KYBERNETIKA — VOLUME 20 (1984), NUMBER 4

SPECTRAL ANALYSIS OF ARMA PROCESSES BY PRONY'S METHOD

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In this paper a method based on the ARMA process autocovariance coefficients fitting to the exponential model is presented. It is shown that the parameters of the exponential model can be estimated by the extended Prony's algorithm which requires solving two systems of linear equations and usual methods for finding polynomial roots. Futhermore, it is shown that the spectral density of ARMA process can be computed directly from the parameters of the exponential model. Two numerical examples demonstrate that the presented method can give good spectral estimations even in the cases where classical methods based on the estimates of the covariance coefficients give no results.

1. INTRODUCTION

We shall consider an ARMA process defined by

$$\sum_{j=0}^{p} a_j x_{t-j} = \sum_{k=0}^{q} b_k e_{t-k}, \tag{1}$$

where $t = \ldots -1, 0, 1, \ldots, a_0, a_1, \ldots, a_p$ are real autoregressive parameters ($a_0 = 1$, $a_p \neq 0$), $b_0, b_1, \ldots, b_q$ are moving average parameters ($b_0 = 1$, $b_q \neq 0$), $\{e_t, t = \ldots -1, 0, 1, \ldots\}$ are independent random variables with zero mean and the variance $\mathsf{E}(e_t^2) = \sigma^2 > 0$, p and q are integer parameters. Usually we suppose that the polynomials

$$A(z) = \sum_{j=0}^{p} a_j z^{p-j}, \tag{2}$$

$$B(z) = \sum_{k=0}^{q} b_k z^{q-k} \tag{3}$$

have all their zeros inside the unit circle. Without any substantial loss of generality we shall further suppose that the polynomials $A(z)$ and $B(z)$ have neither identical nor multiple roots, and that $p > q$. The spectral density of the ARMA process is

$$f(\lambda) = \frac{\sigma^2}{2\pi} \frac{|B(e^{i\lambda})|^2}{|A(e^{i\lambda})|^2}, \tag{4}$$

where $-\pi < \lambda \leqq \pi$. Obviously $f(-\lambda) = f(\lambda)$. Let $c(l) = \mathsf{E}(x_t x_{t-l})$ be autocovariance coefficients of the ARMA process (1), where $c(-l) = c(l)$, $l = 0, 1, \ldots$. From (1) it can be easily derived that (see [1])

$$\begin{bmatrix} c(q), & c(q-1), \ldots, c(q-p+1) \\ c(q+1), & c(q), \ldots, c(q-p+2) \\ \vdots & \vdots \\ c(q+p-1), & c(q+p-2), \ldots, c(q) \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_p \end{bmatrix} = - \begin{bmatrix} c(q+1) \\ c(q+2) \\ \vdots \\ c(q+p) \end{bmatrix} \tag{5}$$

Thus we can obtain autoregressive parameters by solving the system of linear equations with the Toeplitz matrix (5). If we know the parameters $a_1, a_2, \ldots, a_p$, we can define the process $\{y_t, t = \ldots -1, 0, 1, \ldots\}$ by

$$y_t = \sum_{j=0}^{p} a_j x_{t-j} \,. \tag{6}$$

From (1) we get

$$y_t = \sum_{k=0}^{q} b_k e_{t-k} \,. \tag{7}$$

Thus the process y_t is an MA process (Moving Average process). Putting $c_y(l) = \mathsf{E}(y_t y_{t-l})$ we can easily get

$$c_y(l) = \begin{cases} \sigma^2 \sum_{k=0}^{q-l} b_k b_{k+l} & \text{for} \quad 0 \leqq l \leqq q \\ 0 \quad \text{for} \quad l > q \,. \end{cases} \tag{8}$$

Classical identification of ARMA parameters from the autocovariance coefficients is based on solving equations (5) and (8) where the autocovariance coefficients are replaced by their estimates computed as

$$\hat{c}(l) = \frac{1}{N-l} \sum_{t=1}^{N-l} x_t x_{t+l} \quad \text{for} \quad (5) \tag{9}$$

and

$$\hat{c}_y(l) = \frac{1}{N-p-l} \sum_{t=1}^{N-p-l} y_{t+p} y_{t+p+l} \quad \text{for} \quad (8) \tag{10}$$

where $l = 0, 1, \ldots L-1$, $L \leqq N$, $\{x_t, t = 1, 2, \ldots N\}$ is the given realization of the process (1). Nonlinear system of equations (8) can be solved, for example, by Wilson's method [2]. Spectral analysis is finished by substitution of the obtained parameters into theoretical relation (4) for the spectral density.

2. SPECTRAL ANALYSIS BY PRONY'S METHOD

We shall start from the classical definition of the spectral density of the stationary process

$$f(\lambda) = \frac{1}{2\pi} \sum_{l=-\infty}^{+\infty} c(l)\, e^{-i\lambda l}, \tag{11}$$

where $-\pi < \lambda \leqq \pi$. Autocovariance coefficients can be expressed by the relation

$$c(l) = \int_{-\pi}^{+\pi} f(\lambda)\, e^{i\lambda l}\, d\lambda. \tag{12}$$

Substituting (4) into (12) we have

$$c(l) = \int_{-\pi}^{+\pi} \frac{\sigma^2 |B(e^{i\lambda})|^2}{2\pi |A(e^{i\lambda})|^2} e^{i\lambda l}\, d\lambda = \frac{\sigma^2}{2\pi i} \oint_{|z|=1} \frac{B(z)\, B(1/z)}{A(z)\, A(1/z)} z^{l-1}\, dz \tag{13}$$

where $z = e^{i\lambda}$.

Let us put

$$A_1(z) = z^p A(1/z) = \sum_{j=0}^{p} a_j z^j$$

$$A_2(z) = \frac{dA_1(z)}{dz} = \sum_{j=0}^{p-1} a_j (p - j)\, z^{p-j-1} \tag{14}$$

$$B_1(z) = z^q B(1/z) = \sum_{k=0}^{q} b_k z^k.$$

Then

$$c(l) = \frac{\sigma^2}{2\pi i} \oint_{|z|=1} \frac{B(z)\, B_1(z)}{A(z)\, A_1(z)} z^{l+p-q-1}\, dz. \tag{15}$$

If we suppose that the polynomials $A(z)$ and $B(z)$ have neither identical nor multiple roots and that $p > q$, only zeros z_m of $A(z)$ contribute to the contour integral and thus

$$c(l) = \sigma^2 \sum_{m=1}^{p} \frac{B(z_m)\, B_1(z_m)}{A_2(z_m)\, A_1(z_m)} z_m^{l+p-q-1} \tag{16}$$

where $l = 0, 1, \ldots$. Relation (16) can be rewritten by

$$c(l) = \sum_{m=1}^{p} d_m z_m^l \tag{17}$$

where d_m, $m = 1, 2, \ldots, p$ are complex parameters and z_m, $m = 1, 2, \ldots, p$ are the roots of $A(z)$. By the backward substitution (17) into (11) we get

$$f(\lambda) = \frac{1}{2\pi} \sum_{m=1}^{p} d_m \frac{(1 - z_m^2)}{(1 - z_m e^{i\lambda})(1 - z_m e^{-i\lambda})}. \tag{18}$$

If we know the parameters d_m, z_m, $m = 1, 2, \ldots, p$ of the exponential model (17), the spectral density can be calculated from the relation (18).

Now we consider the problem of the identification of parameters of the exponential model from the estimates of autocovariance coefficients $\hat{c}(l)$, $l = 0, 1, \ldots, L - 1$ by extended Prony's method (see [3]). It can be easily shown that

$$\sum_{j=0}^{p} a_j \, c(t - j) = 0 \tag{19}$$

where $t = p, p + 1, \ldots$. Thus the parameters $a_1, \ldots, a_p$ can be obtained from the autocovariance coefficients by solving the system of linear equations (19). After finding the roots of $A(z)$, the parameters $d_1, d_2, \ldots, d_p$ can be obtained by solving the system of linear equations (17). In practical situations we have only estimates $\hat{c}(l)$, $l = 0, 1, \ldots, L - 1$ and therefore we shall minimize

$$\sum_{t=p}^{L-1} \varepsilon_t^2 \tag{20}$$

where

$$\varepsilon_t = \sum_{j=0}^{p} a_j \hat{c}(t - j) \,. \tag{21}$$

Thus we solve the system of linear equations

$$\sum_{j=0}^{p} a_j \Big(\sum_{t=p}^{L-1} \hat{c}(t - j) \, \hat{c}(t - k) \Big) = 0 \tag{22}$$

where $k = 1, 2, \ldots, p$. The parameters $d_1, \ldots, d_p$ can be obtained by minimizing

$$\sum_{l=0}^{L-1} \Big(\hat{c}(l) - \sum_{m=1}^{p} d_m z_m^l \Big)^2 \,, \tag{23}$$

which leads to the solution of the system of linear equations

$$\mathbf{d} = (\boldsymbol{\Phi}^{\mathrm{H}} \boldsymbol{\Phi})^{-1} \, \boldsymbol{\Phi}^{\mathrm{H}} \hat{\mathbf{c}} \tag{24}$$

where

$$\mathbf{d}^{\mathrm{T}} = (d_1, d_2, \ldots, d_p) \,, \quad \hat{\mathbf{c}}^{\mathrm{T}} = (\hat{c}(0), \hat{c}(1), \ldots, \hat{c}(L - 1)) \,,$$

$^{\mathrm{H}}$ denotes complex conjugative transpose and $\boldsymbol{\Phi}$ is the $L \times p$ matrix

$$\boldsymbol{\Phi} = \begin{bmatrix} 1, & 1, & \ldots, & 1 \\ z_1, & z_2, & \ldots, & z_p \\ \vdots & & \vdots & \\ z_1^{L-1}, & z_2^{L-1}, & \ldots, & z_p^{L-1} \end{bmatrix} . \tag{25}$$

If $\gamma_{k,l}$ is the element of matrix $\boldsymbol{\Phi}^{\mathrm{H}} \boldsymbol{\Phi}$ which is in the kth row and in the lth column, then

$$\gamma_{k,l} = \frac{(z_k^* z_l)^L - 1}{z_k^* z_l - 1} \,. \tag{26}$$

Relation (26) reduces the computational burden of (24).

3. NUMERICAL RESULTS

The methods presented in this paper was applied to the first $N = 100$ values of realizations of two ARMA processes which are represented in Fig. 1 and Fig. 4. The sequence $\{e_t, t = \dots -1, 0, 1, \dots\}$ was chosen as an uncorrelated Gaussian random sequence with $\mathsf{E}(e_t) = 0$ and $\mathsf{E}(e_t^2) = 1$. Theoretical spectral densities of the given

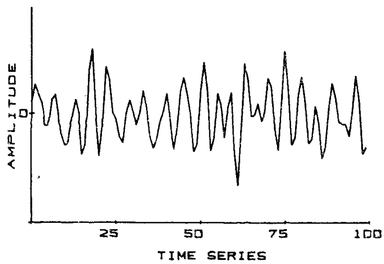

Fig. 1. A realization of the ARMA process with parameters $p = 4$, $q = 3$, $a_1 = -0{\cdot}874$, $a_2 = 1{\cdot}728$, $a_3 = -0{\cdot}677$, $a_4 = 0{\cdot}625$, $b_1 = 0{\cdot}327$, $b_2 = 0{\cdot}524$, $b_3 = 0{\cdot}722$.

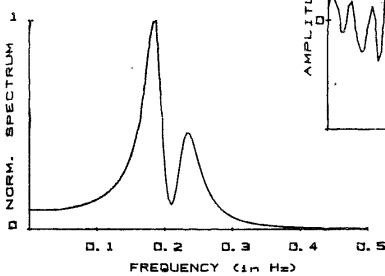

Fig. 2. True spectrum of the ARMA process from Fig. 1.

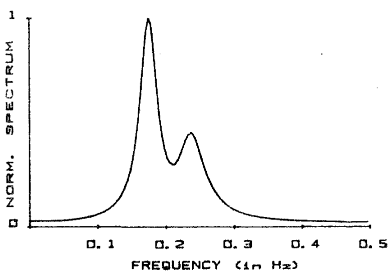

Fig. 3. Spectral estimate by Prony's method of the ARMA process from Fig. 1.

Table 1.

m	a_m	$\lvert z_m \rvert$	f_m	A_m	φ_m
1	−0·945	0·862	0·237	0·990	0·160
2	1·682	0·862	−0·237	0·990	−0·160
3	−0·714	0·907	0·177	1·549	−0·073
4	0·613	0·907	−0·177	1·549	0·073

two processes are in Fig. 2 and Fig. 5 (the horizontal axis corresponds to the frequency $f = \lambda/2\pi$). First $L = 20$ autocovariance coefficients were calculated according to (9). The parameters $a_1 = -0{\cdot}505$, $a_2 = 3{\cdot}199$, $a_3 = -1{\cdot}137$, $a_4 = 1{\cdot}770$ for the first ARMA process were obtained as a solution of the Toeplitz system (5). In this case $A(z)$ has two roots outside the unit circle and the classical method cannot be used. In Table 1 the parameters obtained by Prony's method for the first ARMA process

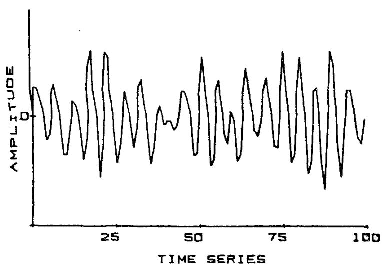

Fig. 4. A realization of the ARMA process with parameters $p = 2$, $q = 1$, $a_1 = -0{\cdot}556$, $a_2 = 0{\cdot}810$, $b_1 = 0{\cdot}800$.

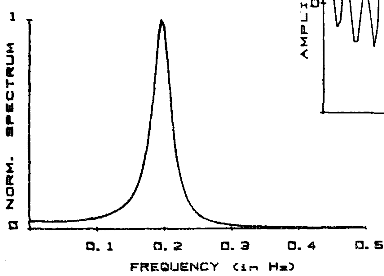

Fig. 5. True spectrum of the ARMA process from Fig. 4.

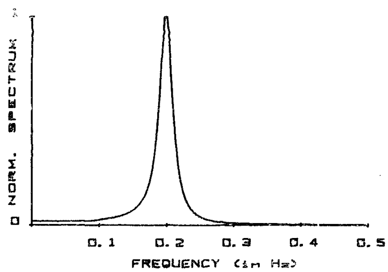

Fig. 6. Spectral estimate by Prony's method of the ARMA process from Fig. 4.

are presented. The parameters a_m were obtained by solving the linear system (22), the parameters z_m are the roots of $A(z)$ and $z_m = |z_m| \, e^{i2\pi f_m}$, $d_m = A_m \, e^{i\varphi_m}$, $m = 1, 2, 3, 4$. The spectral density calculated by (13) is in Fig. 3.

By solving the Toeplitz system (5) for the second ARMA process we obtained the parameters $a_1 = -0{\cdot}559$, $a_2 = 0{\cdot}878$ and by (10) we obtained $\hat{c}_y(0) = 1{\cdot}754$,

$\hat{c}_y(1) = 0{\cdot}912$. But for these values of autocovariance coefficients the system (8) has no real solution (see, for example, [4]). In Table 2 the parameters obtained by Prony's method for the second ARMA process are presented. The spectral density calculated by (13) is in Fig. 6.

Table 2.

	a_m	$\|z_m\|$	f_m	A_m	φ_m
1	−0·561	0·929	0·201	5·079	−0·082
2	0·863	0·929	−0·201	5·079	0·082

4. DISCUSSION

Numerical results clearly indicate that our method can give good spectral estimates even if a classical method cannot be used. This is caused by using all L autocovariance coefficients in the calculation of parameters and by using a relatively simple numerical algorithm. Finally we can mention some disadvantages of our method: Difficulties with the determination of the integer parameter p, the restricting condition $p > q$ and the production a negative spectral density in some cases.

(Received August 29, 1983.)

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