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REGRESSION QUANTILES AND TRIMMED LEAST SQUARES ESTIMATOR UNDER A GENERAL DESIGN

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For the regression quantiles introduced by Koenker and Bassett [16], the Bahadur-type representation up to the remaining term $O_p(n^{-3/4})$ is derived under a general design and for generally asymmetric distribution. This is then applied to derive the representation and the asymptotic distribution of the trimmed least squares estimator under general conditions.

1. INTRODUCTION

Let us consider the linear model

$$\mathbf{X} = \mathbf{C}\boldsymbol{\beta} + \mathbf{E} \tag{1.1}$$

where $\mathbf{X} = (X_1, \ldots, X_n)'$ is the vector of independent observations, $\mathbf{C}$ is the $n \times p$ design matrix, $\boldsymbol{\beta} = (\beta_1, \ldots, \beta_p)'$ is the vector of unknown parameters and $\mathbf{E} = (E_1, \ldots, E_n)'$ where $E_1, \ldots, E_n$ are independent and identically distributed (*i.i.d.*) random variables with a continuous distribution function (*d.f.*) F. Our main interest is in robust estimating the parameter $\boldsymbol{\beta}$.

For the location submodel, three broad classes of robust estimators, M-, R- and L-estimators, were introduced and intensively studied. Their finite-sample as well as asymptotic properties can be found, e.g., in Huber's monograph [6]. From these classes of estimators, the M- and R-estimators were extended in a straightforward way to the linear model. This was not the case of L-estimators, though they are computionally appealing in the location submodel. Comparing with a host of papers on M- and R-estimators of regression parameters, it was apparently only Bickel's paper [4] until recently which dealt with an extension of L-estimators to the regression model. Koenker and Bassett [16] introduced the concept of regression quantile as an extension of the sample quantile to the linear model. This concept seems to provide a reasonable basis not only for the construction of robust L-estimators of regression parameters but also for the construction of robust tests of the linear hypothesis and for the robust analysis of variance. Koenker and Bassett [16] also suggested the trimmed least squares estimator (trimmed *LSE*) as an extension of the trimmed mean to the linear model.

The regression quantiles and the trimmed *LSE* were later-on studied by Ruppert and Carroll [19], who derived the Bahadur-type representation of both up to the order $o_p(n^{-1/2})$ as well as the asymptotic distribution of the trimmed *LSE*. Jurečková [12] studied the asymptotic relation of the trimmed *LSE* to the Huber *M*-estimator. Both [19] and [12] considered the design matrix satiyfying

$$c_{i1} = 1\,, \quad i = 1, \ldots, n\,; \quad \sum_{i=1}^{n} c_{ij} = 0\,, \quad j = 2, \ldots, p\,. \tag{1.2}$$

Ruppert and Carroll showed that under such design and for asymmetric F, the ambiguity about the parameter being estimated involves only the intercept and none of the slope parameters. Such design was also considered by Bassett and Koenker ([2], [3]) and by Portnoy [18] in the studies of the empirical distribution and of the empirical quantile functions based on regression quantiles and also by Jurečková [13] in the study of Winsorized *LSE* and by Jurečková and Sen [14] in the construction of an adaptive scale-equivariant *M*-estimator.

The problem of interest is what is the behaviour of the regression quantiles and of the trimmed *LSE* in the case of a more general design not satisfying (1.2) and in the special case of design without an intercept,

$$\sum_{i=1}^{n} c_{ij} = 0\,, \quad j = 1, \ldots, p\,. \tag{1.3}$$

In the case of general design satisfying neither (1.2) nor (1.3), the regression quantiles would be asymptotically biased but their asymptotic bias would not have a natural population counterpart. As an alternative, we suggest to supplement the model by an additional dummy intercept and to estimate it simultaneously with the other components of the parameter. We also suggest to calculate the trimmed *LSE* for the extended model in the Koenker and Bassett manner and to consider its marginal vector as an estimator of $\boldsymbol{\beta}$ in the original model. It turns out that, even in this case, the trimmed *LSE* represents an extension of the trimmed mean to the linear model. Its asymptotic relation to Huber's estimator and to an appropriate *R*-estimator will be also considered.

On the other hand, under the design without an intercept (1.3), the regression quantiles are asymptotically undistinguishable from each other and thus the trimmed *LSE* is not well-defined. It means that, even if the intercept is known to be zero, the extended model mentioned above is preferable to (1.3).

The Bahadur-type representation of regression quantiles up to the order $O_p(n^{-3/4})$ is derived in Section 2. The trimmed *LSE* is defined in Section 3 which also contains the asymptotic representation and the asymptotic distribution of the same. Section 4 brings some concluding remarks.

2. BAHADUR'S REPRESENTATION OF REGRESSION QUANTILES

Let $X_1, \ldots, X_n$ be independent observations, X_i distributed according to the *d.f.* $F(x - \sum_{j=1}^{p} c_{ij}\beta_j)$, $i = 1, \ldots, n$, where F is an absolutely continuous d.f., $\mathbf{C} = \mathbf{C}_n = \|c_{ij}\|_{i=1,\ldots,n}^{j=1,\ldots,p}$ is a known design matrix; we assume that $\mathbf{C}$ satisfies the following set of conditions (A):

$$\max_{1 \leq i \leq n, 1 \leq j \leq p} n^{-1/4}|c_{ij}| = O(1), \quad \text{as} \quad n \to \infty . \tag{A.1}$$

$$\lim_{n \to \infty} \frac{1}{n} \mathbf{C}_n'\mathbf{C}_n = \mathbf{Q}^* , \quad \text{where } \mathbf{Q}^* \text{ is a positively definite } p \times p \text{ matrix} . \tag{A.2}$$

$$\frac{1}{n} \sum_{i=1}^{n} c_{ij}^4 = O(1) , \quad \text{as} \quad n \to \infty , \quad j = 1, \ldots, p . \tag{A.3}$$

Denote $\mathbf{c}_i'$ the ith row of $\mathbf{C}$, $i = 1, \ldots, n$, and $\bar{\mathbf{c}} = \bar{\mathbf{c}}_n = (\bar{c}_{n1}, \ldots, \bar{c}_{np})'$ with $\bar{c}_{nj} = n^{-1} \sum_{i=1}^{n} c_{ij}, j = 1, \ldots, p$, the vector the column averages of $\mathbf{C}$. We shall assume that

$$\lim_{n \to \infty} \bar{c}_{nj} = q_{0j} , \quad j = 1, \ldots, p . \tag{A.4}$$

Let $\mathbf{D}_n$ denote the $n \times (p + 1)$ matrix

$$\mathbf{D}_n = \|\mathbf{1} \,\vdots\, \mathbf{C}_n\| \tag{2.1}$$

where $\mathbf{1} = (1, \ldots, 1)'$ is an $n \times 1$ vector. Denoting $\mathbf{d}_i'$ the ith row of $\mathbf{D}_n$, $i = 1, \ldots, n$, we have

$$\mathbf{d}_i' = (d_{i0}, d_{i1}, \ldots, d_{ip}) = (1, c_{i1}, \ldots, c_{ip}) , \quad i = 1, \ldots, n . \tag{2.2}$$

By (A.2) and (A.4),

$$\lim_{n \to \infty} \frac{1}{n} \mathbf{D}_n'\mathbf{D}_n = \mathbf{Q} \tag{2.3}$$

where $\mathbf{Q} = \|q_{jk}\|_{j,k=0,\ldots,p}$ is a $(p + 1) \times (p + 1)$ matrix.

We shall consider an extended model

$$\mathbf{X} = \mathbf{D}_n\boldsymbol{\beta} + \mathbf{E} \tag{2.4}$$

with $\boldsymbol{\beta} = (\beta_0, \beta_1, \ldots, \beta_p)'$; then (2.4) coincides with the original model (1.1) if $\beta_0 = \mathbf{0}$. Following Koenker and Bassett [16], we shall consider the α-regression quantile $\hat{\boldsymbol{\beta}}(\alpha)$ $(0 < \alpha < 1)$ for the model (2.4) as any value of $\mathbf{t} \in \mathbb{R}^{p+1}$ which solves

$$\sum_{i=1}^{n} \varrho_\alpha(X_i - \mathbf{d}_i'\mathbf{t}) := \min \tag{2.5}$$

where

$$\varrho_\alpha(x) = x \, \varphi_\alpha(x) , \quad x \in \mathbb{R}^1 \tag{2.6}$$

and

$$(2.7) \qquad \varphi_\alpha(x) = \alpha - I[x < 0]\,, \quad x \in \mathbb{R}^1\,.$$

If the density f of F is continuous and positive in a neighbourhood of $F^{-1}(\alpha)$, we get from Ruppert and Carroll [19], under (1.3), (A.2), (A.3) and under the condition

$$(2.8) \qquad \max_{1 \leqq i \leqq n, 1 \leqq j \leqq p} |c_{ij}| = o(n^{1/2})\,, \quad \text{as} \quad n \to \infty\,,$$

the following asymptotic representation of $\hat{\boldsymbol{\beta}}(\alpha)$:

$$(2.9) \quad \hat{\boldsymbol{\beta}}(\alpha) - \boldsymbol{\beta}(\alpha) = n^{-1}[f(F^{-1}(\alpha))]^{-1}\,\mathbf{Q}^{-1}\sum_{i=1}^{n}\mathbf{d}_i\,\varphi_\alpha(E_i - F^{-1}(\alpha)) + o_p(n^{-1/2})$$

where

$$(2.10) \qquad \boldsymbol{\beta}(\alpha) = (F^{-1}(\alpha), \beta_1, \ldots, \beta_p)'$$

is a $(p + 1) \times 1$ vector and

$$(2.11) \qquad E_i = X_i - \mathbf{c}_i'\boldsymbol{\beta}$$

is the ith residual, $i = 1, \ldots, n$. (2.9) further implies the asymptotic normality of $n^{1/2}(\hat{\boldsymbol{\beta}}(\alpha) - \boldsymbol{\beta}(\alpha))$ as $n \to \infty$:

$$(2.12) \qquad \mathscr{L}\{n^{1/2}(\hat{\boldsymbol{\beta}}(\alpha) - \boldsymbol{\beta}(\alpha))\} \to$$
$$\to \mathscr{N}_{p+1}(\mathbf{0}, \{\alpha(1 - \alpha)/f^2(F^{-1}(\alpha))\}\,\mathbf{Q}^{-1})\,.$$

The subvector $\hat{\boldsymbol{\beta}}^*(\alpha) = (\hat{\beta}_1(\alpha), \ldots, \hat{\beta}_p(\alpha))'$ of $\hat{\boldsymbol{\beta}}(\alpha)$ is then a consistent estimator of $\boldsymbol{\beta}$.

The following theorem gives an extension of the representation (2.9) to one with the remaining term of the order $O_p(n^{-3/4})$; the matrix $\mathbf{C}$ is assumed to satisfy (A.1) to (A.4) but not necessarily (1.2).

Theorem 2.1. Let $X_1, \ldots, X_n$ be independent random variables, X_i distributed according to the *d.f.* $F(x - \mathbf{c}_i'\boldsymbol{\beta})$, $i = 1, \ldots, n$, where F is absolutely continuous and its density f is positive and finite and f' is bounded in a neighborhood of $F^{-1}(\alpha)$, $0 < \alpha < 1$. Let $\mathbf{C}_n$ satisfy (A.1)–(A.4) and let the matrix $\mathbf{Q}$ of (2.3) be positively definite. Then

$$(2.13) \quad \hat{\boldsymbol{\beta}}(\alpha) - \boldsymbol{\beta}(\alpha) = n^{-1}[f(F^{-1}(\alpha))]^{-1}\,\mathbf{Q}^{-1}\sum_{i=1}^{n}\mathbf{d}_i\,\varphi_\alpha(E_i - F^{-1}(\alpha)) + R_n$$

where

$$(2.14) \qquad R_n = O_p(n^{-3/4})\,, \quad \text{as} \quad n \to \infty\,.$$

Theorem 2.1 will be proved with the aid of two lemmas.

Lemma 2.1. Let $Y_1, \ldots, Y_n$ be *i.i.d.* random variables with the *d.f.* F and the density f such that $0 < f(y) < \infty$ and $f'(y)$ is bounded in a neighborhood of $F^{-1}(\alpha)$, $0 < \alpha < 1$. Denote

$$(2.15) \quad S_j(\mathbf{t}) = n^{-1/4}\sum_{i=1}^{n} d_{ij}[\varphi_\alpha(Y_i - F^{-1}(\alpha) - n^{-1/2}\mathbf{d}_i'\mathbf{t}) - \varphi_\alpha(Y_i - F^{-1}(\alpha))]\,,$$

$j = 0, 1, \ldots, p$; $\mathbf{t} \in \mathbb{R}^{p+1}$. Then, under the conditions (A.1)–(A.4),

$$\max_{0 \leqq j \leqq p} \sup_{\|\mathbf{t}\| \leqq K} \left|S_j(\mathbf{t}) + n^{-3/4} f(F^{-1}(\alpha)) \sum_{i=1}^{n} \sum_{k=0}^{p} d_{ij} d_{ik} t_k\right| = O_p(1) \tag{2.16}$$

for every fixed $K > 0$, as $n \to \infty$.

Proof. Denote

$$Z_i = Y_i - F^{-1}(\alpha), \quad i = 1, \ldots, n \tag{2.17}$$

and

$$S_j^0(\mathbf{t}) = S_j(\mathbf{t}) - \mathsf{E}\, S_j(\mathbf{t}), \quad j = 0, \ldots, p. \tag{2.18}$$

Then Z_i's are *i.i.d.* with the *d.f.* $F^*(z) = F(z + F^{-1}(\alpha))$ and with the density $f^*(x) = f(z + F^{-1}(\alpha))$; then $F^{*-1}(\alpha) = 0$. For $\mathbf{t}, \mathbf{u} \in \mathbb{R}^{p+1}$, we shall write $\mathbf{u} \leqq \mathbf{t}$ if $u_j \leqq t_j$ for $j = 0, \ldots, p$. We could easily verify

$$\begin{aligned} \mathsf{E}\left|S_j^0(\mathbf{t}) - S_j^0(\mathbf{u})\right|^4 &\leqq n^{-1}\{11 \sum_{i=1}^{n} d_{ij}^4 \left|F^*(n^{-1/2}\mathbf{d}_i'\mathbf{t}) - F^*(n^{-1/2}\mathbf{d}_i'\mathbf{u})\right| + \\ &+ \sum_{\substack{i=1 \\ i \neq k}}^{n} \sum_{k=1}^{n} d_{ij}^2 d_{kj}^2 \left|F^*(n^{-1/2}\mathbf{d}_i'\mathbf{t}) - F^*(n^{-1/2}\mathbf{d}_i'\mathbf{u})\right| \\ &\quad . \left|F^*(n^{-1/2}\mathbf{d}_k'\mathbf{t}) - F^*(n^{-1/2}\mathbf{d}_k'\mathbf{u})\right|\} \leqq \\ &\leqq K_1 n^{-3/2} \sum_{i=1}^{n} d_{ij}^4 \left|\mathbf{d}_i'(\mathbf{t} - \mathbf{u})\right| + K_2 n^{-2}[\sum_{i=1}^{n} d_{ij}^2 \mathbf{d}_i'(\mathbf{t} - \mathbf{u})]^2 = \\ &= O(n^{-1/4})\|\mathbf{t} - \mathbf{u}\| + O(1)\, \|\mathbf{t} - \mathbf{u}\|^2, \quad \text{as} \quad n \to \infty \end{aligned} \tag{2.19}$$

and, by Chebyshev and Schwarz inequalities, we get for $\mathbf{u} \leqq \mathbf{v} \leqq \mathbf{t}$ and for $\lambda > 0$

$$\begin{aligned} &\mathsf{P}\{\left|S_j^0(\mathbf{t}) - S_j^0(\mathbf{v})\right| \geqq \lambda, \left|S_j^0(\mathbf{v}) - S_j^0(\mathbf{u})\right| \geqq \lambda\} \leqq \\ &\leqq \lambda^{-4}(K_3\|\mathbf{t} - \mathbf{u}\|^2 + K_4 n^{-1/4}\|\mathbf{t} - \mathbf{u}\|), \quad \text{for} \quad n \geqq n_0. \end{aligned} \tag{2.20}$$

Using the extension of Theorem 12.1 of Billingsley [5] to the vector arguments (see Jurečková and Sen [14], proof of Lemma 3.1, for details), we get from (2.20)

$$\max_{0 \leqq j \leqq p} \sup_{\|\mathbf{t}\| \leqq K} \left|S_j^0(\mathbf{t})\right| = O_p(1), \quad \text{as} \quad n \to \infty. \tag{2.21}$$

It remains to prove

$$\sup_{\|\mathbf{t}\| \leqq K} \left|\mathsf{E}S_j(\mathbf{t}) + n^{-3/4} f(F^{-1}(\alpha)) \sum_{i=1}^{n} \sum_{k=0}^{p} d_{ij} d_{ik} t_k\right| = O(1). \tag{2.22}$$

Denote $\delta_i = n^{-1/2}\mathbf{d}_i'\mathbf{t}$. Then

$$\begin{aligned} &\left|\mathsf{E}\, S_j(\mathbf{t}) + n^{-1/4} \sum_{i=1}^{n} d_{ij} \delta_i f(F^{-1}(\alpha))\right| = \\ &= n^{-1/4}\left|\sum_{i=1}^{n} d_{ij}[F^*(0) - F^*(\delta_i) + f(F^{-1}(\alpha))\, \delta_i]\right| \leqq \end{aligned} \tag{2.23}$$

$$\leqq n^{-1/4} \sum_{i=1}^{n} |d_{ij}| \int_0^{\delta_i} \int_0^{z} |f'(F^{-1}(\alpha) + u)| \,\mathrm{d}u \,\mathrm{d}z \leqq$$

$$\leqq K_5 n^{-5/4} \sum_{i=1}^{n} |d_{ij}| \left(\sum_{k=0}^{p} d_{ik} t_k \right)^2 = O(n^{-1/4}) \|\mathbf{t}\|^2 ,$$

as $n \to \infty$; the integration bounds should be written in the reverse order if necessary.

□

The following lemma studies the stochastic order of $\hat{\boldsymbol{\beta}}(\alpha) - \boldsymbol{\beta}(\alpha)$.

Lemma 2.2. Assume that the conditions of Theorem 2.1 are satisfied. Then

$$(2.24) \qquad n^{1/2}(\hat{\boldsymbol{\beta}}(\alpha) - \boldsymbol{\beta}(\alpha)) = O_p(1) , \quad \text{as} \quad n \to \infty .$$

Proof. Let $\hat{\boldsymbol{\beta}}(\alpha)$ be a solution of the minimization (2.5). Then it follows from (A.1) and from Lemma A.2 of Ruppert and Carroll [19] that

$$(2.25) \qquad n^{-1/2} \sum_{i=1}^{n} d_{ij} \varphi_\alpha(X_i - \mathbf{d}_i' \hat{\boldsymbol{\beta}}(\alpha)) = O_p(n^{-1/4}), \quad j = 0, \dots, p .$$

Moreover, φ_α given by (2.7) is nondecreasing and

$$(2.26) \qquad n^{-1/2} \sum_{i=1}^{n} d_{ij} \varphi_\alpha(E_i - F^{-1}(\alpha)) = O_p(1) , \quad j = 0, \dots, p .$$

Proceeding analogously as in the proof of Lemma 5.2 of Jurečková [10], we get from (2.26) and from Lemma 2.1 that, to every $\varepsilon > 0$, there exist $K > 0$, $\eta > 0$ and n_1 such that

$$(2.27) \qquad \mathsf{P}\{ \inf_{\|\mathbf{t}\| \geqq K} n^{-1/2} \left| \sum_{i=1}^{n} d_{ij} \varphi_\alpha(E_i - F^{-1}(\alpha) - n^{-1/2} \mathbf{d}_i' \mathbf{t}) \right| < \eta \} < \varepsilon$$

for $n \geqq n_1$. By (2.25) and (2.27), there exist $K > 0$, $\eta > 0$ and n_1 to every $\varepsilon > 0$ so that

$$(2.28) \qquad \mathsf{P}\{ n^{1/2} \|\hat{\boldsymbol{\beta}}(\alpha) - \boldsymbol{\beta}(\alpha)\| \geqq K \} \leqq \mathsf{P}\{ n^{1/2} \|\hat{\boldsymbol{\beta}}(\alpha) - \boldsymbol{\beta}(\alpha)\| \geqq K ,$$

$$(2.28) \qquad n^{-1/2} \left| \sum_{i=1}^{n} d_{ij} \varphi_\alpha(E_i - F^{-1}(\alpha) - \mathbf{d}_i'(\hat{\boldsymbol{\beta}}(\alpha) - \boldsymbol{\beta}(\alpha))) \right| < \eta \} +$$

$$+ \mathsf{P}\{ n^{-1/2} \left| \sum_{i=1}^{n} d_{ij} \varphi_\alpha(X_i - d_i' \hat{\boldsymbol{\beta}}(\alpha)) \right| \geqq \eta \} < 2\varepsilon$$

for $n \geqq n_1$.

□

Proof of Theorem 2.1. It follows from Lemma 2.1. that

$$(2.29) \qquad n^{-1/2} \sum_{i=1}^{n} d_{ij} [\varphi_\alpha(E_i - F^{-1}(\alpha) - n^{-1/2} \mathbf{d}_i' \mathbf{T}_n) - \varphi_\alpha(E_i - F^{-1}(\alpha)) +$$

$$+ n^{-1/2} \mathbf{d}_i' \mathbf{T}_n f(F^{-1}(\alpha))] = O_p(n^{-1/4}) , \quad j = 0, \dots, p$$

for every sequence of random vectors such that $\|\mathbf{T}_n\| = O_p(1)$. Hence, (2.13) follows from (2.29), (2.24) and (2.25).

□

Corollary 1. (i) Under the conditions of Theorem 2.1,

$$(2.30) \qquad \mathscr{L}\{n^{1/2}(\hat{\boldsymbol{\beta}}(\alpha) - \boldsymbol{\beta}(\alpha))\} \to \mathscr{N}_{p+1}(\mathbf{0}, \{\alpha(1-\alpha)/f^2(F^{-1}(\alpha))\}\, \mathbf{Q}^{-1}),$$

as $n \to \infty$.

(ii) If $\hat{\boldsymbol{\beta}}^*(\alpha) = (\hat{\beta}_1(\alpha), \ldots, \hat{\beta}_p(\alpha))'$, then

$$(2.31) \qquad \mathscr{L}\{n^{1/2}(\hat{\boldsymbol{\beta}}^*(\alpha) - \boldsymbol{\beta})\} \to \mathscr{N}_p(\mathbf{0}, \{\alpha(1-\alpha)/f^2(F^{-1}(\alpha))\}\, \bar{\mathbf{Q}}^{-1})$$

as $n \to \infty$, where the $p \times p$ matrix $\bar{\mathbf{Q}} = \|\bar{q}_{jk}\|_{j,k=1}^{p}$ is given by

$$(2.32) \qquad \bar{q}_{jk} = \lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} (c_{ij} - \bar{c}_j)(c_{ik} - \bar{c}_k)\,.$$

Proof. (2.30) follows directly from (2.13). Concerning $\hat{\boldsymbol{\beta}}^*(\alpha)$, consider the block decomposition of $\mathbf{Q}^{-1}$:

$$(2.33) \qquad \mathbf{Q}^{-1} = \begin{array}{c} \left\| \begin{matrix} \mathbf{Q}^{11} & \mathbf{Q}^{12} \\ \underbrace{\mathbf{Q}^{21}}_{1} & \underbrace{\mathbf{Q}^{22}}_{p} \end{matrix} \right\| \begin{matrix} \}\,1 \\ \}\,p\,. \\ \end{matrix} \end{array}$$

It follows from (2.13) that

$$(2.34) \qquad n^{1/2}(\hat{\boldsymbol{\beta}}^*(\alpha) - \boldsymbol{\beta}) = n^{-1/2}[f(F^{-1}(\alpha))]^{-1}\, \|\mathbf{Q}^{21} \mid \mathbf{Q}^{22}\| \sum_{i=1}^{n} \mathbf{d}_i \varphi_\alpha(E_i - F^{-1}(\alpha)) + \\ + O_p(n^{-1/4})$$

and this further implies

$$\mathscr{L}\{n^{1/2}(\hat{\boldsymbol{\beta}}^*(\alpha) - \boldsymbol{\beta})\} \to \mathscr{N}(\mathbf{0}, \{\alpha(1-\alpha)/f^2(F^{-1}(\alpha))\}\, \mathbf{Q}^{22})\,.$$

Moreover,

$$(2.35) \qquad \mathbf{Q}^{22} = (\mathbf{Q}_{22} - \mathbf{Q}_{21}\mathbf{Q}_{12})^{-1} = \bar{\mathbf{Q}}^{-1}$$

where $\mathbf{Q}_{21} = \mathbf{Q}'_{12}$ and $\mathbf{Q}_{22} = \mathbf{Q}^*$ (see (A.2)) are the blocks in the decomposition

$$(2.36) \qquad \mathbf{Q} = \left\| \begin{matrix} \mathbf{Q}_{11} & \mathbf{Q}_{12} \\ \underbrace{\mathbf{Q}_{21}}_{1} & \underbrace{\mathbf{Q}_{22}}_{p} \end{matrix} \right\| \begin{matrix} \}\,1 \\ \}\,p \\ \end{matrix}$$

of the matrix $\mathbf{Q}$. This completes the proof of (2.31). □

Theorem 2.1 and its corollary apply also to the special case of $\mathbf{C}_n$ satisfying (1.3). Let us now assume that, for $\mathbf{C}_n$ satisfying (1.3), the regression quantile $\hat{\boldsymbol{\beta}}(\alpha)$ is defined through the minimization

$$(2.37) \qquad \sum_{i=1}^{n} \varrho_\alpha(X_i - \mathbf{c}_i'\mathbf{t}) := \min$$

with respect to $\mathbf{t} \in \mathbb{R}^p$, instead of minimization (2.5). The following theorem shows that, in such case, the regression quantiles corresponding to different α's are asymptotically undistinguishable from each other and gives their common asymptotic distribution.

Theorem 2.2. Let $X_1, \ldots, X_n$ be independent random variables, X_i distributed according to the *d.f.* $F(x - \mathbf{c}_i'\boldsymbol{\beta})$, $i = 1, \ldots, n$, where F is absolutely continuous and such that its density f is positive and finite and has a bounded derivative in a neighborhood of 0. Assume that $\mathbf{C}_n$ satisfies (1.3), (A.1)–(A.4). Let $\hat{\boldsymbol{\beta}}(\alpha)$ be the α-regression quantile defined as a solution of the minimization (2.37), $0 < \alpha < 1$. Then

$$\hat{\boldsymbol{\beta}}(\alpha) - \boldsymbol{\beta} = n^{-1}(f(0))^{-1}\, \mathbf{Q}^{*-1} \sum_{i=1}^{n} \mathbf{c}_i\, \varphi_{\alpha_0}(E_i) + O_p(n^{-3/4}) \tag{2.38}$$

as $n \to \infty$, where $\alpha_0 = F(0)$, and

$$\mathscr{L}\{n^{1/2}(\hat{\boldsymbol{\beta}}(\alpha) - \boldsymbol{\beta})\} \to \mathscr{N}_p(\mathbf{0}, [F(0)(1 - F(0))/f^2(0)]\, \mathbf{Q}^{*-1}) \tag{2.39}$$

as $n \to \infty$.

Proof. It follows from (A.1) and from Lemma A.2 of Ruppert and Carroll [19] that

$$n^{-1/2} \sum_{i=1}^{n} c_{ij} \varphi_{\alpha_0}(X_i - \mathbf{c}_i'\, \hat{\boldsymbol{\beta}}(\alpha)) = O_p(n^{-1/4}), \quad j = 1, \ldots, p\,. \tag{2.40}$$

Moreover,

$$n^{-1/2} \sum_{i=1}^{n} c_{ij}\, \varphi_{\alpha_0}(E_i) = O_p(1), \quad j = 1, \ldots, p\,. \tag{2.41}$$

The rest of the proof follows from Lemma 2.1 analogously as in the proof of Theorem 2.1. □

3. TRIMMED *LSE* AND ITS ASYMPTOTIC DISTRIBUTION

Let α_1, α_2 be fixed, $0 < \alpha_1 < \alpha_2 < 1$. Let $X_1, \ldots, X_n$ be independent observations, X_i distributed according to the *d.f.* $F(x - \mathbf{c}_i'\boldsymbol{\beta})$, $i = 1, \ldots, n$. The following set of conditions (B) is assumed to hold for the *d.f.* F:

(B.1) F is absolutely continuous with the density f.

(B.2) $0 < f(x) < \infty$ for $F^{-1}(\alpha_1) - \varepsilon < x < F^{-1}(\alpha_2) + \varepsilon$, $\varepsilon > 0$.

(B.3) The derivative f' of f exists and is bounded in neighbourhoods of $F^{-1}(\alpha_1)$ and $F^{-1}(\alpha_2)$.

Assume that the matrix $\mathbf{C}_n$ satisfies (A.1)–(A.4) and that the matrix $\mathbf{Q}$ of (2.3) is positively definite.

Let $\hat{\boldsymbol{\beta}}(\alpha_1)$ and $\hat{\boldsymbol{\beta}}(\alpha_2)$ be the regression quantiles, defined through the minimization (2.5) for the extended model (2.4). The solution of the minimization (2.5) is generally not uniquely determined; suppose that a rule is given which selects a unique solution of (2.5) for $\alpha = \alpha_1, \alpha_2$.

Let $\mathbf{A}$ be the diagonal $n \times n$ matrix with the diagonal

$$a_{ii} = a_i = \begin{cases} 0 & \text{if } X_i \leqq \mathbf{d}_i'\, \hat{\boldsymbol{\beta}}(\alpha_1) \text{ or } X_i \geqq \mathbf{d}_i'\, \hat{\boldsymbol{\beta}}(\alpha_2) \\ 1 & \text{otherwise}\,, \quad i = 1, \ldots, n\,. \end{cases} \tag{3.1}$$

Denote $\mathbf{L}_n(\alpha_1, \alpha_2)$ the trimmed *LSE* corresponding to the extended model (2.3). According to Koenker and Bassett [16], $\mathbf{L}_n(\alpha_1, \alpha_2)$ is calculated as the ordinary *LSE* after trimming-off X_i with $a_i = 0$, $i = 1, \ldots, n$, i.e.,

$$\mathbf{L}_n(\alpha_1, \alpha_2) = (\mathbf{D}'\mathbf{A}\mathbf{D})^{-} (\mathbf{D}'\mathbf{A}\boldsymbol{X}) \tag{3.2}$$

where $\boldsymbol{X} = (X_1, \ldots, X_n)'$. Notice that $\mathbf{L}_n(\alpha_1, \alpha_2) = (L_0, L_1, \ldots, L_p)'$ is, in fact, an estimator of $\bar{\boldsymbol{\beta}} = (0, \beta_1, \ldots, \beta_p)'$. We suggest $\mathbf{L}_n^*(\alpha_1, \alpha_2) = \mathbf{L}_n^* = (L_1, \ldots, L_p)'$ as an estimate of $\boldsymbol{\beta} = (\beta_1, \ldots, \beta_p)'$ and we shall call it the trimmed *LSE* of $\boldsymbol{\beta}$. The following theorem gives an asymptotic representation and the asymptotic distribution of $\mathbf{L}_n^*(\alpha_1, \alpha_2)$.

Theorem 3.1. Let $X_1, \ldots, X_n$ be independent observations, X_i distributed according to the *d.f.* $F(x - \mathbf{c}_i'\boldsymbol{\beta})$, $i = 1, \ldots, n$, where $\boldsymbol{\beta} \in \mathbb{R}^p$ is an unknown parameter. Assume that the design matrix $\mathbf{C}_n$ satisfies the conditions (A.1)–(A.4) and that the matrix $\mathbf{Q}$ of (2.3) is positively definite. Assume that the *d.f.* F satisfies the conditions (B.1)–(B.3). Then (i)

$$\mathbf{L}_n^*(\alpha_1, \alpha_2) - \boldsymbol{\beta} = n^{-1}(\alpha_2 - \alpha_1)^{-1} \left\| \mathbf{Q}^{21} \,\vdots\, \mathbf{Q}^{22} \right\| \sum_{i=1}^{n} \mathbf{d}_i \, \psi(E_i) + O_p(n^{-3/4}) \tag{3.3}$$

as $n \to \infty$, where the matrix $\left\| \mathbf{Q}^{21} \,\vdots\, \mathbf{Q}^{22} \right\|$ is given by the block decomposition (2.33) of $\mathbf{Q}^{-1}$ and

$$\psi(x) = \begin{cases} F^{-1}(\alpha_1) & \text{if} \quad x < F^{-1}(\alpha_1) \\ x & \text{if} \quad F^{-1}(\alpha_1) \leqq x \leqq F^{-1}(\alpha_2) \\ F^{-1}(\alpha_2) & \text{if} \quad F^{-1}(\alpha_2) < x\,. \end{cases} \tag{3.4}$$

(ii)

$$\mathscr{L}\{n^{1/2}(\mathbf{L}_n^*(\alpha_1, \alpha_2) - \boldsymbol{\beta})\} \to \mathscr{N}_p(\boldsymbol{0}, \sigma^2(\alpha_1, \alpha_2, F)\, \bar{\mathbf{Q}}^{-1}) \tag{3.5}$$

where the matrix $\bar{\mathbf{Q}}$ is given by (2.32) and

$$\begin{aligned} \sigma^2(\alpha_1, \alpha_2, F) = (\alpha_2 - \alpha_1)^{-2} \Big\{ & \int_{\alpha_2}^{\alpha_1} (F^{-1}(u) - \delta)^2 \, \mathrm{d}u + \\ & + \alpha_1(F^{-1}(\alpha_1) - \delta)^2 + (1 - \alpha_2)(F^{-1}(\alpha_2) - \delta)^2 - \\ & - [\alpha_1(F^{-1}(\alpha_1) - \delta) + (1 - \alpha_2)(F^{-1}(\alpha_2) - \delta)]^2 \Big\} \end{aligned} \tag{3.6}$$

where

$$\delta = \delta(\alpha_1, \alpha_2, F) = (\alpha_2 - \alpha_1)^{-1} \int_{\alpha_1}^{\alpha_2} F^{-1}(u) \, \mathrm{d}u\,. \tag{3.7}$$

Theorem 3.1 will be proved with the aid of two lemmas.

Lemma 3.1. Let $Y_1, \ldots, Y_n$ be independent random variables, identically distributed according to the absolutely continuous *d.f.* F; assume that $0 < f(x) < \infty$ and $f'(x)$ exists and is bounded in a neighbourhood of $F^{-1}(\alpha)$, $0 < \alpha < 1$. Denote

$$T_j(\mathbf{t}) = n^{-1/2} \sum_{i=1}^{n} d_{ij} Y_i I[Y_i \leqq F^{-1}(\alpha) + n^{-1/2}\mathbf{d}_i'\mathbf{t}]\,, \tag{3.8}$$

$j = 0, \dots, p$; $\mathbf{t} \in \mathbb{R}^{p+1}$. Then, under the conditions (A.1)–(A.4),

$$(3.9) \quad \sup_{\|\mathbf{t}\| \leqq K} \left| T_j(\mathbf{t}) - T_j(\mathbf{0}) - n^{-1} F^{-1}(\alpha) f(F^{-1}(\alpha) \sum_{i=1}^{n} \sum_{k=0}^{p} d_{ij} d_{ik} t_k \right| = O_p(n^{-1/4}), \quad j = 0, \dots, p$$

for every fixed $K > 0$, as $n \to \infty$.

Proof. The proof is analogous to that of Lemma 2.1: Denote

$$(3.10) \qquad T_j^0(\mathbf{t}) = T_j(\mathbf{t}) - \mathsf{E}\, T_j(\mathbf{t}), \quad j = 0, \dots, p$$

and

$$(3.11) \qquad \delta_i(\mathbf{t}) = F^{-1}(\alpha) + n^{-1/2} \mathbf{d}_i' \mathbf{t}, \quad i = 1, \dots, n.$$

Then, for $\mathbf{u}, \mathbf{t} \in R^{p+1}$, $\mathbf{u} \leqq \mathbf{t}$,

$$(3.12) \qquad \mathsf{E}(n^{1/4} |T_j^0(\mathbf{t}) - T_j^0(\mathbf{u})|)^4 \leqq n^{-1} \left\{ \sum_{i=1}^{n} d_{ij}^4 \int_{\delta_i(\mathbf{u})}^{\delta_i(\mathbf{t})} x^4 \,\mathrm{d}F(x) + \right.$$

$$\left. + \sum_{\substack{i=1 \\ i \neq l}}^{n} \sum_{l=1}^{n} d_{ij}^2 d_{lj}^2 \int_{\delta_i(\mathbf{u})}^{\delta_i(\mathbf{t})} x^2 \,\mathrm{d}F(x) \int_{\delta_l(\mathbf{u})}^{\delta_l(\mathbf{t})} x^2 \,\mathrm{d}F(x) \right\} \leqq$$

$$\leqq n^{-1} \{ 11 \sum_{i=1}^{n} d_{ij}^4 n^{-1/2} \mathbf{d}_i'(\mathbf{t} - \mathbf{u}) (F^{-1}(\alpha))^4 f(F^{-1}(\alpha)) +$$

$$+ [\sum_{i=1}^{n} d_{ij}^2 n^{-1/2} \mathbf{d}_i'(\mathbf{t} - \mathbf{u})]^2 (F^{-1}(\alpha))^4 f^2(F^{-1}(\alpha)) \} +$$

$$+ o(n^{-1/4}) \|\mathbf{t} - \mathbf{u}\| = O(1) \|\mathbf{t} - \mathbf{u}\|^2 + O(n^{-1/4}) \|\mathbf{t} - \mathbf{u}\|,$$

as $n \to \infty$, where the integral bounds should be written in the reverse order if necessary. Again, using the extension of Theorem 12.1 of Billingsley [5] to the vector arguments, we get from (3.12)

$$(3.13) \qquad \max_{0 \leqq j \leqq p} \sup_{\|\mathbf{t}\| \leqq K} |T_j^0(\mathbf{t})| = O_p(n^{-1/4}).$$

It remains to prove

$$(3.14) \quad \sup_{\|\mathbf{t}\| \leqq K} n^{1/4} \left| \mathsf{E}[T_j(\mathbf{t}) - T_j(\mathbf{0})] - n^{-1} F^{-1}(\alpha) f(F^{-1}(\alpha) \sum_{i=1}^{n} \sum_{k=0}^{p} d_{ij} d_{ik} t_k \right| = O(1), \quad \text{as} \quad n \to \infty.$$

It will be proved in the following steps:

$$(3.15) \qquad \left| \mathsf{E}[T_j(\mathbf{t}) - T_j(\mathbf{0})] - n^{-1} F^{-1}(\alpha) f(F^{-1}(\alpha)) \sum_{i=1}^{n} d_{ij} \mathbf{d}_i' \mathbf{t} \right| \leqq$$

$$n^{-1/2} \sum_{i=1}^{n} |d_{ij}| \left| \int_{F^{-1}(\alpha)}^{\delta_i(\mathbf{t})} [x f(x) - F^{-1}(\alpha) f(F^{-1}(\alpha))] \,\mathrm{d}x \right| \leqq$$

$$\leqq n^{-1/2} \sum_{i=1}^{n} |d_{ij}| \int_{F^{-1}(\alpha)}^{\delta_i(\mathbf{t})} \int_{F^{-1}(\alpha)}^{x} |f(y) + y f'(y)| \,\mathrm{d}y \,\mathrm{d}x \leqq$$

$$\leqq K_1 n^{-3/2} \sum_{i=1}^{n} |d_{ij}| \, |\mathbf{d}_i' \mathbf{t}|^2 = O(n^{-1/2}) \|\mathbf{t}\|^2$$

where, again, the integral bounds should be eventually reversed. □

Lemma 3.2. Denote

$$U_{jk}(\mathbf{t}) = n^{-1} \sum_{i=1}^{n} d_{ij} d_{ik} I[Y_i \leqq F^{-1}(\alpha) + n^{-1/2} \mathbf{d}_i' \mathbf{t}] \,, \tag{3.16}$$

$j, k = 0, \ldots, p$ and $\mathbf{t} \in R^{p+1}$. Then, under the conditions of Lemma 3.1,

$$\sup_{\|\mathbf{t}\| \leqq K} |U_{jk}(\mathbf{t}) - \alpha q_{jk}| \xrightarrow{p} 0 \tag{3.17}$$

for every fixed $K > 0$ and $j, k = 0, \ldots, p$, as $n \to \infty$; q_{jk} are the elements of matrix $\mathbf{Q}$ of (2.3).

Proof. The proof is quite analogous to that of Lemma 3.1.

Proof of Theorem 3.1. It follows from Lemma 3.2

$$n^{-1} \mathbf{D}' \mathbf{A} \mathbf{D} = (\alpha_2 - \alpha_1) \mathbf{Q} + o_p(1) \tag{3.18}$$

as $n \to \infty$. By (3.1),

$$\begin{aligned} \mathbf{D}'\mathbf{A}\mathbf{E} = \sum_{i=1}^{n} d_i \{ & I[E_i - F^{-1}(\alpha_2) < \mathbf{d}_i'(\hat{\boldsymbol{\beta}}(\alpha_2) - \boldsymbol{\beta}(\alpha_2))] - \\ & - I[E_i - F^{-1}(\alpha_1) < \mathbf{d}_i'(\hat{\boldsymbol{\beta}}(\alpha_1) - \boldsymbol{\beta}(\alpha_1))]\} \, E_i \end{aligned} \tag{3.19}$$

hence, it follows from Lemma 3.1 and Lemma 2.2,

$$\begin{aligned} n^{-1/2} \mathbf{D}'\mathbf{A}\mathbf{E} = n^{-1/2} \sum_{i=1}^{n} \mathbf{d}_i \{ & \mathbf{d}_i'[(\hat{\boldsymbol{\beta}}(\alpha_2) - \boldsymbol{\beta}(\alpha_2)) \, F^{-1}(\alpha_2) f(F^{-1}(\alpha_2)) - \\ & - (\hat{\boldsymbol{\beta}}(\alpha_1) - \boldsymbol{\beta}(\alpha_1)) \, F^{-1}(\alpha_1) f(F^{-1}(\alpha_1)] + \\ & + E_i I[F^{-1}(\alpha_1) \leqq E_i < F^{-1}(\alpha_2)]\} + O_p(n^{-1/4}) \end{aligned} \tag{3.20}$$

and this, combined with (2.13) of Theorem 2.1 and with (3.18) gives

$$n^{1/2}(\mathbf{L}_n(\alpha_1, \alpha_2) - \bar{\boldsymbol{\beta}}) = n^{-1/2}(\alpha_2 - \alpha_1)^{-1} \, \mathbf{Q}^{-1} \sum_{i=1}^{n} \mathbf{d}_i(\psi(E_i) - \gamma) + O_p(n^{-1/4}) \tag{3.21}$$

with

$$\gamma = \alpha_1 \, F^{-1}(\alpha_1) + (1 - \alpha_2) \, F^{-1}(\alpha_2) \tag{3.22}$$

and this further implies (3.3). The asymptotic distribution of $\mathbf{L}_n^*(\alpha_1, \alpha_2)$ then follows from (3.3). □

4. CONCLUDING REMARKS

Under a general design satisfying the conditions (A.1)–(A.4), the regression model (1.1) was extended by a dummy intercept $\beta_0(=0)$. The first component $\hat{\beta}_0(\alpha)$ of the α-regression quantile was then shown to be a consistent estimator of $F^{-1}(\alpha)$ while $(\hat{\beta}_1(\alpha), \ldots, \hat{\beta}_p(\alpha))'$ was shown to be a consistent estimator of $\boldsymbol{\beta} = (\beta_1, \ldots, \beta_p)'$.

The first component L_0 of the trimmed *LSE* $\mathbf{L}_n(\alpha_1, \alpha_2)$ for the extended model is a consistent estimator of $\delta(\alpha_1, \alpha_2, F)$ of (3.8) while $\mathbf{L}_n^*(\alpha_1, \alpha_2) = (L_1, \ldots, L_p)'$ was shown to be a consistent estimator of $\boldsymbol{\beta} = (\beta_1, \ldots, \beta_p)'$ with the asymptotic covariance matrix $\sigma^2(\alpha_1, \alpha_2, F)\,\mathbf{Q}^{-1}$.

It follows from Jurečková [10] that $\mathbf{L}_n^*(\alpha_1, \alpha_2)$ is asymptotically equivalent to the M-estimator $\mathbf{M}_n$ which is defined as a solution of the system of equations

$$\sum_{i=1}^{n} (c_{ij} - \bar{c}_j)\,\psi(X_i - \mathbf{c}_i'\mathbf{t}) = 0\,, \quad j = 1, \ldots, p \tag{4.1}$$

with respect to $\mathbf{t} \in \mathbb{R}^p$.

On the other hand, let $\mathbf{R}_n$ denote an R-estimator of $\boldsymbol{\beta}$ based on ranks, generated by a score-function $\varphi(u)$, $0 < u < 1$, as suggested by the author in [8] or in a modified form by Jaeckel [7] and by Koul [17]. Again, it follows from [10] that $\mathbf{R}_n$ is asymptotically equivalent to $\mathbf{L}_n^*(\alpha_1, \alpha_2)$, provided

$$\varphi(u) = \begin{cases} F^{-1}(\alpha_1) & \text{if} \quad u < \alpha_1 \\ F^{-1}(u) & \text{if} \quad \alpha_1 \leqq u \leqq \alpha_2 \\ F^{-1}(\alpha_2) & \text{if} \quad \alpha_2 < u\,. \end{cases} \tag{4.2}$$

Because the ranks are translation invariant, $\mathbf{R}_n$ is not able to estimate the intercept even if this is non-zero (the intercept should be estimated separately, cf. Jurečková [9]). The trimmed *LSE* always needs to have an intercept involved in the model, possibly an artificial one, to yield an asymptotically unbiased estimator of $\boldsymbol{\beta}$. The M-estimator works in the model without as well as with the intercept; the choice of the matrix $\|(c_{ij} - \bar{c}_j)\|$ $(i = 1, \ldots, n;\ j = 1, \ldots, p)$ guarantees the asymptotic unbiasedness of $\mathbf{M}_n$ even if both F and ψ are asymmetric.

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