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ASYMPTOTIC EFFICIENCY AND ROBUSTNESS OF D-ESTIMATORS

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Asymptotic normality of standard, weak, and directed D-estimators investigated in preceding two issues of Kybernetika is established and influence curves are derived, all under the assumption of vector-valued parameter spaces. Asymptotic variance matrices of estimators under consideration are expressed as variances of the corresponding multidimensional influence curves. Conditions of asymptotic efficiency are established as well.

1. PRELIMINARIES

This paper is a direct continuation of [9, 10]. It is assumed that the reader is familiar with notation and basic concepts presented there.

We consider in this paper parameter spaces $\Theta \subset \mathbb{R}^m$ with non-empty interiors Θ^0 in the $\mathbb{R}^m$-topology and with no isolated points. The interior Θ^0 is assumed σ-compact with increasing sequence of subsets $\Theta_j \subset \Theta^0$ compact in $\mathbb{R}^m$ and tending to Θ^0 in the set-theoretic sense (cf. (1.2) in [9]).

We say that a *rate of convergence* of a well-defined estimator $T : \mathscr{P}(T) \to \Theta$ to the parameter of a family $\mathscr{Q}_\Theta$ is n^{-a} if

$$n^a(T(P_n) - \theta) \xrightarrow{\{Q_\theta\}} No(0, V_\theta(T)) \quad \text{for all} \quad \theta \in \Theta^0 , \tag{1.1}$$

where $\xrightarrow{\{Q_\theta\}}$ denotes the weak convergence w.r.t. Q_θ^∞ as $n \to \infty$ and $V_\theta(T)$ denotes a finite non-zero *asymptotic variance* matrix of T at $Q_\theta \in \mathscr{Q}_\Theta$.

If for some T and Q there exists $\varepsilon_x > 0$ such that $(1 - \varepsilon)\, Q + \varepsilon 1_{\{x\}} \in \mathscr{P}(T)$ for all $0 < \varepsilon < \varepsilon_x$, $x \in \mathscr{X}$, and

$$\Omega_Q(x) = \lim_{\varepsilon \downarrow 0} \frac{T((1 - \varepsilon)\, Q + \varepsilon\, 1_{\{x\}}) - T(Q)}{\varepsilon} \tag{1.2}$$

exists for all $x \in \mathscr{X}$, then $\Omega_Q : \mathscr{X} \to \mathbb{R}^m$ is called *influence curve* of T at Q. By (6.1) in [9], for every equivariant estimator of location parameter from $\Theta = \mathbb{R}$ and every $\mathcal{Q}_\Theta$ generated by a parent $Q \in \mathcal{P}$, Ω_{Q_θ} for $\theta \in \Theta$ exist iff Ω_Q exists and

$$\Omega_{Q_\theta}(x) = \Omega_Q([\theta]^{-1}(x)) \quad \text{for all} \quad (\theta, x) \in \Theta \times \mathcal{X} . \tag{1.3}$$

The influence curve has originally been introduced by Hampel [5] as a characteristics of robustness of estimators of location. We shall see later that in general $V_\theta(T) = \mathsf{E}_Q \Omega_Q^{\mathrm{T}} \circ \Omega_Q$. Hence the influence curves $\{\Omega_{Q_\theta} : \theta \in \Theta\}$ represent an operative characteristics of both asymptotic efficiency and robustness of estimators of arbitrary parameters. Note that the robustness means in this paper a limited sensitivity of asymptotic estimates* $T(Q_\theta) = \theta$ to replacements of the "assumed" generating probabilities Q_θ by generating probabilities from ε-neighborhoods $U_\varepsilon(Q_\theta) = \{(1-\varepsilon) \,.\, Q_\theta + \varepsilon P^* : P^* \in \mathcal{P}\}$ of the former ones. We shall see later that in general $[T((1-\varepsilon) \,.\, Q_\theta + \varepsilon P^*) - T(Q_\theta)]/\varepsilon \to \mathsf{E}_{P^*}\Omega_{Q_\theta}$ as $\varepsilon \downarrow 0$ for all $P^* \subset \mathcal{P}$.

If $\mathcal{X} \subset \mathbb{R}^k$ then, following Hampel [5], the triple

$$\sigma_{GE}(T) = \sup_{\mathcal{X}} \|\Omega_{Q_\theta}(x)\|_m , \quad \sigma_{LS}(T) = \sup_{\mathcal{X}} \left\| \frac{\mathrm{d}\Omega_{Q_\theta}(x)}{\mathrm{d}x} \right\|_{mk} \tag{1.4}$$

$$\varrho_\varepsilon(T) = \inf_{\mathcal{X}} \{ \|x\|_k : \sup_{\|\tilde{x}\| > \|x\|} \|\Omega_{Q_\theta}(\tilde{x})\|_m \leqq \varepsilon \} \tag{1.5}$$

will be considered as a simple intensional descriptor of robustness of T at $Q \in \mathcal{P}$ (here and in the sequel $\|\cdot\|_m$ denotes the usual Euclidean $\mathbb{R}^m$-norm). The components $\sigma_{GE}(T)$, $\sigma_{LS}(T)$, $\varrho_\varepsilon(T)$ are called correspondingly *gross-error sensitivity*, *local-shift sensitivity*, and *radius of ε-negligibility* (radius of rejection).

Now we clarify a non-asymptotic meaning of influence curves under consideration. Following Tukey [7] we call Ω_{P_n} a *sensitivity curve* of T at $P_n \in \mathcal{P}_e$. If $\mathcal{X} = \mathbb{R}$ and $P_n^{(\theta)}$ denotes an empirical probability given by (1.1) in [9] for a sample vector $\boldsymbol{x}^{(\theta)} = (\mathsf{E}_{Q_\theta^n} X_{(1)}, \ldots, \mathsf{E}_{Q_\theta^n} X_{(n)}) \in \mathcal{X}^n$ of expectations of order statistics of a r. v. $X = (X_1, \ldots, X_n)$ with sample space $(\mathcal{X}^n, \mathcal{B}^n, Q_\theta^n)$, then the sensitivity curves $\{\Omega_{P_n^{(\theta)}} : \theta \in \Theta\}$ are suitable sample-size-n alternatives to $\{\Omega_{Q_\theta} : \theta \in \Theta\}$. Obviously, under certain regularity conditions, one of these systems of curves approximates the other. Note that for location

$$\Omega_{P_n^{(\theta)}}(x) = \Omega_{P_n^{(e)}}([\theta]^{-1}(x)) \quad \text{for all} \quad (\theta, x) \in \Theta \times \mathcal{X} , \tag{1.6}$$

where $e = 0 \in \mathbb{R}$ and $\mathsf{E}_{Q^n} X_{(i)} = G^{-1}(i/(n+1))$ where G is the d.f. of a parent Q of $\mathcal{Q}_\Theta$. Thus by (1.1) in [9]

$$P_n^{(e)}(E) = \frac{1}{n} \sum_{i=1}^{n} 1_E \left(G^{-1} \left(\frac{i}{n+1} \right) \right) \quad \text{for} \quad E \in \mathcal{B} \tag{1.7}$$

and the curves $\{\Omega_{P_n^{(\theta)}} : \theta \in \mathbb{R}\}$ can explicitly be evaluated by (1.6), (1.7).

This paper is essentially based on elementary results of mathematical analysis

* If T is Fisher consistent for $\mathcal{Q}_\Theta$ then $T(Q_\theta) = \theta$ on Θ and if T is moreover consistent for $\mathcal{Q}_\Theta$ then $T(P_n) \overset{Q_\theta}{\to} T(Q_\theta) = \theta$ on Θ (cf. Sec. 1 in [10]).

formulated in detail below. Let $g = g(x, y) : \mathscr{X} \times \mathscr{Y} \to \mathbb{R}^k$ ($\mathscr{Y}$ is a topological space) be $\mathscr{B}$-measurable for all $y \in \mathscr{Y}$ and let λ be a σ-finite measure on $(\mathscr{X}, \mathscr{B})$. The g is said locally uniformly λ-integrable at $y \in \mathscr{Y}$ if there exists an open neighborhood $U(y)$ and a function $\tilde{g} : \mathscr{X} \to \mathbb{R}^k$ such that $\mathsf{E}_\lambda \|\tilde{g}\|_k < \infty$ and $\|g(x, \tilde{y})\|_k \leqq \|\tilde{g}(x)\|_k$ for all $x \in \mathscr{X}$, $\tilde{y} \in U(y)$.

We say that g is *λ-regular* if it is continuous on $\mathscr{Y}$ for every $x \in \mathscr{X}$, and locally uniformly λ-integrable at each $y \in \mathscr{Y}$.

Lemma 1.1. The expectation $\mathsf{E}_\lambda\, g(\cdot, y)$ of any λ-regular function g is continuous on $\mathscr{Y}$ in the sense $\mathsf{E}_\lambda \|g(\cdot, y) - g(\cdot, \tilde{y})\|_k \to 0$ for $\tilde{y} \to y$.

Proof. Clear from the Lebesque dominated convergence theorem. □

Lemma 1.2. (A mean value theorem.) Let $\mathscr{Y}$ be an open subset of $\mathbb{R}^m$ and $g : \mathscr{Y} \to \mathbb{R}^k$ be differentiable. Then for all $y, y' \in \mathscr{Y}$ with sufficiently small norm $\|y - y'\|_m$ there exists $y^* = \{y^1, \ldots, y^k\} \subset \mathscr{Y}$ such that $g(y) - g(y') = (y - y') \circ g'(y^*)$ where $g'(y^*)$ is the $m \times k$ matrix $g' = (\mathrm{d}/\mathrm{d}y)^{\mathrm{T}} \circ g$ with an argument $y = y^s$ in the s-th column and

$$\|y - y^*\|_m = \max_s \|y - y^s\|_m \leqq \|y - y'\|_m \,.$$

If g' is continuous at $y \in \mathscr{Y}$ then (in the $\mathbb{R}^{km}$-norm) $g'(y^*) \to g'(y)$ as $y' \to y$.

Proof. For the s-th coordinate of g the desired equality together with the inequality $\|y - y^s\|_m \leqq \|y - y'\|_m$ follow from the Lagrange mean value theorem for real-valued function of real variable. It suffices to suppose that the segment connecting y and y' lies in $\mathscr{Y}$ and to parametrize this segment by $t \in [0, 1]$. □

We say that $g : \mathscr{X} \times \mathscr{Y} \to \mathbb{R}^k$ is *strongly λ-regular* if $\mathscr{Y}$ is an open subset of R^m and the derivative $g'(x, y) = (\mathrm{d}/\mathrm{d}y)\, g(x, y)$ is λ-regular.

Lemma 1.3. The expectation $D(y) = \mathsf{E}_\lambda g(\cdot, y)$ of a strongly λ-regular function g is differentiable, $D'(y) = (\mathrm{d}/\mathrm{d}y)\, D(y) = \mathsf{E}_\lambda g'(\cdot, y)$, and $D'(y)$ is continuous on $\mathscr{Y}$.

Proof. Cf. Lemmas 1.1, 1.2, and the Lebesgue dominated convergence theorem. □

Lemmas 1.1–1.3 will be applied mainly to $\mathscr{Y} = \Theta^0 \subset \mathbb{R}^m$. The elements of $\mathbb{R}^m$ will be considered as row vectors ($1 \times m$ matrices), including the differential element $(\mathrm{d}/\mathrm{d}\theta) = (\partial/\partial\theta_1, \ldots, \partial/\partial\theta_m)$. Through the paper we write

$$p'_\theta = \frac{\mathrm{d}}{\mathrm{d}\theta}\, p_\theta\,, \quad p''_\theta = \left(\frac{\mathrm{d}}{\mathrm{d}\theta}\right)^{\mathrm{T}} \circ p'_\theta\,.$$

At several places we consider divergences for non-probabilistic measures – they are defined by the respective formulas of Sec. 2 of [9] with probabilistic measures replaced by the non-probabilistic ones.

In order to keep the extent of this paper limited we illustrate the main results by simple examples only. More complex applications are presented in separate papers.

2. EFFICIENCY AND ROBUSTNESS OF STANDARD D-ESTIMATORS

This section is a continuation of Section 2 of [10]: we consider well-defined standard D-estimators $T \triangleq \mathscr{P}_\Theta / D_f$ with projection families $\mathscr{P}_\Theta$ and (not necessarily identical) sample generating families $\mathscr{Q}_\Theta \subset \mathscr{P}$ on a discrete $\mathscr{X}$. We also consider for all $Q \in \mathscr{P}$ the function $D_Q(\theta) = D_f(P_\theta, Q)$ on parameter spaces under consideration. In addition to the assumptions of Section 1 we assume the following:

(i) Θ can be compactified in the sense that there exists a set $\bar{\Theta} \subset [-\infty, \infty]^m$ containing Θ and containing a cluster point of each sequence $\{\theta_j\} \subset \Theta$.

(ii) $\mathscr{X}$ is finite, λ denotes the counting measure on $\mathscr{X}$.

(iii) $\mathscr{P}_\Theta \ll \lambda$, $p_\theta = \mathrm{d}P_\theta/\mathrm{d}\lambda$ are twice continuously differentiable on Θ^0 for every $x \in \mathscr{X}$.

(iv) $\mathscr{Q}_\Theta \equiv \lambda$.

(v) f is twice continuously differentiable on $(0, \infty)$, $f''(1) \neq 0$.

(vi) If $\mathscr{Q}_\Theta \neq \mathscr{P}_\Theta$ then D_f is a metric on $\mathscr{P}$.

(vii) There exists $\mathscr{P}_{\bar{\Theta}}$ containing probabilities $P_\theta \in \mathscr{P}_\Theta$ for $\theta \in \Theta$ and probabilities or measures P_θ for $\theta \in \bar{\Theta} - \Theta$ such that $p_\theta(x) \to p_{\tilde{\theta}}(x)$ as $\theta \to \tilde{\theta} \in \bar{\Theta} - \Theta$, $\theta \in \bar{\Theta}$, for all $x \in \mathscr{X}$.

(viii) It holds $D_{Q_\theta}(\theta) < D_{Q_\theta}(\tilde{\theta})$ for all $\theta \in \Theta$, $\tilde{\theta} \in \bar{\Theta}$, $\theta \neq \tilde{\theta}$.

Lemma 2.1. (viii) with $\bar{\Theta}$ replaced by Θ is equivaent with the Fisher consistency of T for $\mathscr{Q}_\Theta$. (i) (ii), (vii), (viii) imply D_f-compatibility of $\mathscr{Q}_\Theta$ with $\mathscr{P}_\Theta$ and (i). (ii), (vi)–(viii) imply strong consistency of T for $\mathscr{Q}_\Theta$.

Proof. The first assertion is clear from (1.1) in [10]. If $\mathscr{Q}_\Theta$ is not D_f-compatible with $\mathscr{P}_\Theta$ then there exists $\theta \in \Theta$ and a sequence $\{\theta_j\} \subset \Theta$ such that θ is not among cluster points of $\{\theta_j\}$ and $D_{Q_\theta}(\theta_j) \to D_{Q_\theta}(\theta)$. Since by (i) there exists a cluster point $\tilde{\theta} \in \bar{\Theta}$ of $\{\theta_j\}$ and, by (ii) and (vii) and by the continuity of f, $D_{Q_\theta}(\theta_j) \to D_{Q_\theta}(\tilde{\theta})$, we get a contradiction with (viii). The third assertion of Lemma 2.1 follows from the second one and from Corollaries 2.1, 2.2 in [10]. □

Corollary 2.1. For all $\theta \in \Theta^0$ it holds $\lim_{n\to\infty} Q_\theta(E_n(T)) = 1$ where $E_n(T) = \{\boldsymbol{x} \in \mathscr{X}^n : T(P_n) \in \Theta^0\}$.

Lemma 2.2. For every $Q \equiv \lambda$ and $\theta \in \Theta^0$ (ii)–(v) imply

$$D'_Q(\theta) = \frac{\mathrm{d}}{\mathrm{d}\theta} D_Q(\theta) = \mathsf{E}_\lambda \psi_{\theta/Q}\,, \quad D''_Q(\theta) = \left(\frac{\mathrm{d}}{\mathrm{d}\theta}\right)^{\mathrm{T}} \circ D'_Q(\theta) = \mathsf{E}_\lambda \psi'_{\theta/Q}$$

where

$$\psi_{\theta/Q} = f'\left(\frac{p_\theta}{q}\right) p'_\theta\,, \quad \psi'_{\theta/Q} = f''\left(\frac{p_\theta}{q}\right) \frac{(p'_\theta)^{\mathrm{T}} \circ p'_\theta}{q} + f'\left(\frac{p_\theta}{q}\right) p''_\theta$$

and $D_Q(\theta)$, $D'_Q(\theta)$, $D''_Q(\theta)$ are continuous on Θ^0.

Proof. Clear. □

Lemma 2.3. (ii)–(v) imply that, in the product topology $\|\cdot\| \times \chi^1$ on $\Theta \times \mathscr{P}$,

$$\dot{\psi}_{\theta/Q} = -f''\left(\frac{p_\theta}{q}\right)\frac{p_\theta p'_\theta}{q^2}$$

is continuous on the interior $\Theta^0 \times \mathscr{P}^0$ for every $x \in \mathscr{X}$ and $Q \equiv \lambda$.

Proof. Clear. □

If in (ii) the finiteness of $\mathscr{X}$ is replaced by countability then the equivalence $\mathscr{Q}_\Theta \equiv \lambda$ in (iv) has to be replaced by a locally uniform summability (λ-integrability) of functions $\{\psi_{\theta/Q}, \psi'_{\theta/Q} : Q \in \mathscr{Q}_{\Theta^0}\}$ at each $\theta \in \Theta^0$ and of the function $\dot{\psi}_{\theta/Q}$ at each $(\theta, Q) \in \in (\Theta^0, \mathscr{P}^0)$. Modifying (ii)–(v) in this manner Lemma 2.2 still holds (cf. Lemmas 1.1, 1.3) and Lemma 2.3 can be replaced by an analogue of Lemma 3.3 in the next section.

Theorem 2.1. If $D''_{Q_\theta}(\theta)$ is positive definite on Θ^0 then influence curves Ω_{Q_θ} of T at $\mathscr{Q}_{\Theta^0}$ exist,

$$\Omega_{Q_\theta} = (\xi - \mathsf{E}_{Q_\theta}\xi) \circ D''_{Q_\theta}(\theta)^{-1}, \quad \text{where} \quad \xi = -\psi_{\theta/Q_\theta}, \tag{2.1}$$

and for every $P^* \in \mathscr{P}$

$$\lim_{\varepsilon \downarrow 0} \frac{T((1-\varepsilon)\,Q + \varepsilon P^*) - T(Q_\theta)}{\varepsilon} = \mathsf{E}_{P^*}\Omega_{Q\theta} \quad \text{on} \quad \Theta^\circ. \tag{2.2}$$

If moreover $q_\theta = \mathrm{d}Q_\theta/\mathrm{d}\lambda$ is differentiable on Θ^0 then

$$D''_{Q_\theta}(\theta) = -\mathsf{E}_{Q_\theta}\left(\frac{q'_\theta}{q_\theta}\right)^{\mathrm{T}} \circ (\xi - \mathsf{E}_{Q_\theta}\xi). \tag{2.3}$$

Proof. (I) Suppose $\lambda \equiv Q \in \mathscr{P}(T)$, $\mathbb{T}(Q) = \{T(Q)\} \subset \Theta^0$, $D_Q(\bar\theta) < D_Q(T(Q))$ for $\bar\theta \in \bar\Theta$, $\bar\theta \neq T(Q)$, and $D''_Q(T(Q))$ positive definite. Let $P^* \in \mathscr{P}$, $\theta \in \Theta$ be arbitrary fixed and define for all $\varepsilon \in [0, 1)$, $(\theta, P) \in \Theta \times \mathscr{P}$, $Q_\varepsilon = (1-\varepsilon)\,Q_\theta + \varepsilon P^*$, $g_{\theta/P} = f(p_\theta/p) - f'(p_\theta/p)\,p_\theta/p$. By the Lagrange mean value theorem it holds $|D_{Q_\varepsilon}(\theta) - D_Q(\theta)| \leqq \varepsilon\,\mathsf{E}_\lambda|g_{\theta/Q_\varepsilon^*}|$ where $\chi^1(Q, Q_\varepsilon^*) \leqq \chi^1(Q, Q_\varepsilon) \leqq \varepsilon$. Since for every $x \in \mathscr{X}\,|g_{\theta/P}(x)|$ is continuous on $\mathscr{P}^0$ (cf. Lemma 2.3) and $Q \in \mathscr{P}^0$, there exists an open neighborhood $U(Q) \subset \mathscr{P}^0$ of Q such that $|g_{\theta/P}|$ is bounded on $U(Q) \times \mathscr{X}$. Therefore $D_{Q_\varepsilon}(\theta) \to D_Q(\theta)$ as $\varepsilon \downarrow 0$ for all $\theta \in \Theta$.

(II) Since $T(Q) \in \Theta^0$, there exists i such that $T(Q)$ is an interior point of the compact subset $\Theta_i \subset \Theta^0$. Since $D_{Q_\varepsilon}(\theta)$ is continuous on Θ^0 (cf. Lemma 2.2), the set $\mathbb{T}_i(Q_\varepsilon)$ of parameters minimizing $D_{Q_\varepsilon}(\theta)$ on Θ_i is non-empty compact. Further, by (I), $\theta_\varepsilon \to T(Q)$ as $\varepsilon \downarrow 0$ for all $\theta_\varepsilon \in \mathbb{T}_i(Q_\varepsilon)$.

(III) Now we prove $\mathbb{T}_i(Q_\varepsilon) = \mathbb{T}(Q_\varepsilon)$ for all sufficiently small $\varepsilon > 0$. If the contrary holds, there exist sequences $\{\theta_j\} \subset \Theta$ and $\varepsilon(j) \downarrow 0$ such that $T(Q)$ is not a cluster point of $\{\theta_j\}$ and $D_{Q_{\varepsilon(j)}}(\theta_j) \leqq D_{Q_{\varepsilon(j)}}(T(Q))$. Therefore

$$\liminf_{j\to\infty} D_{Q_{\varepsilon(j)}}(\theta_j) \leqq \lim_{j\to\infty} D_{Q_{\varepsilon(j)}}(T(Q)) = D_Q(T(Q)).$$

On the other hand, by (i) there exists a cluster point $\bar{\theta} \in \bar{\Theta}$ of $\{\theta_j\}$ for which the last inequality implies $D_Q(\bar{\theta}) \leqq D_Q(T(Q))$, which contradicts the assumptions of (I) and the desired result holds. This result and (II) yield $Q_\varepsilon \in \mathscr{P}(T)$, $T(Q_\varepsilon) \in \Theta^0$ for all sufficiently small $\varepsilon > 0$ and $T(Q_\varepsilon) \to T(Q)$ as $\varepsilon \downarrow 0$.

(IV) For all sufficiently small $\varepsilon > 0$ the Lagrange mean value theorem yields $D'_Q(T(Q_\varepsilon)) - D'_{Q_\varepsilon}(T(Q_\varepsilon)) = \varepsilon \, \mathsf{E}_\lambda \psi_{T(Q_\varepsilon)/Q_\varepsilon^*}(q - p^*)$, where $\chi^1(Q, Q_\varepsilon^*) \leqq \chi^1(Q, Q_\varepsilon) \leqq \varepsilon$. This together with obvious identities $D'_{Q_\varepsilon}(T(Q_\varepsilon)) = D'_Q(T(Q)) = 0$ implies the identity

$$D'_Q(T(Q_\varepsilon)) - D'_Q(T(Q)) = \varepsilon \, \mathsf{E}_\lambda \psi_{T(Q_\varepsilon)/Q_\varepsilon^*}(q - p^*) \, .$$

The last inequality, Lemma 2.3, and (III) yield

$$\lim_{\varepsilon \downarrow 0} \mathsf{E}_\lambda \psi_{T(Q_\varepsilon)/Q_\varepsilon^*}(q - p^*) = \mathsf{E}_Q \psi_{T(Q)/Q} - \mathsf{E}_{P^*} \psi_{T(Q)/Q} \, . \tag{2.4}$$

On the other hand, applying Lemma 1.2 to the left side of the identity above we get

$$(T(Q_\varepsilon) - T(Q)) \circ D''_Q(\theta_\varepsilon^*) = \varepsilon \, \mathsf{E}_\lambda \psi_{T(Q_\varepsilon)/Q_\varepsilon^*}(q - p^*)$$

where (cf. Lemma 2.2 and (III) $D''_Q(\theta_\varepsilon^*) \to D''_Q(T(Q))$ as $\varepsilon \downarrow 0$. This together with (2.4) and the assumptions of (I) yields

$$\lim_{\varepsilon \downarrow 0} \frac{T(Q_\varepsilon) - T(Q)}{\varepsilon} = -(\mathsf{E}_{P^*} \psi_{T(Q)/Q} - \mathsf{E}_Q \psi_{T(Q)/Q}) \circ D''_Q(T(Q))^{-1} \, .$$

(V) The assumptions of (I) hold for every $Q = Q_\theta \in \mathscr{Q}_{\Theta^0}$, and $T(Q_\theta) = \theta$ on Θ^0 (cf. Lemma 2.1). Therefore (2.2) for Ω_{Q_θ} given by (2.1) follows from (IV). Further, (2.1) follows from (2.2) with $P^* = 1_{\{x\}}$, $x \in \mathscr{X}$. Applying the operator $(\mathrm{d}/\mathrm{d}\theta)^{\mathrm{T}}$ to the identity $D'_{Q_\theta}(T(Q_\theta)) = \mathsf{E}_\lambda \psi_{\theta/Q_\theta} = 0$ on Θ^0 we get

$$\mathsf{E}_\lambda \left[q_\theta \left(\frac{q'_\theta}{q_\theta} \right)^{\mathrm{T}} \circ \xi + \psi'_{\theta/Q_\theta} \right] = 0 \quad \text{on} \quad \Theta^0 \, .$$

From here and from the obvious identity $\mathsf{E}_\lambda q'_\theta = 0$ on Θ^0 it follows (2.3). □

Corollary 2.2. If the Fisher information

$$I(\theta \mid \mathscr{P}_\Theta) = \mathsf{E}_{P_\theta} \left(\frac{p'_\theta}{p_\theta} \right)^{\mathrm{T}} \circ \left(\frac{p'_\theta}{p_\theta} \right) \tag{2.5}$$

is positive definite on Θ^0 then influence curves of T at $\mathscr{P}_{\Theta^0}$ are given by $\Omega_{P_\theta} = -(p'_\theta/p_\theta) \circ I(\theta \mid \mathscr{P}_\Theta)^{-1}$.

Proof. By the definition of $\psi_{\theta/Q}$, it holds $\xi = -\psi_{\theta/P_\theta} = f''(1)\, p'_\theta/p_\theta$ in (2.1) with $Q_\theta = P_\theta$ and the rest follows from (2.3) and from the identity $\mathsf{E}_\lambda p'_\theta = 0$ on Θ^0. □

Lemma 2.4. If $Q \in \mathscr{P}$ and $\xi : \mathscr{X} \to \mathbb{R}^m$ then the r.v. $Y_n = n^{1/2}\, \mathsf{E}_\lambda \xi(p_n - q)$ defined on $(\mathscr{X}^n, Q^n)$ satisfies the relation $Y_n \xrightarrow{(Q)} No(0, \mathsf{D}_Q \xi)$, where $\mathsf{D}_Q \xi = \mathsf{E}_Q(\xi - \mathsf{E}_Q \xi)^{\mathrm{T}} \circ (\xi - \mathsf{E}_Q \xi)$.

Proof. By (1.1) in [9], $p_n = \mathrm{d}P_n/\mathrm{d}\lambda$ is a function of r.v. $X = (X_1, \ldots, X_n)$ with sample probability space $(\mathscr{X}^n, Q^n)$. The function is defined by $p_n(x) = n^{-1}$. $. (1_{\{X_1\}}(x) + \ldots + 1_{\{X_n\}}(x))$ for all $x \in \mathscr{X}$. Therefore $Y_n = n^{-1/2}(Z_1 + \ldots + Z_n)$ where $Z_i = \mathsf{E}_\lambda \xi (1_{\{X_i\}} - q)$ are i.i.d. with expectations

$$\mathsf{E}_{Q^n} Z_1 = \mathsf{E}_\lambda \xi \, \mathsf{E}_{Q^n}(1_{\{X_1\}} - q) = \mathsf{E}_\lambda \xi (q - q) = 0 ,$$

and variance matrices

$$\mathsf{E}_{Q^n}(Z_1 - \mathsf{E}_{Q^n} Z_1)^{\mathrm{T}} \circ (Z_1 - \mathsf{E}_{Q^n} Z_1) = \mathsf{E}_{Q^n}(\mathsf{E}_\lambda \xi \, 1_{\{X_1\}})^{\mathrm{T}} \circ \mathsf{E}_\lambda \xi \, 1_{\{X_1\}} -$$
$$- (\mathsf{E}_\lambda \xi q)^{\mathrm{T}} \circ \mathsf{E}_\lambda \xi q = \mathsf{E}_Q \xi^{\mathrm{T}} \circ \xi - \mathsf{E}_Q \xi)^{\mathrm{T}} \circ \mathsf{E}_Q \xi = \mathsf{D}_Q \xi .$$

The rest follows from the multidimensional central limit theorem. □

Theorem 2.2. If $D''_{Q_\theta}(\theta)$ is positive definite on Θ^0 then the rate of convergence of T to the parameter of $\mathscr{Q}_\Theta$ is $n^{-1/2}$ or, more precisely, (1.1) holds with $V_\theta(T) = \mathsf{E}_{Q_\theta} \Omega^{\mathrm{T}}_{Q_\theta} \circ \circ \Omega_{Q_\theta}$ where Ω_{Q_θ} is given by (2.1). Moreover, if $\mathscr{Q}_\Theta$ satisfies the condition of Theorem 2.1 and the Fisher information $I(\theta \mid \mathscr{Q}_\Theta)$ is positive definite on Θ^0, then $V_\theta(T) - - I(\theta \mid \mathscr{Q}_\Theta)^{-1}$ is positive definite unless there exist constants $c_i \in \mathbb{R}$ such that $\Omega_{Q_\theta} = = c_1(q'_\theta/q_\theta) + c_2$ on Θ^0 in which case $V_\theta(T) = I(\theta \mid \mathscr{Q}_\Theta)^{-1}$ on Θ^0.

Proof. Let $Q_\theta \in \mathscr{Q}_{\Theta 0}$ be arbitrary fixed. By Lemma 2.1

$$T(P_n) \xrightarrow{[Q_\theta]} T(Q_\theta) = \theta . \tag{2.6}$$

Since $Q = Q_\theta$ satisfies all assumptions of (I) in the proof of Theorem 2.1, it can be proved analogically as in part (IV) of that proof

$$(T(P_n) - \theta) \circ D''_Q(\theta^*_n) = \mathsf{E}_\lambda \psi_{T(P_n)/P_n^*}(q_\theta - p_n)$$

or

$$n^{1/2}(T(P_n) - \theta) = n^{1/2}(\mathsf{E}_\lambda \xi (q_\theta - p_n)) \circ D''_Q(\theta^*_n)^{-1} + n^{1/2} \tilde{X}_n \circ D''_{Q_\theta}(\theta^*_n)^{-1} \tag{2.7}$$

for all $x \in E_n(T)$ (cf. Corollary 2.1), where $\tilde{X}_n = \mathsf{E}_\lambda(\psi_{T(P_n)/P_n} - \psi_{\theta/Q_\theta}) . (q_\theta - p_n)$ and $\chi^1(Q_\theta, P^*_n) \leqq \chi^1((Q_\theta, P_n)$. By Corollary 2.1 and the Cramér-Slutskij theorem (Theorem 10, Chap. 10 of Anděl [1]) we can assume that (2.7) holds for all $x \in \mathscr{X}^n$. Moreover, by (2.6) and Lemmas 1.2, 2.2, $D''_{Q_\theta}(\theta^*_n) \xrightarrow{Q_\theta} D''_{Q_\theta}(\theta)$, i.e. $D''_{Q_\theta}(\theta^*_n)^{-1} \xrightarrow{Q_\theta} D''_{Q_\theta}(\theta)^{-1}$. By Lemma 2.1 in [10], $\chi^1(Q_\theta, P_n) \xrightarrow{[Q_\theta]} 0$ which together with (2.6) and Lemmas 2.2, 2.4 implies $n^{1/2} \tilde{X}_n \xrightarrow{Q_\theta} 0$. Therefore by (2.1), (2.7), Lemma 2.4, and the Cramér-Slutskij theorem [1], (1.1) with $V_\theta(T) = \mathsf{E}_{Q_\theta} \Omega^{\mathrm{T}}_{Q_\theta} \circ \Omega_{Q_\theta}$ holds. The rest obviously follows from this result and from (2.1) and (2.3). □

Corollary 2.3. If $I(\theta \mid \mathscr{P}_\Theta)$ is positive definite on Θ^0 then (1.1) holds with $\mathscr{Q}_\Theta = = \mathscr{P}_\Theta$, $V_\theta(T) = I(\theta \mid \mathscr{P}_\Theta)^{-1}$ on Θ^0, i.e. all standard D-estimators $T \triangleq \mathscr{P}_\Theta / D_f$ under consideration are *BAN* (best asymptotically normal) for $\mathscr{Q}_\Theta = \mathscr{P}_\Theta$.

Example 2.1. Let $\Theta = [0, 1] \subset \mathbb{R}$, $\mathscr{X} = \{0, 1, \ldots, k\}$, $k > 0$, and let $\mathscr{P}_{[0,1]}$ supported by $S = \{0, k\} \subset \mathscr{X}$ be composed of probabilities P_θ with densities

$$p_\theta(x) = \begin{cases} 1 - p(\theta) & \text{for } x = 0, \\ p(\theta) & \text{for } x = k, \end{cases} \quad p(\theta) = \frac{[\theta/(1-\theta)]^k}{1 + [\theta/(1-\theta)]^k} \quad \text{for} \quad \theta \in [0, 1].$$

Finally let $Q_\theta \in \mathscr{Q}_{[0,1]}$ be the binomial probability $Bi(\theta, k)$ and T the Hellinger-distance estimator $T \triangleq \mathscr{P}_{[0,1]}/D^{1/2}$ yield by a convex function $f(u) = (1 - u^{1/2})/\frac{1}{2}$ (cf. (2.4) in [9]).

Since $f''(u) = u^{3/2}/2$ and $D_Q(\theta) = 1 - [(1 - p(\theta))\, q(0)]^{1/2} - [p(\theta)\, q(k)]^{1/2}$ for all $\theta \in [0, 1]$, $Q \in \mathscr{P}$, all assumptions considered in this section hold with $\bar{\Theta} = \Theta = [0, 1]$. Obviously $\mathscr{P}(T) = \mathscr{P}$ and

$$T(Q) = \frac{[q(k)/q(0)]^{1/k}}{1 + [q(k)/q(0)]^{1/k}} \quad \text{for all} \quad N \in \mathscr{P}.$$

By (2.1) $\xi(x) = 0$ for $x \in \mathscr{X} - S$ and

$$\xi(x) = \begin{cases} -\dfrac{1}{2}\left[\dfrac{1 - p(\theta)}{(1-\theta)^k}\right]^{1/2} \dfrac{p(\theta)'}{1 - p(\theta)} & \text{for} \quad x = 0 \\ \dfrac{1}{2}\left[\dfrac{p(\theta)}{\theta^k}\right]^{1/2} \dfrac{p(\theta)'}{p(\theta)} & \text{for} \quad x = k. \end{cases}$$

This result together with (2.1) permits to evaluate the influence curves Ω_{Q_θ} of T at the binomial family $\mathscr{Q}_{(0,1)}$ and asymptotic variance $V_\theta(T)$ for all $\theta \in (0, 1)$. Asymptotic behaviour of T when θ takes on the extreme values 0 or 1 are clear as well since $T(P_n) = \theta$ a.s. $[Q_\theta]$ there.

3. EFFICIENCY AND ROBUSTNESS OF WEAK D-ESTIMATORS

This section is a continuation of Section 3 of [10]: we consider a sample space $(\mathscr{X}, \mathscr{B})$ with a sufficient class $\mathscr{E} = \{E_x : x \in \mathscr{X}\}$, well-defined weak D-estimators $T \triangleq \mathscr{P}_\Theta/\mathscr{W}_\Theta D_f$ with projection families $\mathscr{P}_\Theta \subset \mathscr{P}$ and families of weights $\mathscr{W}_\Theta$, and sample-generating families $\mathscr{Q}_\Theta \subset \mathscr{P}$. We also consider for all $Q \in \mathscr{P}$ the functions $D_Q(\theta) = \mathsf{E}_{W_\theta} \Phi(F_\theta, G)\, (F_\theta(x) = P_\theta(E_x), G(x) = Q(E_x), \ldots$ are d.f.'s of P_θ, $Q, \ldots$ (cf. Sec. 1 in [9])) and, for reason clarified in Remark 3.1 below, we write in this section $\Phi(u, v)$ instead of $d_f(u, v)$ (cf. Corollary 2.1 in [9]). In addition to what has been supposed in Section 1, we suppose the following:

(i) $\mathscr{W}_\Theta$ is absolutely continuous w.r.t. a σ-finite measure λ on $(\mathscr{X}, \mathscr{B})$ (Lebesgue measure if $\mathscr{X} = \mathbb{R}^k$), $w_\theta = \mathrm{d}\, W_\theta/\mathrm{d}\lambda$, and $W_\theta(\mathscr{X}) \in \mathbb{R}$ is constant on Θ.

(ii) The function Φ and its derivatives Φ'_u, Φ'_v, Φ''_{uu}, Φ''_{uv} are continuous on $[0, 1]^2$ and $f''(1) \neq 0$.

(iii) The functions $w_\theta F'_\theta$, w'_θ, $w_\theta(F'_\theta)^{\mathrm{T}} \circ F'_\theta$, $(w'_\theta)^{\mathrm{T}} \circ F'_\theta$, $w_\theta F''_\theta$, and w''_θ are λ-regular on $\mathscr{X} \times \Theta^0$, where

$$F'_\theta = \frac{\mathrm{d}}{\mathrm{d}\theta} F_\theta\,, \quad w'_\theta = \frac{\mathrm{d}}{\mathrm{d}\theta} w_\theta\,, \quad F''_\theta = \left(\frac{\mathrm{d}}{\mathrm{d}\theta}\right)^{\mathrm{T}} \circ F'_\theta\,, \quad w''_\theta = \left(\frac{\mathrm{d}}{\mathrm{d}\theta}\right)^{\mathrm{T}} \circ w'_\theta\,.$$

(iv) If $\mathscr{Q}_\Theta \neq \mathscr{P}_\Theta$, then the weak divergence $\mathsf{E}_W \Phi(F, G)$ is a metric on $\mathscr{P}$.

(v) $\mathscr{Q}_\Theta$ is WD_f-compatible with $\mathscr{P}_\Theta$ (cf. Sec. 3 of [10]).

Lemma 3.1. T is strongly consistent as well as Fisher consistent for $\mathscr{Q}_\Theta$.

Proof (ii) implies $\|f\| < \infty$ so that, by part (a) of Lemma 3.1 in [10], $\mathscr{Q}_\Theta$ is strongly WD_f-regular. The rest follows from Theorems 3.1, 3.2 in [10]. □

Corollary 3.1. For all $\theta \in \Theta^0$ it holds $\lim_{n\to\infty} Q_\theta(E_n(T)) = 1$ where $E_n(T) = \{x \in \mathscr{X}^n : T(P_n) \notin \Theta^0\}$.

Define for each $\theta \in \Theta^0$, $Q \in \mathscr{P}$ the following functions on $\mathscr{X}$

$$\psi_{\theta/G} = \frac{\mathrm{d}}{\mathrm{d}\theta} (\Phi(F_0, G)\, w_\theta) = w_\theta\, \Phi'_u(F_\theta, G)\, F'_\theta + \Phi(E_\theta, G)\, w'_\theta\,, \tag{3.1}$$

$$\psi'_{\theta/G} = \left(\frac{\mathrm{d}}{\mathrm{d}\theta}\right)^{\mathrm{T}} \circ \psi_{\theta/G} = w_\theta\, \Phi''_{uu}(F_\theta, G)\, (F'_\theta)^{\mathrm{T}} \circ F'_\theta + \tag{3.2}$$

$$+ \Phi'_u(F_\theta, G) \left[(w'_\theta)^{\mathrm{T}} \circ F'_\theta + (F'_\theta)^{\mathrm{T}} \circ w'_\theta\right] + w_\theta\, \Phi'_u(F_\theta, G)\, F''_\theta + \Phi(F_\theta, G)\, w''_\theta\,,$$

$$\dot\psi_{\theta/G} = \frac{\mathrm{d}}{\mathrm{d}G} \psi_{\theta/G} = w_\theta\, \Phi''_{uv}(F_\theta, G)\, F'_\theta + \Phi'_v(F_\theta, G)\, w'_\theta\,. \tag{3.3}$$

Lemma 3.2. For any $Q \in \mathscr{P}$, $D_Q(\theta)$ is twice differentiable,

$$D'_Q(\theta) = \frac{\mathrm{d}}{\mathrm{d}\theta} D_Q(\theta) = \mathsf{E}_\lambda \psi_{\theta/G}, \quad D''_Q(\theta) = \left(\frac{\mathrm{d}}{\mathrm{d}\theta}\right)^{\mathrm{T}} \circ D'_Q(\theta) = \mathsf{E}_\lambda\, \psi'_{\theta/G} \quad \text{on} \quad \Theta^0$$

and $D_Q(\theta)$, $D'_Q(\theta)$, $D''_Q(\theta)$ are continuous on Θ^0.

Proof. By (ii), all functions cited in (ii) and (3.1)–(3.3) are bounded on the compact $[0, 1]^2$. Hence, by the continuity in (ii), (iii), $\psi_{\theta/G}$, $\psi'_{\theta/G}$ are λ-regular on $\mathscr{X} \times \Theta^0$ for every $Q \in \mathscr{P}$. Applying Lemmas 1.1, 1.3 to the latter two functions we get the desired results. □

Lemma 3.3. $\mathsf{E}_\lambda \dot\psi_{\theta/G}$ is continuous on a topological space $\Theta^0 \times \mathscr{P}$ with the product topology $\|\cdot\|_m \times KS$ (where KS denotes the Kolmogorov-Smirnov distance) in the sense $\mathsf{E}_\lambda \|\dot\psi_{\tilde\theta/\tilde G} - \dot\psi_{\theta/G}\|_m \to 0$ as $(\tilde\theta, \tilde Q) \to (\theta, Q) \in \Theta^0 \times \mathscr{P}$.

Proof. Analogically as in the preceding proof we get from (3.3), (ii) and (iii) that $g(x, (\theta, Q)) = \dot\psi_{\theta/G}(x)$ is λ-regular on $\mathscr{X} \times (\Theta^0 \times \mathscr{P})$ so that the desired result follows from Lemma 1.1. □

Theorem 3.1. If $D''_{Q_\theta}(k)$ is positive definite on Θ^0 then influence curves Ω_{Q_θ} of T at $\mathcal{Q}_{\Theta^0}$ exist,

(3.4) $$\Omega_{Q_\theta} = (\tilde{\xi} - \mathsf{E}_\lambda \xi G_\theta) \circ D''_{Q_\theta}(\theta)^{-1}$$

where

(3.5) $\xi = -\dot{\psi}_{\theta/G_\theta}$, $\tilde{\xi}(x) = \mathsf{E}_\lambda \xi \, 1_{E(x)}$, $E(x) = \{\tilde{x} \in \mathcal{X} : x \in E_{\tilde{x}}\}$ on $\mathcal{X}$,

and for every $P^* \in \mathcal{P}$

(3.6)
$$\lim_{\varepsilon \downarrow 0} \frac{T((1-\varepsilon)\, Q_\theta + \varepsilon P^*) - T(Q_\theta)}{\varepsilon} = \mathsf{E}_{P^*} \Omega_{Q_\theta} = \mathsf{E}_\lambda \xi (G_\theta - F^*) \circ D''_{Q_\theta}(\theta)^{-1} \quad \text{on } \Theta^0 .$$

If moreover $w_\theta(G'_\theta)^{\mathrm{T}} \circ F'_\theta$, $(G'_\theta)^{\mathrm{T}} \circ w'_\theta$ are λ-regular on $\mathcal{X} \times \Theta^0$, where G'_θ are defined analogically as F'_θ in (iii), then

(3.7) $$D''_{Q_\theta}(\theta) = \mathsf{E}_\lambda (G'_\theta)^{\mathrm{T}} \circ \xi .$$

Proof. (I) Suppose $Q \in \mathcal{P}(T)$, $\mathbb{T}(Q) = \{T(Q)\} \subset \Theta^0$, $D''_Q(T(Q))$ positive definite, and $\inf_{\Theta - U(T(Q))} D_Q(\theta) < D_Q(T(Q))$ for all open neighborhoods $U(T(Q))$ of $T(Q)$. Let $P^* \in \mathcal{P}$, $\theta \in \Theta^0$ be arbitrary fixed and define $Q_\varepsilon = (1-\varepsilon)\, Q + \varepsilon P^*$ for $\varepsilon \in [0, 1)$. Since $T(Q) \in \Theta^0$ there exists i such that $T(Q)$ is an interior point of the compact subset $\Theta_i \subset \Theta^0$. Since $D_{Q_\varepsilon}(\theta)$ is continuous on Θ^0 (cf. Lemma 2.2), the set $\mathbb{T}_i(Q_\varepsilon)$ of parameters minimizing $D_{Q_\varepsilon}(\theta)$ on Θ_i is non-empty compact for all $\varepsilon \in [0, 1)$.

(II) By the Lagrange mean value theorem

$$|D_{Q_\varepsilon}(\theta) - D_Q(\theta)| = |\mathsf{E}_\lambda(\Phi'_v(F_\theta, G^*)\, w_\theta \varepsilon (F^* - G))| \leqq \varepsilon K W_\theta(\mathcal{X})$$

where $KS(G, G^*) \leqq KS(F^*, G) \leqq \varepsilon$ and $K = \max_{[0,1]^2} |\Phi'_v|$ is finite by (i), (ii). Hence, by (I), $\theta_\varepsilon \to T(Q)$ as $\varepsilon \downarrow 0$ for all $\theta_\varepsilon \in \mathbb{T}_i(Q_\varepsilon)$.

(III) Now we prove $\mathbb{T}_i(Q_\varepsilon) = \mathbb{T}(Q_\varepsilon)$ for all sufficiently small $\varepsilon > 0$. If the contrary holds, there exist $U(T(Q)) \subset \Theta_i$ and sequences $\theta_j \notin \Theta_i$, $\varepsilon(j) \downarrow 0$ such that $D_{Q_{\varepsilon(j)}}(\theta_j) \leqq$ $\leqq D_{Q_{\varepsilon(j)}}(T(Q))$. In view of the inequality (II) this implies

$$\inf_{\theta \notin U(T(Q))} D_Q(\theta) \leqq \liminf_{j \to \infty} D_Q(\theta_j) \leqq \lim_{j \to \infty} D_{Q_{\varepsilon(j)}}(T(Q)) = D_Q(T(Q))$$

which contradicts the assumptions of (I). Thus we have proved $Q_\varepsilon \in \mathcal{P}(T)$, $T(Q_\varepsilon) \in \Theta^0$ for all sufficiently small $\varepsilon > 0$ and $T(Q_\varepsilon) \to T(Q)$ as $\varepsilon \downarrow 0$.

(IV) For all sufficiently small $\varepsilon > 0$ the Lagrange mean value theorem yields $D'_Q(T(Q_\varepsilon)) - D'_{Q_\varepsilon}(T(Q_\varepsilon)) = \mathsf{E}_\lambda \dot{\psi}_{T(Q_\varepsilon)/G_\varepsilon^*}(G - G_\varepsilon)$, where $KS(G, G_\varepsilon^*) \leqq KS(G, G_\varepsilon) \leqq$ $\leqq \varepsilon$. This together with obvious identities $D'_{Q_\varepsilon}(T(Q_\varepsilon)) = D'_Q(T(Q)) = 0$ implies the identity

$$D'_Q(T(Q_\varepsilon)) - D'_Q(T(Q)) = \varepsilon\, \mathsf{E}_\lambda \dot{\psi}_{T(Q_\varepsilon)/G_\varepsilon^*}(G - F^*) .$$

The last inequality, Lemma 3.3, and (III) yield

$$0 \leqq \lim_{\varepsilon \downarrow 0} \|\mathsf{E}_\lambda\, \dot{\psi}_{T(Q_\varepsilon)/G_\varepsilon^*}(G - F^*) - \mathsf{E}_\lambda\, \dot{\psi}_{T(Q)/G}(G - F^*)\|_m \leqq$$

$$\leq \lim_{\varepsilon \downarrow 0} \mathsf{E}_\lambda \|\psi_{T(Q_\varepsilon)/G_\varepsilon^*} - \psi_{T(Q)/G}\|_m = 0$$

i.e.

$$\lim_{\varepsilon \downarrow 0} \mathsf{E}_\lambda \dot{\psi}_{T(Q_\varepsilon)/G_\varepsilon^*}(G - F^*) = \mathsf{E}_\lambda \dot{\psi}_{T(Q)/G}(G - F^*) . \tag{3.8}$$

On the other hand, applying Lemma 1.2 to the left side of the last identity, we get

$$(T(Q_\varepsilon) - T(Q)) \circ D''_Q(\theta^*_\varepsilon) = \varepsilon \mathsf{E}_\lambda \dot{\psi}_{T(Q_\varepsilon)/G_\varepsilon^*}(G - F^*)$$

where (cf. Lemma 3.2 and (III)) $D''_Q(\theta^*_\varepsilon) \to D''_Q(T(Q))$ as $\varepsilon \downarrow 0$. These results together with (3.8) yield

$$\lim_{\varepsilon \downarrow 0} \frac{T(Q_\varepsilon) - T(Q)}{\varepsilon} = \mathsf{E}_\lambda \dot{\psi}_{T(Q^\cdot)/G}(G - F^*) \circ D''_Q(T(Q))^{-1} .$$

(V) The assumptions of (I) hold for every $Q \in \mathcal{Q}_{\Theta^0}$ and $T(Q_\theta) = \theta$ on Θ^0 (cf. Lemma 3.1). Therefore (3.6) for Ω_{Q_0} given by (3.1) holds. Further, (3.4) follows from (3.6) with $P^* = 1_{\{x\}}, x \in \mathcal{X}$, if we take into account the obvious identity $F^*(x) = 1_{E(x)}$ valid for these P^*. The expression (3.7) follows from the relation

$$\mathsf{E}_\lambda \left(\frac{\mathrm{d}}{\mathrm{d}\theta}\right)^{\mathrm{T}} \circ \psi_{\theta/G_\theta} = 0 \quad \text{on} \quad \Theta^0 .$$

This relation follows from the identity $D'_{Q_\theta}(\theta) = \mathsf{E}_\lambda \dot{\psi}_{\theta/G_\theta} = 0$ on Θ^0 and from the fact that, for $\mathcal{Q}_{\Theta^0}$ satisfying assumptions of Theorem 3.1, the expectation E_λ and the differential operator $(\mathrm{d}/\mathrm{d}\theta)^{\mathrm{T}}$ are exchangeable in this identity (compare Lemma 3.2). □

If $\mathcal{X} = \mathbb{R}^k$ then we denote by $(-\infty, x)$ the product of semi-bounded intervals upper-bounded by the respective coordinates of $x \in \mathbb{R}^k$.

Theorem 3.2. If $D''_{Q_\theta}(\theta)$ is positive definite on Θ^0, $\mathcal{X} = \mathbb{R}$, and $\mathcal{Q}_\Theta \ll \lambda$, then functions $\bar{\xi}$ primitive to ξ (cf. (3.5)) in the sense $\bar{\xi}(x) = \mathsf{E}_\lambda \xi \, 1_{(-\infty, x)}$ on $\mathbb{R}$ exist and the influence curves (3.4) are given by

$$\Omega_{Q_\theta} = -(\bar{\xi} - \mathsf{E}_{Q_\theta} \bar{\xi}) \circ D''_{Q_\theta}(\theta)^{-1} . \tag{3.9}$$

Proof. The sets $E(x), x \in \mathbb{R}$, defined in (3.5) are equal to $(-\infty, x)$. By Lemma 3.3, ξ is absolutely integrable in the sense $\mathsf{E}_\lambda \|\xi\|_m < \infty$ on Θ^0 and $\bar{\xi}(-\infty) = 0, \bar{\xi}(\infty) \in \mathbb{R}^m$. This implies for all $P \ll \lambda$ or for all $P \in \mathcal{P}_e$

$$\mathsf{E}_\lambda \xi F = \bar{\xi}(\infty) - \mathsf{E}_P \bar{\xi} . \tag{3.10}$$

Applying this result to (3.4) with $F = G_\theta$ and taking into account that $\bar{\xi}(x) = \mathsf{E}_\lambda \xi - \mathsf{E}_\lambda \xi \, 1_{(-\infty, x)} = \bar{\xi}(\infty) - \bar{\xi}(x)$ on $\mathbb{R}^m$ we get (3.9). □

For $\mathcal{Q}_\Theta = \mathcal{P}_\Theta$ it follows from (3.2), (3.5) that $\xi = f''(1)\, w_\theta F'_\theta / [F'_\theta(1 - F_\theta)]$ and that $\mathcal{Q}_\Theta$ satisfies the conditions of Theorem 3.1. Thus the following corollary holds.

Corollary 3.1. Let $\tilde{\mathscr{W}}_\Theta \ll \lambda$ be arbitrary such that $\mathscr{W}_\Theta \triangleq \tilde{\varphi}\tilde{\mathscr{W}}_\Theta$ for $\tilde{\varphi}(u, v) = u(1 - u)$, let $\mathscr{Q}_\Theta = \mathscr{P}_\Theta$ satisfy (i)–(v), and let $\mathsf{E}_\lambda(F'_\theta)^{\mathrm{T}} \circ \xi$ be positive definite on Θ^0. Then influence curves of $T \triangleq \mathscr{P}_\Theta|\tilde{\varphi}\tilde{\mathscr{W}}_\Theta D_f$ at $\mathscr{P}_{\Theta^0}$ are given by (3.9) with $\xi = \tilde{w}_\theta F'_\theta$ and $D''_{Q_0}(\theta) = \mathsf{E}_\lambda(F'_\theta)^{\mathrm{T}} \circ \xi$.

Example 3.1. Let $\Theta = \mathbb{R}$ be the location parameter space and P, W be arbitrary such that the generated families* $\mathscr{P}_R$, $\tilde{\mathscr{W}}_R$ satisfy assumptions of Corollary 3.1. If $\mathsf{E}_\lambda p\tilde{w} \in (0, \infty)$, $p^0 = p\tilde{w}/\mathsf{E}_\lambda p\tilde{w}$, F^0 is the d.f. of p^0, and $\mathsf{E}_P p^0 > 0$, then influence curves of $T \triangleq P/\tilde{\varphi}\tilde{\mathscr{W}}D_f$ are given by (1.3) with $\Theta = \mathbb{R}$, $Q = P$, and it holds

$$\Omega_P = \frac{F^0 - \mathsf{E}_P F^0}{\mathsf{E}_P p^0} = -\frac{F^0 - \mathsf{E}_P F^0}{\mathsf{E}_P(F^0 - \mathsf{E}_P F^0)\,\dfrac{p'}{p}}$$

This statement follows from Corollary 3.1, from the fact that $F'_\theta = -p([\theta]^{-1})$ here, and the identity $\mathsf{E}_\lambda p' F^0 = -\mathsf{E}_\lambda p p^0$ (cf. (3.10). Notice that if $\mathsf{E}_\lambda p\tilde{w} = 0$ or ∞ then the influence curves are zero or unbounded respectively. Further, for any $\tilde{W}, f$ under consideration these curves are monotone on $\mathbb{R}$ (thus never "redescending") and symmetric about 0 provided $p\tilde{w}$ is symmetric (in this case $\mathsf{E}_P F^0 = \frac{1}{2}$).

Remark 3.1. All results of this section apply to estimators $T \triangleq \mathscr{P}_\Theta/\varphi_1\varphi_2\tilde{\mathscr{W}}_\Theta D_f$ with $\Phi = d_f\varphi_1$ satisfying (ii), (iv) and with $\varphi_2(u, v) = u(1 - u)$. This conclusion has already been used in formulations of Corollary 3.1 and Example 3.1 (with $\varphi_1 = 1$, $\varphi_2 = \tilde{\varphi}$). Another examples are provided by the function $f(u) = (1 - u)^2$ yielding the weak χ^2-divergence. Here $d_f(u, v) = (u - v)^2/v(1 - v)$ is not satisfying (ii) and (iv) while $\Phi(u, v) = d_f(u, v)\, v(1 - v)$ is. The results of Example 3.1 for the particular weak χ^2-estimators of location $T \triangleq P/\varphi_1\varphi_2\tilde{W}D_f$, $\varphi_1(u, v) = v(1 - v)$, have formerly been obtained by Boos [3].

In the rest of this section we assume, in addition to (i)–(v), the following condition (vi): $\mathscr{X} = \mathbb{R}^k$ and $\mathscr{Q}_\Theta \ll \lambda$.

Lemma 3.4. If $Q \in \mathscr{P}$, $Q \ll \lambda$, and $\xi : \mathbb{R}^k \to \mathbb{R}^m$ is absolutely Q-integrable with finite variance matrix $\mathsf{D}_Q\hat{\xi} = \mathsf{E}_Q(\hat{\xi} - \mathsf{E}_Q\hat{\xi})^{\mathrm{T}} \circ (\hat{\xi} - \mathsf{E}_Q\hat{\xi})$ of the respective primitive function $\hat{\xi}(x) = \mathsf{E}_\lambda \xi\, 1_{(-\infty, x)}$, $x \in R^k$, then the r.v. $Y_n = n^{1/2}\mathsf{E}_\lambda\xi(G - F_n)$ defined on $(\mathbb{R}^n, \mathscr{B}^n, Q^n)$ satisfies the relation $Y_n \xrightarrow{(Q)} No(0, \mathsf{D}_Q\hat{\xi})$.

Proof. By (3.10) it holds for each $\boldsymbol{x} \in (\mathbb{R}^k)^n$

$$Y_n = n^{1/2}(\mathsf{E}_{P_n}\hat{\xi} - \mathsf{E}_Q\hat{\xi}) = n^{-1/2}\sum_{i=1}^{n}(\hat{\xi}(x_i) - \mathsf{E}_Q\hat{\xi}) \quad \text{(cf. (1.1) in [9])}$$

and the rest is clear. □

* Hereafter, from typographical reasons, we use R instead of $\mathbb{R}$ in subscripts.

Theorem 3.3. If $D''_{Q_\theta}(\theta)$ is positive definite and $\mathsf{E}_{Q_\theta}\Omega^{\mathrm{T}}_{Q_\theta} \circ \Omega_{Q_\theta}$ finite on Θ^0 then the rate convergence of T to the parameter of $\mathcal{Q}_\Theta$ is $n^{-1/2}$ or, more precisely, (1.1) holds with $V_\theta(T) = \mathsf{E}_{Q_\theta}\Omega^{\mathrm{T}}_{Q_\theta} \circ \Omega_{Q_\theta}$.

Proof. Let $Q_\theta \in \mathcal{Q}_{\Theta^0}$ be arbitrary fixed. Analogically as in (2.7), we can assume for all $x \in (\mathbb{R}^k)^n$ the identity

$$n^{1/2}(T(P_n) - \theta) = n^{1/2}\,\mathsf{E}_\lambda \xi(G - F_n) \circ D''_Q(\theta^*_n)^{-1} + n^{1/2}\tilde{X}_n \circ D''_Q(\theta^*_n)^{-1}\,,$$

where

$$n^{1/2}\|\tilde{X}_n\|_m = n^{1/2}\|\mathsf{E}_\lambda \dot{\psi}_{T(P_n)/F_n*}(G_\theta - F_n) - \mathsf{E}_\lambda \dot{\psi}_{\theta/G_\theta}(G_\theta - F_n)\|_m \leqq$$
$$\leqq \mathsf{E}_\lambda \|\dot{\psi}_{T(P_n)/F_n*} - \dot{\psi}_{\theta/G_\theta}\|_m\, n^{1/2} KS(G_\theta, F_n)\,,$$

ξ is given by (3.5), $KS(F^*_n, G_\theta) \leqq KS(F_n, G_\theta) \xrightarrow{[Q_\theta]} 0$ (cf. Glivenko theorem), $T(P_n) \xrightarrow{[Q_\theta]} T(Q_\theta) = \theta$ cf. Lemma 3.1), and $D''_{Q_\theta}(\theta^*_n) \xrightarrow{[Q_\theta]} D''_{Q_\theta}(\theta)$ (cf. Lemmas 3.1, 3.2, and $T(P_n) \xrightarrow{[Q_\theta]} \theta$). By the Glivenko theorem, the above stated inequality, and Lemma 3.3, it holds $n^{1/2}\tilde{X}_n \xrightarrow{Q_\theta} 0$. Therefore the identity above, together with the Gramér-Slutskij theorem and Lemma 3.4, imply (1.1) with

$$V_\theta(T) = \mathsf{E}_{Q_\theta}[(\xi - \mathsf{E}_{Q_\theta}\xi) \circ D''_{Q_\theta}(\theta)^{-1}]^{\mathrm{T}} \circ [(\xi - \mathsf{E}_{Q_\theta}\xi) \circ D''_{Q_\theta}(\theta)^{-1}]\,.$$

The rest follows from (3.9). □

Example 3.2. If T is as in Example 3.1 and $\mathsf{E}_P\Omega^2_P < \infty$ then the asymptotic variances $V_\theta(T)$ of T at $P_\theta \in \mathcal{P}_R$ are equal to $\mathsf{E}_P\Omega^2_P$ and $\mathsf{E}_P\Omega^2_P \geqq 1/I(P) = 1/\mathsf{E}_P(p'/p)^2$ with equality iff there exist $c_i \in \mathbb{R}$ such that $F^0 = c_1 p'/p + c_2$.

4. EFFICIENCY AND ROBUSTNESS OF DIRECTED D-ESTIMATORS

This section is a continuation of Section 4 in [10]: we consider well-defined directed D^α-estimators $T^\alpha \triangleq \mathcal{P}_\Theta // W$, $\alpha \in (0, 1]$, with projection families $\mathcal{P}_\Theta \ll W$ and sample-generating families $\mathcal{Q}_\Theta$ on a sample space $(\mathcal{X}, \mathcal{B})$. We also consider the functions $D_Q(\theta) = \mathsf{E}_Q p^\alpha_\theta$ on Θ for all $Q \in \mathcal{P}$ where $p_\theta = \mathrm{d}P_\theta/\mathrm{d}W$. In addition to what has been supposed in Section 1, we suppose the following:

(i) $\mathcal{X}$ is a pseudo-metric space and the functions p^α_θ,

$$\psi_\theta = \frac{\mathrm{d}}{\mathrm{d}\theta}\, p^\alpha_\theta = \alpha p^{\alpha-1}_\theta p'_\theta\,, \quad \psi'_\theta = \left(\frac{\mathrm{d}}{\mathrm{d}\theta}\right)^{\mathrm{T}} \circ \psi_\theta = \alpha(\alpha - 1)\, p^{\alpha-2}_\theta (p'_\theta)^{\mathrm{T}} \circ p'_\theta + \alpha p^{\alpha-1}_\theta p''_\theta$$

are bounded and continuous on $\mathcal{X}$ uniformly for all $\theta \in \Theta^0$ and continuous on Θ for all $x \in \mathcal{X}$.

(ii) T^α is consistent for $\mathcal{Q}_\Theta$.

(iii) $\mathcal{Q}_\Theta$ is α-compatible with $\mathcal{P}_\Theta$ (cf. Sec. 4 and the footnote in Sec. 5 of [10]).

Lemma 4.1. For every $Q \in \mathscr{P}$, $D_Q(\theta)$ is twice differentiable,

$$D'_Q(\theta) = \frac{\mathrm{d}}{\mathrm{d}\theta} D_Q(\theta) = \mathsf{E}\theta\psi_\theta\,, \quad D''_Q(\theta) = \left(\frac{\mathrm{d}}{\mathrm{d}\theta}\right)^{\mathrm{T}} \circ D'_Q(\theta) = \mathsf{E}_Q\psi'_\theta \quad \text{on} \quad \Theta^0$$

and $D_Q(\theta)$, $D'_Q(\theta)$, $D''_Q(\theta)$ are continuous on Θ^0.

Proof. By (i), p^α_θ, ψ_θ, ψ'_θ are Q-regular for all $Q \in \mathscr{P}$. The rest is clear from Lemmas 1.1, 1.3. □

Theorem 4.1. If $D''_{Q_0}(\theta)$ is negative definite on Θ^0 then influence curves Ω_{Q_θ} of T^α at $\mathscr{Q}_\Theta$ exist,

$$\Omega_{Q_\theta} = -(\psi_\theta - \mathsf{E}_Q\psi_\theta) \circ D''_{Q_\theta}(1)^{-1}\,. \tag{4.1}$$

and for every $P^* \in \mathscr{P}$

$$\lim_{\varepsilon\downarrow 0} \frac{T^\alpha((1-\varepsilon)\,Q_\theta + \varepsilon P^*) - T^\alpha(Q_\theta)}{\varepsilon} = \mathsf{E}_{P^*}\Omega_{Q_\theta} \quad \text{on} \quad \Theta^0\,. \tag{4.2}$$

If moreover $\mathscr{Q}_\Theta \ll W$, $q_\theta = \mathrm{d}Q_\theta/\mathrm{d}W$ is differentiable, $\mathsf{E}_W q'_\theta = 0$, and $\psi'_\theta q_\theta + (q'_\theta)^{\mathrm{T}} \circ \psi_\theta$ is W-regular on $\mathscr{X} \times \Theta^0$, then

$$D''_{Q_\theta}(\theta) = \mathsf{E}_{Q_\theta}\left(\frac{q'_\theta}{q_\theta}\right)^{\mathrm{T}} \circ (\psi_\theta - \mathsf{E}_{Q_\theta}\psi_\theta)\,. \tag{4.3}$$

Proof. (I) Suppose $Q \in \mathscr{P}(T^\alpha)$, $\mathbb{T}^\alpha(Q) = \{T^\alpha(Q)\} \subset \Theta^0$, $D''_Q(T^\alpha(Q))$ negative definite, and

$$\sup_{\theta\in\Theta - U(T^\alpha(Q))} D_Q(\theta) < D_Q(T^\alpha(Q))$$

for all open neighborhoods $U(T^\alpha(Q))$ of $T^\alpha(Q)$. Let $P^* \in \mathscr{P}$, $\theta \in \Theta^0$ be arbitrary fixed, and define $Q_\varepsilon = (1-\varepsilon)\,Q + \varepsilon P^*$ for $\varepsilon \in [0, 1)$. It holds

$$|D_{Q_\varepsilon}(\theta) - D_Q(\theta)| = \varepsilon|D_{P^*}(\theta) - D_Q(\theta)| \leqq \varepsilon K \chi^1(P^*, Q)$$

where $K = \sup_{\mathscr{X}} p^\alpha_\theta < \infty$ (cf. (iii)). Hence $D_{Q_\varepsilon}(\theta) \to D_Q(\theta)$ as $\varepsilon \downarrow 0$ where $D_{Q_\varepsilon}(\theta)$ is continuous on Θ^0 (cf. Lemma 4.1). Therefore, analogically as in the proofs of Theorems 2.1, 3.1, we can employ (iii) to prove by contradiction that $Q_\varepsilon \in \mathscr{P}(T^\alpha)$, $T^\alpha(Q_\varepsilon) \in \Theta^0$ for all sufficiently small $\varepsilon > 0$ and $T^\alpha(Q_\varepsilon) \to T^\alpha(Q)$ as $\varepsilon \downarrow 0$.

(II) For all sufficiently small $\varepsilon > 0$ the Lagrange mean value theorem yields $D'_{Q_\varepsilon}(T(Q)) - D'_Q(T(Q)) = \varepsilon(D'_{P^*}(T(Q)) - D'_Q(T(Q))$. This and the obvious identities $D'_{Q_\varepsilon}(T(Q_\varepsilon)) = D'_Q(T(Q)) =$ imply

$$D'_{Q_\varepsilon}(T(Q)) - D'_{Q_\varepsilon}(T(Q_\varepsilon)) = \varepsilon\, D'_{P^*}(T(Q))\,.$$

Thus, by Lemmas 1.2, 4.1,

$$(T(Q_\varepsilon) - T(Q)) \circ D''_{Q_\varepsilon}(\theta^*_\varepsilon) = -\varepsilon\, D'_{P^*}(T(Q))$$

where $\|T(Q) - \theta^*_\varepsilon\|_m \leqq \|T(Q) - T(Q_\varepsilon)\|_m$. On the other hand

$$\|D''_{Q_\varepsilon}(\theta^*_\varepsilon) - D''_Q(T(Q))\|_{m^2} \leqq \|D''_{Q_\varepsilon}(\theta^*_\varepsilon) - D''_Q(\theta^*_\varepsilon)\|_{m^2} + \|D''_Q(\theta^*_\varepsilon) - D''_Q(T(Q))\|_{m^2}$$

where $\|D''_{Q_\varepsilon}(\theta^*_\varepsilon) - D''_Q(\theta^*_\varepsilon)\|_{m^2} \leqq K\chi^1(Q_\varepsilon, Q)$ for $K = \sup_{\mathscr{X}} \|\psi'_\theta\|_{m^2} < \infty$. Since it holds $\|T(Q) - \theta^*_\varepsilon\|_m \to 0$ as $\varepsilon \downarrow 0$ (cf. (I) and an inequality above) and since $D''_Q(\theta)$ is continuous on Θ^0 (cf. Lemma 4.1), it holds

$$\lim_{\varepsilon \downarrow 0} \frac{T(Q_\varepsilon) - T(Q)}{\varepsilon} = -D'_{P^*}(T(Q)) \circ D''_Q(T(Q))^{-1} .$$

(III) As all $Q \in \mathscr{Q}_{\Theta^0}$ satisfy the assumptions of (I), (4.2) with Ω_{Q_θ} given by (4.1) follows from (II) and (4.1) follows from (1.2) and (4.2) with $P^* = 1_{\{x\}}$, $x \in \mathscr{X}$. Finally applying the operator $(\mathrm{d}/\mathrm{d}\theta)^\mathrm{T}$ to the identity $D'_{Q_\theta}(T(Q_\theta)) = \mathsf{E}_W q_\theta \psi_\theta = 0$ on Θ^0 and interchanging the operators $(\mathrm{d}/\mathrm{d}\theta)^\mathrm{T}$ and E_W (cf. assumptions of Theorem 4.1 and Lemma 1.3) we get the identity $\mathsf{E}_W(\psi'_\theta q_\theta + (q'_\theta)^\mathrm{T} \circ \psi_\theta) = 0$ on Θ^0. This identity together with the assumption $\mathsf{E}_{Q\theta}(q'_\theta/q_\theta) = 0$ and Lemma 4.1 yields (4.3). □

Example 4.1. If a location estimator $T^\alpha \triangleq P//\lambda$ and a parent Q of a sample generating location family $\mathscr{Q}_R$ satisfy the assumptions of Theorem 4.1, then influence curves of T^α at $\mathscr{Q}_R$ are given by (1.3) with

$$\Omega_Q = -\frac{p^{\alpha-1}p' - \mathsf{E}_Q p^{\alpha-1}p'}{\mathsf{E}_Q(p^{\alpha-1}p' - \mathsf{E}_Q p^{\alpha-1}p')\dfrac{q'}{q}} = -\frac{p^\alpha \dfrac{p'}{p} - \mathsf{E}_Q p^\alpha \dfrac{p'}{p}}{\mathsf{E}_Q p^\alpha \dfrac{p'q'}{pq}} .$$

If Q is not satisfying the assumptions of Theorem 4.1 then the last expression should be replaced by

$$\Omega_Q = -\frac{p^\alpha \dfrac{p'}{p} - \mathsf{E}_Q p^\alpha \dfrac{p'}{p}}{(1-\alpha)\,\mathsf{E}_Q p^\alpha \dfrac{q'}{p} - \mathsf{E}_Q p^\alpha \dfrac{p''}{p}} .$$

These conclusions follow from the fact that in the location case $p'_\theta = -p'([\theta]^{-1})$ for all $\theta \in R$. They agree with Huber's formulas for influence curves of M-estimators of location with loss function $D(x) = C - p(x)^\alpha$ (cf. the well-motivated M-estimators of location in Sec. 3 of [8] or Sec. 5 of [9]). □

Theorem 4.2. If $D''_{Q_\theta}(\theta)$ is negative definite and $\mathsf{E}_{Q_\theta}\Omega^\mathrm{T}_{Q_\theta} \circ \Omega_{Q_\theta}$ finite on Θ^0 then the rate of convergence of T^α to the parameter of $\mathscr{Q}_\Theta$ is $n^{-1/2}$ or, more precisely, (1.1) holds with $V_\theta(T^\alpha) = \mathsf{E}_{Q_\theta}\Omega^\mathrm{T}_{Q_\theta} \circ \Omega_{Q_\theta}$. If moreover $\mathscr{Q}_\Theta$ satisfies the conditions of Theorem 4.1 and the Fisher information $I(\theta \mid \mathscr{Q}_\Theta)$ is positive definite on Θ^0, then $V_\lambda(T^\alpha) - I(\theta \mid \mathscr{Q}_\Theta)^{-1}$ is positive definite on Θ^0 unless there exist constants $c_i \in R$ such that $\Omega_{Q_\theta} = c_1 q'_\theta/q_\theta + c_2$ a.s. $[Q_\theta]$ on Θ^0, in which case $V_\theta(T^\alpha) = I(\theta \mid \mathscr{Q}_\Theta)^{-1}$.

Proof. (I) Taking into account (ii) and Lemma 1.2, we can assume analogically as in the proofs of Theorems 2.2, 3.3 the identity $D'_{P_n}(\theta) - D'_{P_n}(T(P_n)) = (\theta - T(P_n)) \circ$

$\circ\, D''_{P_n}(\theta_n^*)$ for all $x \in \mathscr{X}^n$, $\theta \in \Theta^0$, where the r. v. $\|\theta - \theta_n^*\|_m$ defined on $(\mathscr{X}^n, \mathscr{B}^n, Q^n)$ satisfies the relation $\|\theta - \theta_n^*\|_m \xrightarrow{Q_\theta} 0$.

(II) Now we prove $D''_{P_n}(\theta_n^*) \xrightarrow{Q_\theta} D''_{Q_\theta}(\theta)$. It holds

$$\|D''_{P_n}(\theta_n^*) - D''_{Q_\theta}(\theta)\|_{m^2} \leqq \|D''_{P_n}(\theta_n^*) - D''_{Q_\theta}(\theta_n^*)\|_{m^2} + \|D''_{Q_\theta}(\theta_n^*) - D''_{Q_\theta}(\theta)\|_{m^2}\,.$$

By Lemma 4.1 and (I) the second right term tends to zero in the Q_θ^∞-probability. As to the first right term, using the uniform boundedness and continuity of ψ'_θ on $\mathscr{X} \times \Theta^0$ (cf. (ii)) and the method employed in part (II) of the proof of Theorem 4.1 in [10], we get

$$\|D''_{P_n}(\theta_n^*) - D''_{Q_\theta}(\theta_n^*)\|_{m^2} = \|\mathsf{E}_{P_n}\psi'_{\theta_n*} - \mathsf{E}_{Q_\theta}\psi'_{\theta_n*}\|_{m^2} \xrightarrow{Q_\theta} 0\,.$$

(III) By (I), $n^{1/2}(T(P_n) - \theta) = -n^{1/2}D'_{P_n}(\theta) \circ D''_{P_n}(\theta_n^*)$, i.e.

$$n^{1/2}(T(P_n) - \theta) = (-n^{-1/2}\sum_{i=1}^{n}\psi_\theta(X_i)) \circ D''_{P_n}(\theta_n^*)^{-1}$$

where $X = (X_1, \ldots, X_n)$ is the r. v. with sample probability space $(\mathscr{X}^n, \mathscr{B}^n, Q^n)$. This together with (II), the multidimensional central limit theorem, and the Cramér-Slutskij theorem [1] yields that (1.1) holds with

$$V_\theta(T^\alpha) = \mathsf{E}_{Q_\theta}[(\psi_\theta - \mathsf{E}_Q\psi_\theta) \circ D''_{Q_\theta}(\theta)^{-1}] \circ [(\psi_\theta - \mathsf{E}_Q\psi_\theta) \circ D''_Q(\theta)^{-1}]\,.$$

The first assertion of Theorem 4.2 follows from here and (4.1). The rest is clear from (4.1), (4.3). □

Example 4.2. Asymptotic variances of a location estimator $T^\alpha \triangleq P//\lambda$ considered in Example 4.1 at $\mathscr{Q}_R$ with $\mathsf{E}_Q\Omega_Q^2 < \infty$ are given by $V_\theta(T^\alpha) = \mathsf{E}_Q\Omega_Q^2$ on $\mathbb{R}$. If Q satisfies the assumptions of Theorem 4.1 then T^α is *BAN* for $\mathscr{Q}_R$ iff there exist constants $c_i \in \mathbb{R}$ such that $p^\alpha p'/p = c_1 q'/q + c_2$ a.s. $[Q]$. □

Example 4.3. If $P = No(0, 1)$ on $\mathscr{X} = \mathbb{R}$ and $\mathscr{Q}_R = \mathscr{P}_R$, then all assumptions of Theorem 4.1 hold for all location estimators $T^\alpha \triangleq No(0, 1)//\lambda$. By Example 4.2, influence curves of these estimators at $\mathscr{P}_R$ are given by (1.3) with $Q = P$, and it holds

$$\Omega_P(x) = (1 + \alpha)^{3/2} x\, \mathrm{e}^{-\alpha x^2/2} \quad \text{for all} \quad x \in \mathbb{R}\,. \tag{4.4}$$

It is easily verified that the influence curve of the *MLE* $T^0 \triangleq No(0, 1)$ (the sample mean) is given by (4.4) with $\alpha = 0$. The curves (4.4) are the smooth curves shown in Fig. 4.1 (the piecewise linear function of Fig. 4.1 is explained in Remark 4.1 below).

By (4.4) and Example 4.2, $V_\theta(T^\alpha)$ is in this case increasing with increasing $\alpha \in [0, 1]$ uniformly for $\theta \in \mathbb{R}$. Further, by (1.5), (1.6) and (4.4), the gross-error sensitivity $\sigma_{GE}(T)$ at P decreases with increasing $\alpha \in [0, 0{\cdot}5]$ and slightly increases with increasing $\alpha \in [0{\cdot}5, 1]$. On the other hand, the local shift sensitivity $\sigma_{LS}(T)$ at P slightly increases with increasing $\alpha \in [0, 1]$. The radius of ε-negligibility at P decreases for $\varepsilon \leqq 10^{-2}$ from $\varrho_\varepsilon(T^0) = \infty$ to $\varrho_\varepsilon(T^1) \doteq 3$ with increasing $\alpha \in [0, 1]$. The whole descriptor of robustness $(\sigma_{GE}(T^\alpha), \sigma_{LS}(T^\alpha), \varrho_\varepsilon(T^\alpha))$ together with $V_\theta(T^\alpha)$ seem to achieve most favourable values for $\alpha \in (0{\cdot}1, 0{\cdot}3)$. □

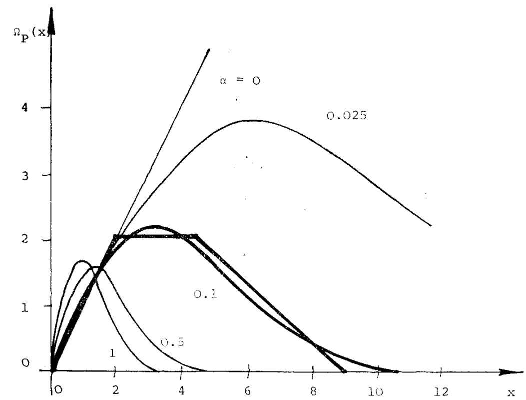

Fig. 4.1.

Remark 4.1. It is interesting that the influence curves (4.4) with $\alpha \in (0.1, 0.3)$ almost coincide with the curves of estimators A 17 – A 25 and AMT which emerged as most promising from the Princeton experimental study [2]. Fig. 4.1 compares (4.4) for $\alpha = 0{\cdot}1$ (the thick curve) with an A-type estimator with influence-curve breakpoints $(a; b; c) = (2; 4{\cdot}5; 9)$ (cf. A 12 with $(a; b; c) = (1{\cdot}2; 3{\cdot}5; 8)$ up to A 25 with $(a; b; c) = (2{\cdot}5; 4{\cdot}5; 9{\cdot}5)$). Since none of the regular weak D-estimators of location analysed in Section 3 possesses a redescending influence curve at $Q = P$ (cf. Example 3.1), $T^{0.2} \triangleq No(0, 1)//\lambda$ seems to be most promising among all D-estimators of location considered in our papers [8–10]. An extensive analysis of asymptotic performances of estimators $T^{0.1} - T^{0.3}$ of structural as well as nonstructural parameters with discrete as well as continuous projection families $\mathscr{P}_\Theta$ carried out so far for various sample-generating contaminated families $\mathscr{Q}_\Theta = (1 - \varepsilon)\,\mathscr{P}_\Theta + \varepsilon\mathscr{P}_\Theta$ dislosed that these estimators quite universally combine a negligible inefficiency at $\varepsilon = 0$ with a reasonably limited bias and inefficiency at $\varepsilon \in (0, 0{\cdot}25)$. These observations together with the analyticity of the respective influence curves are qualifying the estimators $T^{0.1} - T^{0.3}$ as most promising practical robust alternatives to maximum likelihood estimators T^0. Note that numerical algorithms for evaluation of estimates $T^\alpha(P_n)$ and a possibility to use local maxima of functions $\mathsf{E}_{P_n} p_\theta^\alpha$ for a simultaneous analysis of homogenity of the corresponding data $\{x_1, \ldots, x_n\}$ and their clustering have been described by Grim [41].

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REFERENCES

[1] J. Anděl: Mathematical Statistics (in Czech). SNTL - Alfa, Praha—Bratislava 1978.

[2] D. F. Andrews, P. J. Bickel, F. R. Hampel, P. J. Huber, W. H. Rogers and J. W. Turkey: Robust Estimates of Location. Princeton Univ. Press, Princeton 1972.

[3] D. D. Boos: Minimum distance estimators for location and goodness of fit. J. Amer. Statist. Assoc. *76* (1981), 663—670.

[4] J. Grim: An algorithm for maximizing a finite sum of positive functions and its application to cluster analysis. Problems Control Inform. Theory *10* (1981), 427—437.

[5] F. Hampel: The influence curve and its role in robust estimation. J. Amer. Statist. Assoc. *69* (1974), 383—393.

[6] P. J. Huber: Robust estimation of a location parameter. Ann. Math. Statist. *35* (1964), 73—101.

[7] J. W. Tukey: Exploratory Data Analysis. Addison-Wesley, Reading 1970.

[8] I. Vajda: Minimum divergence principle in statistical estimation. Statistics and Decisions *2* (1984), to appear.

[9] I. Vajda: Motivation, existence and equivariance of *D*-estimators. Kybernetika *20* (1984), 3, 189—208.

[10] I. Vajda: Consistency of *D*-estimators. Kybernetika *20* (1984), 4, 283—303.

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