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EXTREME SYMMETRY AND THE DIRECTED DIVERGENCE IN INFORMATION THEORY

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The authors have characterized the directed divergence axiomatically using extreme symmetry, a concept weaker than symmetry in the strict sense.

1. INTRODUCTION

Let

$$\Gamma_n = \{(p_1, p_2, \ldots, p_n);\ p_i \geqq 0,\ i = 1, 2, \ldots, n;\ \sum_{i=1}^{n} p_i = 1\}, \quad n = 2, 3, \ldots$$

denote the set of all n-component discrete probability distributions. Let S_n, $n = 2, 3, \ldots$ denote the set of all $2n$-tuples of the form $(p_1, p_2, \ldots, p_n; q_1, q_2, \ldots, q_n)$ with $(p_1, p_2, \ldots, p_n) \in \Gamma_n$, $(q_1, q_2, \ldots, q_n) \in \Gamma_n$ such that $p_i = 0$ for all those indices i for which $q_i = 0$, $1 \leqq i \leqq n$.

S. Kullback and R. A. Leibler [7] proposed the quantity (with $E_n : S_n \to \mathbb{R} =]-\infty, +\infty[$, $n = 2, 3, \ldots$)

$$E_n(p_1, p_2, \ldots, p_n; q_1, q_2, \ldots, q_n) = \sum_{i=1}^{n} p_i \log_2 \frac{p_i}{q_i} \tag{1}$$

where $0 \log 0/q = 0$, $q \geqq 0$, and named it as a *minimum discrimination information function*. Later on, it has also been called the directed divergence between $(p_1, p_2, \ldots, p_n) \in \Gamma_n$ and $(q_1, q_2, \ldots, q_n) \in \Gamma_n$ with $(p_1, p_2, \ldots, p_n; q_1, q_2, \ldots, q_n) \in S_n$. Several researchers have characterized (1) axiomatically. A detailed account of some of these characterizations may be found in chapter 7 and the bibliography given at the end of the book of J. Aczél and Z. Daróczy [3].

A. Hobson [1], L. L. Campbell [4] etc., while characterizing (1) axiomatically assumed the following as a postulate:

Postulate $\mathbf{I}_n$ (Symmetry). $E_n : S_n \to \mathbb{R}$ is symmetric under the simultaneous permutations of p_k and q_k, $k = 1, 2, \ldots, n$, that is,

$$E_n(p_1, p_2, \ldots, p_n; q_1, q_2, \ldots, q_n) = \tag{2}$$
$$= E_n(p_{k(1)}, p_{k(2)}, \ldots, p_{k(n)}; q_{k(1)}, q_{k(2)}, \ldots, q_{k(n)})$$

where k is an arbitrary permutation of $1, 2, \ldots, n$.

The object of this paper is to weaken the symmetry Postulate I_n in the strict sense and then characterize (1) axiomatically.

2. WEAKENING OF SYMMETRY

Postulate I_n is quite intuitive. It tells us that the amount of directed divergence between $(p_1, p_2, \ldots, p_n) \in \Gamma_n$ and $(q_1, q_2, \ldots, q_n) \in \Gamma_n$ does not depend upon the order in which the paired events (p_k, q_k), $k = 1, 2, \ldots, n$ occur. For a fixed n, (2) represents a system of $n!$ equations, a number fairly large as compared with n whenever $n \geqq 3$. Thus, for $n \geqq 3$, Postulate I_n really gives too much freedom to the variables $p_1, p_2, \ldots$ $\ldots, p_n, q_1, q_2, \ldots, q_n$ in connection with their movements within E_n, of course, without disturbing the correspondence between p_k's and q_k's. In a particular situation, one may not need the use of all $n!$ permutations of the indices $1, 2, 3, \ldots, n$. Under such circumstances, it seems desirable not to use Postulate I_n but its some strictly weaker form. Our way of weakening Postulate I_n is based upon this idea. We introduce the following definition:

Definition. Let E be a non-empty set and $E^n = \underbrace{E \times E \times \ldots \times E}_{n-\text{times}}$, $n \geqq 2$ an integer. A function $f: D \to \mathbb{R} =]-\infty, +\infty[$, $D \subset E^n \times E^n$ is said to be an extreme-symmetric function over the domain D if

$$f(x_1, x_2, \ldots, x_{n-1}, x_n; y_1, y_2, \ldots, y_{n-1}, y_n) =$$
$$= f(x_n, x_2, \ldots, x_{n-1}, x_1; y_n, y_2, \ldots, y_{n-1}, y_1)$$

for all

$$(x_1, x_2, \ldots, x_n; y_1, y_2, \ldots, y_n) \in D .$$

For related work concerning the Shannon entropy, see P. Nath and M. M. Kaur [5].

3. SYSTEM OF POSTULATES

Let

$$f(x, y) = E_2(x, 1 - x; y, 1 - y) \tag{3}$$

where f is a real-valued function with domain

$$J =]0, 1[\times]0, 1[\cup \{(0, y) : 0 \leqq y < 1\} \cup \{(1, y') : 0 < y' \leqq 1\} .$$

We assume the following postulates:

Postulate II. The mapping $(x, y) \to f(x, y)$ is continuous at the origin.

Postulate III$_n$. For all probability distribution $(p_1, p_2, \ldots, p_n) \in \Gamma_n$ with $p_1 + p_2 > 0$, $(q_1, q_2, \ldots, q_n) \in \Gamma_n$ such that $(p_1, p_2, \ldots, p_n; q_1, q_2, \ldots, q_n) \in S_n$,

$$
\begin{aligned}
(4) \qquad & E_n(p_1, p_2, \ldots, p_n; q_1, q_2, \ldots, q_n) = \\
& = E_{n-1}(p_1 + p_2, p_3, \ldots, p_n; q_1 + q_2, q_3, \ldots, q_n) + \\
& + (p_1 + p_1) E_2\left(\frac{p_1}{p_1 + p_2}, \frac{p_2}{p_1 + p_2}; \frac{q_1}{q_1 + q_2}, \frac{q_2}{q_1 + q_2}\right), \quad p_1 + p_2 > 0
\end{aligned}
$$

Notice that, in (4) there is no need to mention $q_1 + q_2 > 0$ because, in S_n, $p_1 + p_2 > 0 \Rightarrow q_1 + q_2 > 0$.

Postulate III$_n$ is not applicable when $p_1 + p_2 = 0$, that is, $p_1 = 0$, $p_2 = 0$. In such a situation, we assume the following:

Postulate IV$_n$. For all probability distributions of the form $(0, 0, p_3, \ldots, p_n) \in \Gamma_n$, $(q_1, q_2, q_3, \ldots, q_n) \in \Gamma_n$ with $0 \leqq q_1 + q_2 < 1$, such that $(0, 0, p_3, \ldots, p_n; q_1, q_2, q_3, \ldots, q_n) \in S_n$,

$$
(5) \quad E_n(0, 0, p_3, \ldots, p_n; q_1, q_2, q_3, \ldots, q_n) = E_{n-1}(0, p_3, \ldots, p_n; q_1 + q_2, q_3, \ldots, q_n) .
$$

Postulate IV$_n$ tells us that if, in a certain experiment, each of the first two events is of probability zero, then these may be combined and their corresponding asserted probabilities may be pooled together. In doing so, the average amount of directed divergence does not undergo any change.

Instead of Postulate I$_n$, we assume the following:

Postulate V$_n$ (Extreme-Symmetry). $E_n : S_n \to \mathbb{R}$ is extreme-symmetric over S_n, that is,

$$
\begin{aligned}
(6) \qquad & E_n(p_1, p_2, \ldots, p_{n-1}, p_n; q_1, q_2, \ldots, q_{n-1}, q_n) = \\
& = E_n(p_n, p_2, \ldots, p_{n-1}, p_1; q_n, q_2, \ldots, q_{n-1}, q_1)
\end{aligned}
$$

Postulate V$_n$ says that the value of E_n remains unaltered if the order of finding the probability estimates of the first and the last event is reversed. Also, from (6), it is quite obvious that Postulate V$_n$ makes use of only *two* permutations of the indices $1, 2, \ldots, n - 1, n$, namely, the identity permutation $1, 2, \ldots, n - 1, n$ and the permutation $n, 2, 3, \ldots, n - 1, 1$. *Notice that Postulate I_n allows us to make use of $n!$ permutations of $1, 2, \ldots, n - 1, n$.*

Postulates I_2 and V_2 are equivalent to each other. Hence, it makes no sense to assume V$_n$ for $n = 2$. For $n \geqq 3$, Postulate V$_n$ is weaker than I$_n$ in the strict sense.

Example. Define $F_n : S_n \to \mathbb{R}$, $n = 3, 4, \ldots$ as

$$F_n(p_1, p_2, \ldots, p_n; q_1, q_2, \ldots, q_n) = p_1 q_1 + p_n q_n \,.$$

Then F_n satisfies Postulate V_n but not I_n. Thus, Postulate V_n is weaker than I_n in the strict sense.

Postulate VI. $E_2(1, 0; \frac{1}{2}, \frac{1}{2}) = 1$.

Postulate VII. $E_2(\frac{1}{2}, \frac{1}{2}; \frac{1}{2}, \frac{1}{2}) = 0$.

4. A CHARACTERIZATION THEOREM

The main result of this paper is the following theorem:

Theorem. Let $E_n : S_n \to \mathbb{R}$, $n = 2, 3, \ldots$ satisfy Postulates II, III_n $(n = 3, 4, \ldots)$, IV_n $(n = 4, 5, \ldots)$, V_n $(n = 2m, 2m - 1)$ for some fixed integer $m \geqq 2$, VI and VII. Then, E_n is of the form (1).

The proof of this theorem needs several results which we put in the form of some lemmas. The notation $A \overset{(a)}{\underset{(b)}{}} B$, henceforth, will mean that B is obtained from A by first applying (a) and then (b).

Lemma 1. Postulates III_n $(n = 3, 4, \ldots)$, IV_n $(n = 4, 5, \ldots)$ and V_n $(n = 2m, 2m - 1)$ for a fixed integer $m \geqq 2$, imply

(7) $$E_2(1, 0; 1, 0) = E_2(0, 1; 0, 1) = 0 \,.$$

(8) $$E_{n+j}(p_1, p_2, \ldots, p_n, \underbrace{0, 0, \ldots, 0}_{j-\text{times}}; q_1, q_2, \ldots, q_n, \underbrace{0, 0, \ldots, 0}_{j-\text{times}}) = E_n(p_1, p_2, \ldots, p_n; q_1, q_2, \ldots, q_n) \,,$$

$$p_1 + p_2 > 0 \,, \quad j = 1, 2, 3, \ldots; n = 2, 3, \ldots$$

(9) $$E_{n+j}(p_1, p_2, \ldots, p_k, \underbrace{0, 0, \ldots, 0}_{j-\text{times}}, p_{k+1}, \ldots, p_n; q_1, q_2, \ldots, q_k, \underbrace{0, 0, \ldots, 0}_{j-\text{times}}, q_{k+1}, \ldots, q_n) = E_n(p_1, p_2, \ldots, p_n; q_1, q_2, \ldots, q_n) \,,$$

$$p_1 > 0 \,, \quad n = 2, 3, \ldots; \quad j = 1, 2, \ldots; \quad k = 1, 2, \ldots, n - 1 \,.$$

(10) $$E_{n+j}(\underbrace{0, 0, \ldots, 0}_{j-\text{times}}, 0, p_2, \ldots, p_n; \underbrace{0, 0, \ldots, 0}_{j-\text{times}}, q_1, q_2, \ldots, q_n) = E_n(0, p_2, \ldots, p_n; q_1, q_2, \ldots, q_n) \,, \quad n = 3, 4, \ldots; \quad j = 1, 2, \ldots.$$

(11) $$E_{n+j+l}(\underbrace{0, 0, \ldots, 0}_{j-\text{times}}, p_1, p_2, \ldots, p_k, \underbrace{0, 0, \ldots, 0}_{l-\text{times}}, p_{k+1}, \ldots, p_n;$$
$$\underbrace{0, 0, \ldots, 0}_{j-\text{times}}, q_1, q_2, \ldots, q_k, \underbrace{0, 0, \ldots, 0}_{l-\text{times}}, q_{k+1}, \ldots, q_n) =$$
$$= E_n(p_1, p_2, \ldots, p_n; q_1, q_2, \ldots, q_n),$$

$$p_1 > 0, \quad n = 2, 3, \ldots; \quad j = 1, 2, \ldots; \quad l = 0, 1, 2, \ldots; \quad k = 1, 2, \ldots, n-1.$$

Proof. Fix $m \geqq 2$ arbitrarily. Then, by Postulates IV_n $(n \geqq 3)$ and V_n $(n = 2m, 2m - 1)$,

$$\begin{aligned} E_{2m-1}&(0, 0, \ldots, 0, 1; 0, 0, \ldots, 0, 1) = \\ &\overset{(5)}{=} E_{2m}(0, 0, 0, \ldots, 0, 1; 0, 0, 0, \ldots, 0, 1) \\ &\overset{(6)}{=} E_{2m}(1, 0, 0, \ldots, 0, 0; 1, 0, 0, \ldots, 0, 0) \\ &\overset{(4)}{=} E_{2m-1}(1, 0, \ldots, 0, 0; 1, 0, \ldots, 0, 0) + E_2(1, 0; 1, 0) \\ &\overset{(6)}{=} E_{2m-1}(0, 0, \ldots, 0, 1; 0, 0, \ldots, 0, 1) + E_2(1, 0; 1, 0). \end{aligned}$$

Hence

(12) $$E_2(1, 0; 1, 0) = 0.$$

Also, by Postulate V_{2m},

(13) $$E_{2m}(\tfrac{1}{2}, \tfrac{1}{2}, \underbrace{0, 0, \ldots, 0, 0}_{(2m-2)}; \tfrac{1}{2}, \tfrac{1}{2}, \underbrace{0, 0, \ldots, 0, 0}_{(2m-2)}) =$$
$$= E_{2m}(0, \tfrac{1}{2}, 0, 0, \ldots, 0, \tfrac{1}{2}; 0, \tfrac{1}{2}, 0, 0, \ldots, 0, \tfrac{1}{2}).$$

By applying repeatedly Postulate III_n for $n = 2m, 2m - 1, \ldots, 3$, the LHS of (13) reduces to

$$(2m - 2)\, E_2(1, 0; 1, 0) + E_2(\tfrac{1}{2}, \tfrac{1}{2}; \tfrac{1}{2}, \tfrac{1}{2}).$$

On the other hand, after applying repeatedly Postulate III_n for $n = 2m, 2m - 1, \ldots$ $\ldots, 3$, the RHS of (13) reduces to

$$\frac{2m - 3}{2}\, E_2(1, 0; 1, 0) + \tfrac{1}{2}\, E_2(0, 1; 0, 1) + E_2(\tfrac{1}{2}, \tfrac{1}{2}; \tfrac{1}{2}, \tfrac{1}{2}).$$

Consequently, (13) reduces to

(14) $$(2m - 1)\, E_2(1, 0; 1, 0) = E_2(0, 1; 0, 1).$$

From (12) and (14), we obtain $E_2(0, 1; 0, 1) = 0$. Thus, (7) is proved.

Equations (8) and (9) follow by the successive application of Postulate III_{n+b}, $b = j, j - 1, \ldots, 1$; $n = 2, 3, \ldots$ and (7).

Equation (10) follows by the successive application of Postulate IV_{n+b}, $b = j$, $j - 1, \ldots, 1$; $n = 3, 4, \ldots$.

Equation (11), for $j = 1$ and $l = 0$, is a consequence of Postulate III_n $(n = 3, 4, \ldots)$ and (7). For $j > 1$ and $l \geqq 1$, it follows from Postulate III_n, (8), (9) and (10). □

Lemma 2. Postulates III_n $(n = 3, 4, \ldots)$, IV_n $(n = 4, 5, \ldots)$ and V_n $(n = 2m, 2m - 1)$, for a fixed integer $m \geqq 2$, imply

$$E_2(p_1, p_2; q_1, q_2) = E_2(p_2, p_1; q_2, q_1) \tag{15}$$

$$E_3(p_1, p_2, p_3; q_1, q_2, q_3) = E_3(p_2, p_1, p_3; q_2, q_1, q_3) \tag{16}$$

$$E_3(p_1, p_2, p_3; q_1, q_2, q_3) = E_3(p_1, p_3, p_2; q_1, q_3, q_2) . \tag{17}$$

Proof. To prove (15), we have the following three cases:

Case 1. $p_1 = 1$, $p_2 = 0$. Then

$$\begin{aligned} E_2(p_1, p_2; q_1, q_2) = E_2(1, 0; q_1, q_2) &= \\ &\overset{(11)}{=} E_{2m}(0, 0, \ldots, 0, 1, 0; 0, 0, \ldots, 0, q_1, q_2) \\ &\overset{(6)}{=} E_{2m}(0, 0, \ldots, 0, 1, 0; q_2, 0, \ldots, 0, q_1, 0) \\ &= E_3(0, 1, 0; q_2, q_1, 0) \quad \text{by repeated use of (5)} \\ &\overset{(8)}{=} E_2(0, 1; q_2, q_1) = E_2(p_2, p_1; q_2, q_1) . \end{aligned}$$

Case 2. $p_1 = 0$, $p_2 = 1$. The proof is similar to that of case 1.

Case 3. $0 < p_1 < 1, 0 < p_2 < 1$. In this case, we must have $0 < q_i < 1$, $i = 1, 2$. Now

$$\begin{aligned} E_2(p_1, p_2; q_1, q_2) &\overset{(9)}{=} E_{2m}(p_1, 0, 0, \ldots, 0, p_2; q_1, 0, 0, \ldots, 0, q_2) \\ &\overset{(6)}{=} E_{2m}(p_2, 0, 0, \ldots, 0, p_1; q_2, 0, 0, \ldots, 0, q_1) \\ &\overset{(9)}{=} E_2(p_2, p_1; q_2, q_1) . \end{aligned}$$

To prove (16), the following two cases arise:

Case 1. $p_1 + p_2 = 0$. Then $p_1 = 0$, $p_2 = 0$ and $p_3 = 1$. Consequently $q_2 + q_3 > > 0$ because q_3 must be positive. Now

$$\begin{aligned} E_3(p_1, p_2, p_3; q_1, q_2, q_3) = E_3(0, 0, 1; q_1, q_2, q_3) &= \\ &\overset{(10)}{=} E_{2m}(0, \ldots, 0, 0, 1; 0, \ldots, q_1, q_2, q_3) \\ &\overset{(6)}{=} E_{2m}(1, \ldots, 0, 0, 0; q_3, \ldots, q_1, q_2, 0) \\ &\overset{(8)}{=} E_{2m-1}(1, \ldots, 0, 0; q_3, \ldots, q_1, q_2) \\ &\overset{(6)}{=} E_{2m-1}(0, \ldots, 0, 1; q_2, \ldots, q_1, q_3) \\ &= E_3(0, 0, 1; q_2, q_1, q_3) \quad \text{by the repeated use of (5)} \\ &= E_3(p_2, p_1, p_3; q_2, q_1, q_3). \end{aligned}$$

Case 2. $0 < p_1 + p_2 \leqq 1$. Then, we must have $0 < q_1 + q_2 \leqq 1$ and (16) follows from Postulate III_3 and (15). Now we prove (17). In this case, the following two cases arise:

Case 1. $0 < p_3 \leqq 1$. Then, we must have $0 < q_3 \leqq 1$. Now

$$E_3(p_1, p_2, p_3; q_1, q_2, q_3) = E_{2m}(0, \ldots, p_1, p_2, p_3; 0, \ldots, q_1, q_2, q_3), \quad p_1 \in [0, 1 - p_3]$$
$$\overset{(6)}{=} E_{2m}(p_3, \ldots, p_1, p_2, 0; q_3, \ldots, q_1, q_2, 0)$$
$$\overset{(8)}{=} E_3(p_3, p_1, p_2; q_3, q_1, q_2)$$
$$\overset{(16)}{=} E_3(p_1, p_3, p_2; q_1, q_3, q_2).$$

Case 2. $p_3 = 0$. Then $p_1 + p_2 = 1$. Hence, at least one, out of p_1 and p_2, must be positive.On account of (16), we may assume that $p_1 > 0$. Then $q_1 > 0$. Now

$$E_3(p_1, p_2, p_3; q_1, q_2, q_3) = E_3(p_1, p_2, 0; q_1, q_2, q_3), \quad p_1 > 0, q_1 > 0$$
$$\overset{(16)}{=} E_3(p_2, p_1, 0; q_2, q_1, q_3)$$
$$\overset{(8)}{=} E_{2m}(p_2, p_1, 0, \ldots, 0; q_2, q_1, q_3, \ldots, 0)$$
$$\overset{(6)}{=} E_{2m}(0, p_1, 0, \ldots, p_2; 0, q_1, q_3, \ldots, q_2)$$
$$\overset{(11)}{=} E_3(p_1, 0, p_2; q_1, q_3, q_2)$$
$$= E_3(p_1, p_3, p_2; q_1, q_3, q_2).$$

This completes the proof of Lemma 2. □

Lemma 3. Postulates III_n $(n = 3, 4, \ldots)$, IV_n $(n = 4, 5, \ldots)$ and V_n $(n = 2m, 2m - 1)$, for a fixed integer $m \geqq 2$, imply that E_n is symmetric, in the sense of (2), for $n = 2, 3, 4, \ldots$.

Proof. The symmetry of E_2 follows from (15). Equation (16) and (17) imply the symmetry of E_3. We prove (2), for $n \geqq 4$, by induction on n. We assume that E_j is symmetric, in the sense of (2), for a *fixed* value of j, say $j = n \geqq 3$ and then prove that E_{n+1} is symmetric. To do so, it is enough to prove the following (for $n + 1 \geqq 4$):

(18)
$$E_{n+1}(p_1, p_2, \ldots, p_{n+1}; q_1, q_2, \ldots, q_{n+1}) =$$
$$= E_{n+1}(p_2, p_1, \ldots, p_{n+1}; q_2, q_1, \ldots, q_{n+k})$$

(19)
$$E_{n+1}(p_1, p_2, p_3, \ldots, p_{n+1}; q_1, q_2, q_3, \ldots, q_{n+1}) =$$
$$= E_{n+1}(p_1, p_3, p_2, \ldots, p_{n+1}; q_1, q_3, q_2, \ldots, q_{n+1})$$

(20)
$$E_{n+1}(p_1, p_2, p_3, \ldots, p_{n+1}; q_1, q_2, q_3, \ldots, q_{n+1}) =$$
$$= E_{n+1}(p_1, p_2, p_{\pi(3)}, \ldots, p_{\pi(n+1)}; q_1, q_2, q_{\pi(3)}, \ldots, q_{\pi(n+1)})$$

where π is an arbitrary permutation of $3, 4, \ldots, n + 1$.

Equation (18) is obvious if $p_1 + p_2 = 0$. If $p_1 + p_2 > 0$, then it follows from Postulate III_{n+1} and (15).

Equation (20) follows from Postulate III_{n+1} and the induction hypothesis if $p_1 + p_2 > 0$. It follows from (5) and the induction hypothesis if $p_1 + p_2 = 0$.

To prove (19), the following two cases arise:

Case 1. $0 < p_1 + p_2 \leqq 1$. Then $0 < q_1 + q_2 \leqq 1$. In this case, (19) can be proved by proceeding as on page 60 in the book of J. Aczél and Z. Daróczy [3]. The details are omitted.

Case 2. $p_1 + p_2 = 0$. Then, we must have $0 \leqq q_1 + q_2 < 1$. Now

$$\begin{aligned} E_{n+1}(p_1, p_2, p_3, \ldots, p_{n+1}; q_1, q_2, q_3, \ldots, q_{n+1}) &= \\ &= E_{n+1}(0, 0, p_3, \ldots, p_{n+1}; q_1, q_2, q_3, \ldots, q_{n+1}) \\ &\overset{(5)}{=} E_{n+1}(0, 0, 0, p_3, \ldots, p_{n+1}; 0, q_1, q_2, q_3, \ldots, q_{n+1}) \\ &= E_{n+2}(0, 0, p_3, 0, \ldots, p_{n+1}; 0, q_1, q_3, q_2, \ldots, q_{n+1}) \quad \text{by (20)} \\ &\overset{(5)}{=} E_{n+1}(0, p_3, 0, \ldots, p_{n+1}; q_1, q_3, q_2, \ldots, q_{n+1}) \\ &= E_{n+1}(p_1, p_3, p_2, \ldots, p_{n+1}; q_1, q_3, q_2, \ldots, q_{n+1}) . \end{aligned}$$

This completes the proof of Lemma 3. □

Proof of the main theorem. From (3) and (15), it follows that

$$f(x, y,) = f(1 - x, 1 - y), \quad (x, y) \in J . \tag{21}$$

From (7) and (21), we obtain

$$f(0, 0) = f(1, 1) = 0 . \tag{22}$$

Also, making use of (3), (21) and Lemma 3 (we need only the symmetry of E_3), it is easy to derive the functional equation

$$f(x, y) + (1 - x) f\left(\frac{u}{1 - x}, \frac{v}{1 - y}\right) = f(u, v) + (1 - u) f\left(\frac{x}{1 - u}, \frac{y}{1 - v}\right) \tag{23}$$

$$x, y, u, v \in [0, 1[\quad \text{with} \quad x + y, u + v \in [0, 1] .$$

Defining

$$\Phi : \{(m, p) : m \in \mathbb{N}, p \in \mathbb{N}, p \geqq m\} \to \mathbb{R}, \quad \mathbb{N} = \{1, 2, 3, \ldots\}$$

as

$$\begin{aligned} \Phi(m, p) &= E_p\left(\frac{1}{m}, \frac{1}{m}, \ldots, \frac{1}{m}, 0, 0, \ldots, 0; \frac{1}{p}, \frac{1}{p}, \ldots, \frac{1}{p}\right) && \text{if } p \geqq 2 \\ &= 0 && \text{if } p = 1 \end{aligned}$$

and making use of the papers of A. Hobson [1], D. K. Fadeev [2], P. L. Kannappan and P. N. Rathie [5], the form of $f(x, y)$ for all $(x, y) \in J$ can be found out. Once the form of $f(x, y)$ is known, by making use of Postulates III_n $(n = 3, 4, \ldots)$, IV_n $(n = 4, 5, \ldots)$, VI and VII, equation (1) follows. The details are omitted for the sake of brevity.

COMMENTS

The proof of our theorem makes an extensive use of probability distributions which contain zeros. If, in $E_n(p_1, p_2, \ldots, p_n; q_1, q_2, \ldots, q_n)$, we have $p_i = q_i = 0$, $i = 1, 2, \ldots, j, j \geqq 2$, then exactly $(j - 1)$ zeros can be omitted with the aid of Postulate IV_n; and if $p_1 = q_1 = 0$, $p_2 > 0$ or $p_1 > 0$, $p_2 = q_2 = 0$, then such a 0 can be removed with the aid of Postulate III_n provided we are in a position to prove (7) whose proof involves the use of probability distributions with zero elements. It is, in this way, that Postulates III_n and IV_n enable us to remove (even add) the desired number of zeros at the appropriate places.

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