V be a reflexive Banach space imbedded continuously and densely into a Hilbert space H; $B : U \to V^*$ be uniformly continuous; $A : V \to V^*$ be hemicontinuous and there exist $\alpha > 0$, $C < +\infty$, $1 < p < +\infty$ such that

$$\forall u, v \in V \quad \|Au\|_{V^*} \leqq C\|u\|_V^{p-1}, \langle Au - Av, u - v \rangle \geqq \alpha\|u - v\|_V^p .$$

Further let the control constraints fulfil (4); the state-space constraints $M_x(t)$ be closed in H for every $t \in [0, 1]$; the initial condition $x_0 \in H$ be given and (1) be controllable, i.e. (2) is valid. Then there exists a time-optimal control of (1). The mere continuity of B can be supposed in case U is finite-dimensional.

Proof. The assumptions in Theorem 2 imply $\langle Au, u \rangle \geqq \alpha\|u\|_V^p$ and $\langle Au - Av, u - v \rangle \geqq 0$ used formely. Thus we may apply Lemma 2 and, from it and from the assumption that M_x are closed in H, we obtain the correctness of the considered problem on the space $\mathscr{V}^*$. The imbedding $C(0, 1; V^*) \subset \mathscr{V}^*$ is continuous and, using Remark 2 with $\mathscr{X} = \mathscr{V}^*$, we obtain the required assertion. □

Let us notice that there is a certain reserve in the choice of the function spaces. Of course, it would be sufficient to prove the continuity of the mapping in Lemma 2 only as a mapping $\mathbb{R}_0^+ \times C(0, 1; V^*) \to C(0,1; H)$ and such a proof would be somewhat simpler. The only reason of ours for using the space $\mathscr{V}^*$ was, in fact, that $\mathscr{V}^*$ is the "natural" space widely used in the theory of the parabolic equations.

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