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DUALITY IN VECTOR OPTIMIZATION
Part I. Abstract Duality Scheme

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This is a contribution to the duality theory in optimization theory. A unified approach is presented. The paper is divided into three parts. In the first part an abstract duality scheme is formulated and studied. The well-known duality principles are formulated and proved for this model, too. The second part studies the dual problems in the vector quasiconcave programming. The last part is devoted to the fractional programming.

0. INTRODUCTION

The duality theory of optimization has an extensive literature. This theory may be regarded as the most delicate subject in the optimization theory and its theoretical importance cannot be questioned (e.g. in the theory of prices and markets in economics). In the one-objective optimization there are several approaches to the duality theory: Wolfe's gradient duality [8], the Lagrangian multipliers method, the conjugate function method and the subgradient duality [9]. These approaches are applicable, unfortunately, only for the convex optimization. There are some attempts to extend the duality theory for wider classes of optimization (cf. [10], [11], [12]). Nevertheless, these approaches are not unified and require strong assumptions (convexity, differentiability, constraint, qualification ...). Moreover, most of them are difficult to convert to the vector optimization.

This part of the tripaper presents a unified abstract duality scheme on the basis of which the duality theory for the vector optimization will be built up.

1. OPTIMALITY CONCEPTS

Throughout this work Y denotes the space in which the values of objective operators occur, therefore some optimality concepts for subsets in Y will be defined and discussed in this section.

So let Y be a topological linear space, let $Y_+ \subset Y$ be a convex cone with $(-Y_+) \cap \cap Y_+ = \{\emptyset\}$ and int $Y_+ \neq \emptyset$. We define for any $a, b \in Y$ the following ordering:

$$a > b \quad \text{iff} \quad a - b \in \text{int } Y_+$$

$$a \geqq b \quad \text{iff} \quad a - b \in Y_+$$

$$a \gtrsim b \quad \text{iff} \quad a \geqq b \quad \text{and} \quad a \neq b$$

$$a \mathrel{\overline{\gtrless}} b \quad \text{iff} \quad a = b \quad \text{or} \quad b - a \notin Y_+ .$$

We recall now some elementary notions needed in the further development.

A subset $A \subset Y$ is called to be *bounded from above* if there exists an element $y \in Y$ such that $a \leqq y$ for all $a \in A$; y is then called an *upper bound* for A. An upper bound y^* for A is said to be the *supremum* of A, labelled by sup A, if $y^* \leqq y$ for all upper bounds y for A. Analogously the *boundedness from below*, *lower bounds* and the *infimum* (inf A) are defined. Let $\{x_\alpha\}_{\alpha \in \Lambda}$ (where Λ is an ordered index set) be a net (a generalized sequence) in Y, then the notions $\liminf_\alpha x_\alpha$ and $\limsup_\alpha x_\alpha$ are supposed to be traditionally defined.

Now we are ready to introduce the optimality concepts in question. Given $A \subset Y$ an element $y^* \in A$ is said to be a *strong maximum* of A if $y \gtrsim y^*$ implies $y \notin A$, (the same concept is called *efficient*, *Pareto-optimal* [2] or *extreme points* [3]), a *weak maximum* (*weakly efficient*, *weakly Pareto-optimal* or *Slater optimal* [2]) if $y > y^*$ implies $y \notin A$. The set of all strong (weak) maxima of A will be denoted by $\text{Max}^s A$ ($\text{Max}^w A$).

It is not generally guaranteed that $\text{Max}^s A$ or $\text{Max}^w A$ is nonempty. That is why we take into consideration the following optimality concepts: $y^* \in Y$ is said to be a *strong* (*weak*) *supremal point* of A if (i) $y^* \in \text{Max}^s A (y^* \in \text{Max}^w A)$ or (ii) $y^* \in \bar{A}$ and there is no net $\{y_\alpha\}_{\alpha \in \Lambda} \subset A$ such that $\liminf_\alpha y_\alpha \gtrsim (>) \, y^*$. Let Sup^s ($\text{Sup}^w A$) denote the set of all strong (weak) supremal points of A.

Analogously are defined the *strong* (*weak*) *minimum infimal point* and the sets $\text{Min}^s A$, $\text{Min}^w A$, $\text{Inf}^s A$ and $\text{Inf}^w A$.

The following inclusions are trivial

$$\text{Max}^s A \subset \text{Max}^w A$$

$$\text{Sup}^s A \subset \text{Sup}^w A$$

$$\text{Max}^{s(w)} A \subset \text{Sup}^{s(w)} A$$

$$\text{Min}^s A \subset \text{Min}^w A$$

$$\text{Inf}^s A \subset \text{Inf}^w A$$

$$\text{Min}^{s(w)} A \subset \text{Inf}^{s(w)} A$$

Given now a class of subsets $\{A_\alpha\}_{\alpha \in \Lambda} \subset Y$, where Λ is an index set, it follows immediately

Lemma 1.1.

$$\operatorname{Max}^{s(w)}[\bigcup_{\alpha\in A}\operatorname{Max}^{s(w)} A_\alpha] = \operatorname{Max}^{s(w)}[\bigcup_{\alpha\in A} A_\alpha] \tag{1.1}$$

$$\operatorname{Min}^{s(w)}[\bigcup_{\alpha\in A}\operatorname{Min}^{s(w)} A^s] = \operatorname{Min}^{s(w)}[\bigcup_{\alpha\in A} A_\alpha] \tag{1.2}$$

Lemma 1.2. Suppose that for any $\alpha \in A$, $y \in A_\alpha$ there exists $y' \in \operatorname{Sup}^{s(w)} A_\alpha$ such that $y' \geqq y$, then

$$\operatorname{Sup}^{s(w)}[\bigcup_{\alpha\in A}\operatorname{Sup}^{s(w)} A_\alpha] \subset \operatorname{Sup}^{s(w)}[\bigcup_{\alpha\in A} A_\alpha] \tag{1.3}$$

Proof. Let $y^* \in \operatorname{Sup}^{s(w)}[\bigcup_{\alpha\in A}\operatorname{Sup}^{s(w)} A_\alpha]$, then $y^* \in \overline{\bigcup_{\alpha\in A}\operatorname{Sup}^{s(w)} A_\alpha} \subset \operatorname{Sup}\overline{\bigcup_{\alpha\in A} A_\alpha}$ (the last inclusion can be derived as follows $\operatorname{Sup}^{s(w)} A_\beta \subset \bar{A}_\beta \subset \overline{\bigcup_{\alpha\in A} A_\alpha}\ \forall \beta \in A \Rightarrow \bigcup_{\alpha\in A}\operatorname{Sup}^{s(w)} . A_\alpha \subset \overline{\bigcup_{\alpha\in A} A_\alpha} \Rightarrow \overline{\bigcup_{\alpha\in A}\operatorname{Sup}^{s(w)} A_\alpha} \subset \overline{\bigcup_{\alpha\in A} A_\alpha}$). If $y^* \notin \operatorname{Sup}^{s(w)}[\bigcup_{\alpha\in A} A_\alpha]$ then, by definition, there is a net $\{y_\beta\} \subset \bigcup_{\alpha\in A} A_\alpha$ such that $\liminf_\beta y_\beta \gtrsim (>)\, y^*$. Then according to the assumption of the lemma there exists a $y'_\beta \in \operatorname{Sup}^{s(w)} A_\beta$ with $y'_\beta \geqq y_\beta$ for each β. Obviously $\liminf_\beta y'_\beta \geqq \liminf_\beta y_\beta \gtrsim (>)\, y^*$ and that contradicts the assumption that $y^* \in \operatorname{Sup}^{s(w)}[\bigcup_{\alpha\in A}\operatorname{Sup}^{s(w)} A_\alpha]$. □

Definition. We say that the space Y has the *K-property* if any of its subsets that is bounded from above has a supremum (this property is called K-property in honour of the Soviet mathematician L. V. Kantorovich who pioneeringly studied the so-called Kantorovich's spaces).

Corollary 1. Suppose Y has the K-property and A_α is bounded from above for every $\alpha \in A$. Then the inclusion (1.3) holds.

It is easy to check that spaces of finite dimension have the K-property. So we have

Corollary 2. Suppose Y is of finite dimension and A_α is bounded from above for every $a \in A$. Then the inclusion (1.3) holds.

Lemma 1.3. Suppose $A_\alpha - Y_+ \subset A_\alpha$ for every $\alpha \in A$. Then

$$\operatorname{Sup}^{w}[\bigcup_{\alpha\in A} A_\alpha] \subset \operatorname{Sup}^{w}[\bigcup_{\alpha\in A}\operatorname{Sup}^{w} A_\alpha] \tag{1.4}$$

Proof. Let $y^* \in \operatorname{Sup}^{w}[\bigcup_{\alpha\in A} A_\alpha]$, then there exists a net $\{y_\beta\} \subset \bigcup_{\alpha\in A} A_\alpha$ such that $\lim_\beta y_\beta = y^*$. Without loss of generality we may assume that $y_\beta \leqq y^*$ for all β (otherwise, instead of $\{y_\beta\}$ we choose the net $\{\inf\{y_\beta, y^*\}\} \subset \bigcup_{\alpha\in A} A_\alpha$). Fixing β we consider the segment $[y_\beta, y^*] = \{y_\beta + t(y^* - y_\beta) \mid t \in [0, 1]\}$. Let A_{α_β} be such a set that

$y_\beta \in A_{\alpha_\beta}$ and put

$$y'_\beta = \sup A_{\alpha_\beta} \cap [y_\beta, y^*] \tag{1.5}$$

We assert that $y'_\beta \in \mathrm{Sup}^w A_{\alpha_\beta}$. Indeed, if it is not so, then $y'_\beta \lesssim y^*$ and there is a point $y''_\beta \in A_{\alpha_\beta}$ such that $y''_\beta > y'_\beta$. Then by the assumption of the lemma, $y'_\beta \in \in \mathrm{int}\,\{y \mid y \leqq y''_\beta\} \subset A_{\alpha_\beta}$. Hence the intersection $(y'_\beta, y^*] \cap A_{\alpha_\beta} \neq \emptyset$ that contradicts (1.5).

Evidently $\lim_\beta y'_\beta = y^*$. It remains to prove that there is no net $\{x_\gamma\} \subset \bigcup_{\alpha \in A} \mathrm{Sup}^w A_\alpha$ such that $\liminf_\gamma x_\gamma > y^*$. If it is not so there exists an $x \in \bigcup_{\alpha \in A} \mathrm{Sup}^w A_\alpha$ such that $x > y^*$. Let $x \in \mathrm{Sup}^w A_\alpha$ then there is a net in A_α converging to x. Hence we can choose a point $z \in A_\alpha$ such that $z > y^*$ and this is a contradiction with $y^* \in \mathrm{Sup}^w [\bigcup_{\alpha \in A} A_\alpha]$. □

Lemma 1.4. Suppose that $\bigcup_{\alpha \in A} A_\alpha$ is closed, then the inclusion (1.4) holds.

Proof. Let $y^* \in \mathrm{Sup}^w [\bigcup_{\alpha \in A} A_\alpha]$, then $y^* \in \bigcup_{\alpha \in A} A_\alpha$ for the closedness of $\bigcup_{\alpha \in A} A_\alpha$. Evidently $y^* \in \mathrm{Sup}^w A_\beta \subset \bigcup_{\alpha \in A} \mathrm{Sup}^w A_\alpha$ for some β. The proof that there is no $y \in \bigcup_{\alpha \in A} \mathrm{Sup}^w A_\alpha$ such that $y > y^*$ is the same as that one of Lemma 1.3. □

Remark 1. The same results for minima and weak infimal points may be derived analogously.

Remark 2. The analogous inclusion as (1.4) may not hold for the strong supremal points. We show it by the following example: Assume $Y = \mathbb{R}^2$ we put

$$A_1 = \{(x; y) \mid x < 0 \ \& \ y < 1\}$$

$$A_2 = \{(x; y) \mid x \leqq 0 \ \& \ y \leqq 0\}$$

Then

$$\mathrm{Sup}^w \{A_1 \cup A_2\} = \{(x; y) \mid (x = 0 \ \& \ y \leqq 1) \vee (y = 1 \ \& \ x \leqq 0)\} = = \mathrm{Sup}^w [\mathrm{Sup}^w A_1 \cup \mathrm{Sup}^w A_2] .$$

But

$$\mathrm{Sup}^s [A_1 \cup A_2] = \{(0; 0), (0; 1)\} \neq \{(0; 1)\} = \mathrm{Sup}^s [\mathrm{Sup}^s A_1 \cup \mathrm{Sup}^s A_2] .$$

2. ABSTRACT DUALITY SCHEME

The duality theory in vector optimization, developed by numerous authors (see [2], [3], [4], [5], [6], [7]) concerns exclusively the convex and linear optimization. In [1] Rubinstein presented a new approach with help of which he considerably extended the class of scalar optimization problems, having dual ones. In this section we generalize Rubinstein's approach for the vector optimization. The main results are the strong duality principles for both the weak supremal and the strong maximum problems.

In this section E is a nonempty set, Y is the space introduced in Section 1 and (Λ_*, Λ^*) is an order interval in Y, it means

$$(\Lambda_*, \Lambda^*) = \{y \in Y \mid \Lambda_* < y < \Lambda^*\} .$$

Let us have a system of subsets of E:

$$P, Q_y \quad y \in (\Lambda_*, \Lambda^*)$$

such that

(2.1) $$\forall y', y'' \in (\Lambda_*, \Lambda^*) : y' \leqq y'' \Rightarrow Q_{y''} \subset Q_{y'}$$

We denote

$$Q = \bigcup_{\Lambda_* < y < \Lambda^*} Q_y \quad \text{and} \quad P_0 = P \cap Q$$

Further, given a set $E^* \subset \exp E$ we define

$$P^* = \{F \in E^* \mid P \subset F\}$$

and

$$Q_y = \{F \in E^* \mid Q_{y'} \cap F = \emptyset \ \forall y' \geqq y\} .$$

Obviously

(2.2) $$\forall y', y'' \in (\Lambda_*, \Lambda^*) : y' \leqq y'' \Rightarrow Q^*_{y'} \subset Q^*_{y''}$$

We denote

$$Q^* = \bigcap_{\Lambda_* < y < \Lambda^*} Q^*_y \quad \text{and} \quad P^*_0 = P^* \cap Q^*$$

Further we put

(2.3) $$\mu(a) = \{y \in (\Lambda_*, \Lambda^*) \mid a \in Q_y\}$$

and

(2.4) $$\nu(F) = \{y \in (\Lambda_*, \Lambda^*) \mid F \in Q^*_y\}$$

and we have the following pairs of optimization problems:

Primal Strong (Weak) Supremal Problem $(S_P^{s(w)})$

(2.5) $$\text{find} \quad \text{Sup}^{s(w)} \bigcap_{a \in P_0} \mu(a) = S_P^{s(w)}$$

Dual Strong (Weak) Infimal Problem $(I_d^{s(w)})$

(2.6) $$\text{find} \quad \text{Inf}^{s(w)} \bigcup_{F \in P_0^*} \nu(F) = I_d^{s(w)}$$

Primal Strong (Weak) Maximum Problem $(M_P^{s(w)}$

(2.7) $$\text{find} \quad \text{Max}^{s(w)} \bigcup_{a \in P_0} \mu(a) = M_P^{s(w)}$$

Dual Strong (Weak) Minimum Problem $(M_d^{s(w)})$

(2.8) $$\text{find} \quad \text{Min}^{s(w)} \bigcup_{F \in P_0^*} \nu(F) = M_d^{s(w)}$$

Definition. Points realizing optimum in these problems are called *optimal* and we say that the corresponding problems attain optimum at these optimal points. Nets which realize supremal or infimal optimum in the problems (2.5) and (2.6) are called *optimal nets.*

Theorem 2.1. (Weak Duality Principle.)

$$\forall a \in P_0 \ \forall F \in P_0^* \ \forall y' \in \mu(a) \ \forall y'' \in \nu(F) : y' \gtreqless y''$$

Proof. Let $a \in P_0$, $F \in P_0^*$, $y' \in \mu(a)$ and $y'' \in \nu(F)$, we have then $a \in Q_{y'}$, $a \in P$, $F \in Q_{y''}^*$ and $P \subset F$ which follows that $F \cap Q_y = \emptyset$ for any $y \geqq y''$ and thus $a \notin Q_y$ for any $y \geqq y''$. Hence $y' \gtreqless y''$. □

Corollary 1.

$$[\bigcup_{a \in P_0} \mu(a)] \cap [\bigcup_{F \in P_0^*} \nu(F)] = M_p^s \cap M_d^s = M_p^w \cap M_d^w \tag{2.9}$$

Corollary 2.

$$[\overline{\bigcup_{a \in P_k} \mu(a)}] \cap [\overline{\bigcup_{F \in P_0^*} \nu(F)}] = S_p^w \cap I_d^w$$

Example. Let $E = (-\infty, +\infty)$, $P = [0, 1]$, $Y = \mathbb{R}^2$, $A_* = (0; 0)$, $A^* = (2; 1)$ $y = (y_1; y_2) \in (A_*, A^*)$, $Q_y = [y_1, +\infty)$. $E^* \subset \exp E$ then

$$P_0 = (0, 1], \ Q_y^* = \{F \subset E^* \mid x < y_1 \ \forall x \in F\}$$

and

$$P_0^* = \{F \subset E^* \mid (0, 1] \subset F \ \& \ \sup F < 2\} .$$

Then it is easy to verify that

$$\bigcup_{a \in P_0} \mu(a) = \{y = (y_1; y_2) \mid 0 < y_1 \leqq 1 \ \textit{and} \ 0 < y_2 < 1\} = T_1$$

$$\bigcup_{F \in P_0^*} \nu(F) = \{y = (y_1; y_2) \mid 1 < y_1 < 2 \ \textit{and} \ 0 < y_2 < 1\} = T_2$$

The sets are illustrated in Figure 1.

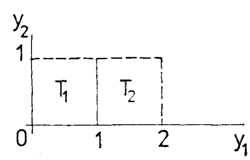

Fig. 1.

We have then

$\mathrm{Max}^s\, T_1 = \emptyset$, $\mathrm{Min}^s\, T_2 = \emptyset$

$\mathrm{Max}^w\, T_1 = \{y = (y_1; y_2) \mid y_1 = 1 \ \textit{and} \ 0 < y_2 < 1\}$

$\mathrm{Min}^w\, T_2 = \emptyset$

$\mathrm{Sup}^s\, T_1 = \{(1; 1)\}$, $\mathrm{Inf}^s\, T_2 = \{(1; 0)\}$

$\mathrm{Sup}^w\, T_1 = \{y = (y_1; y_2) \mid (y_1 = 1 \,\&\, 0 \leqq y_2 \leqq 1) \vee (y_2 = 1 \,\&\, 0 \leqq y_1 \leqq 1)\}$

$\mathrm{Inf}^w\, T_2 = \{y = (y_1; y_2) \mid (y_1 = 1 \,\&\, 0 \leqq y_2 \leqq 1) \vee (y_2 = 0 \,\&\, 1 \leqq y_1 \leqq 2)\}$

We see that

$$\overline{T}_1 \cap \overline{T}_2 = \{y = (y_1; y_2) \mid y_1 = 1 \ and \ 0 \leqq y_2 \leqq 1\} =$$
$$= (\mathrm{Sup}^w \ T_1) \cap (\mathrm{Inf}^w \ T_2) \neq (\mathrm{Sup}^s \ T_1) \cap (\mathrm{Inf}^s T_2) = \emptyset$$

In further development we use the convention

$$\mathrm{Sup}^{s(w)} \emptyset = \Lambda_* \quad \text{and} \quad \mathrm{Inf}^{s(w)} \emptyset = \Lambda^* .$$

Lemma 2.1. If $P \cap [\bigcap_{\Lambda_* < y < \Lambda^*} Q_y] \neq \emptyset$, then

$$S_p^s = I_d^{s(w)} = \{\Lambda^*\}$$

Proof. $P \cap [\bigcap_{\Lambda_* < y < \Lambda^*} Q_y] \neq \emptyset \Rightarrow P_0^* = \emptyset \Rightarrow S_p^s = \{\Lambda^*\} = I_d^{s(w)}$ with regarding to the just made convention. □

Remark. If in the beginning we suppose that y can attain Λ^* then in this case $M_p^s = \{\Lambda\}$ and $M_d^{s(w)} = \emptyset$.

Lemma 2.2. If $P^* \cap [\bigcap_{\Lambda_* < y < \Lambda^*} Q_y^*] \neq \emptyset$, then

$$S_p^{s(w)} = I_d^s = \{\Lambda_*\}$$

Proof. $P^* \cap [\bigcap_{\Lambda_* < y < \Lambda^*} Q_y^*] \neq \emptyset \Rightarrow P_0 = \emptyset \Rightarrow S_p^{s(w)} = I_d^s = \{\Lambda_*\}$

Lemma 2.3.

$$y^* \in M_p^{s(w)} \Leftrightarrow \begin{cases} P \cap Q_{y^*} \neq \emptyset \\ P \cap Q_y \ = \emptyset \ \forall y \gtrsim (>) \, y^* \end{cases}$$

Lemma 2.4.

$$y^* \in M_d^{s(w)} \Leftrightarrow \begin{cases} P^* \cap Q_y^* \neq \emptyset \\ P^* \cap Q_y^* = \emptyset \ \forall y \lesssim (<) \, y^* \end{cases}$$

The proof of Lemma 2.3 and 2.4 is evident

Lemma 2.5.

$$y^* \in S_p^w \Leftrightarrow \begin{cases} P \cap Q_y \neq \emptyset \ \forall y < y^* \\ P \cap Q_y = \emptyset \ \forall y > y^* \end{cases}$$

Proof. The implication $(\Leftarrow)$ is clear. Let now $y^* \in S_p^w$ then obviously $P \cap Q_y = \emptyset$ $\forall y > y^*$. If there is a $y < y^*$ such that $P \cap Q_y = \emptyset$, then there exists a neighbourhood U of y^* such that $[\bigcup_{a \in P_0} \mu(a)] \cap U = \emptyset$. It means $y \notin \overline{\bigcup_{a \in P_0} \mu(a)}$ what contradicts $y^* \in S_p^w$. □

Analogously we can prove

Lemma 2.6.

$$y^* \in I_d^w \Leftrightarrow \begin{cases} P^* \cap Q_y^* \neq \emptyset \quad \forall y > y^* \\ P^* \cap Q_y^* = \emptyset \quad \forall y < y^* \end{cases}$$

Lemma 2.7. Let the following condition be fulfilled

$$[A_1] \qquad \left.\begin{matrix} P \cap Q_y \neq \emptyset \ \forall y < y^* \\ P \cap Q_y = \emptyset \ \forall y > y^* \end{matrix}\right\} \Rightarrow P^* \cap Q_y^* \neq \emptyset \ \forall y > y^*$$

then

$$S_p^w \subset I_d^w$$

Proof. Let $y^* \in S^w$ then by Lemma 2.5 we have

$$P \cap Q_y \neq \emptyset \quad \forall y < y^*$$

$$P \cap Q_y = \emptyset \quad \forall y > y^*$$

Hence in consequence of $[A_1]$ and Corollary 2 we have $y^* \in I_d^w$. □

Lemma 2.8. Suppose

$$[A_2] \qquad P \cap Q_y = \emptyset \Rightarrow P^* \cap Q_{y'}^* \neq \emptyset \quad \forall y' > y .$$

Then

$$I_d^w \subset S_p^w .$$

Proof. Let $y^* \in I_d^w$, then by Lemma 2.6

$$P^* \cap Q_y^* \neq \emptyset \quad \forall y > y^* \tag{2.10}$$

$$P^* \cap Q_y^* = \emptyset \quad \forall y < y^* \tag{2.11}$$

Suppose, on the contrary, that $y^* \notin S_p^w$, then according to Lemma 2.5 we can conclude

(i) there is a $y < y^*$ such that $P \cap Q_y = \emptyset$. Hence by $[A_2]$ there is y' such that $y < y' < y^*$ and $P^* \cap Q_{y'}^* \neq \emptyset$ that contradicts (2.11), or

(ii) there is $y > y^*$ such that $P \cap Q_y \neq \emptyset$ what means $y \in \bigcup_{a \in P_0} \mu(a)$. According to (2.10) we can choose a y' such that $y^* < y' < y$ and $y' \in \bigcup_{F \in P_0^*} \nu(F)$ and that contradicts the weak duality principle. □

For further purpose we formulate the following conditions

$$[A_3] \qquad P \cap Q_y \neq \emptyset \quad \forall y < y^* \Rightarrow P \cap \Big(\bigcap_{y < y^*} Q_y\Big) \neq \emptyset$$

$$[A_4] \qquad P^* \cap Q_y^* \neq \emptyset \quad \forall y > y^* \Rightarrow P^* \cap \Big(\bigcap_{y > y^*} Q_y^*\Big) \neq \emptyset$$

$$[A_5] \qquad P \cap \Big(\bigcap_{\Lambda_* < y < \Lambda^*} Q_y\Big) = \emptyset$$

$$[A_6] \qquad P^* \cap \Big(\bigcap_{\Lambda_* < y < \Lambda^*} Q_y^*\Big) = \emptyset$$

Now we can formulate the celebrated strong dual principle for vector optimization, which is easily proved by Lemma 2.7 Lemma 2.8 and evident considerations.

Theorem 2.2. (Supremal Strong Duality Principle.)

Suppose $P_0 \neq \emptyset$ or $P_0^* \neq \emptyset$, then under conditions $[A_1]$ and $[A_2]$ we have

$$S_p^w = I_d^w$$

If, in addition, $P_0 \neq \emptyset$ & $P_0^* \neq \emptyset$ and conditions $[A_3]$, $[A_4]$ hold, then both problems $[S_p^w]$ and $[I_d^w]$ have optimal solutions.

If conditions $[A_5]$ and $[A_6]$ hold, then in case $P_0 = \emptyset$ or $P_0^* = \emptyset$ there exist no optimal solutions either in $[S_p^w]$ or in $[I_d^w]$.

The following existence theorem and optimality criterion are immediate consequences of Theorem 2.2.

Theorem 2.3. (Existence Theorem).

If conditions $[A_1]-[A_6]$ hold, then the following assertion are equivalent:

1^0 There exist optimal solutions in both problems $[S_p^w]$ and $[I_d^w]$.

2^0 There exist optimal solutions in one of the problems $[S_p^w]$ and $[I_d^w]$.

3^0 There exist feasible solutions in both problems $[S_p^w]$ and $[I_d^w]$ i.e. $P_0 \neq \emptyset$ and $P_0^* \neq \emptyset$.

4^0 $P_0 \neq \emptyset$ and $\mu(a)$ is weakly bounded from above by a value $\beta < \Lambda$ i.e.

$$\forall a \in P_0 \quad \forall y \in \mu(a) \quad y \not> \beta\,.$$

5^0 $P_0^* \neq \emptyset$ and $\nu(F)$ is weakly bounded from below by a value $\alpha < \Lambda$ i.e.

$$\forall F \in P_0^* \quad \forall y \in \nu(F) \quad y \not< \alpha\,.$$

Theorem 2.4. (Optimality Criterion.)

A feasible solution $a \in P_0(F' \in P_0^*)$ is optimal in $[S_p^w]$ (in $[I_d^w]$) if and only if there exists a feasible solution $F \in P_0^*(a' \in P_0)$ such that $\overline{\mu(a)} \cap \overline{\nu(F)} \neq \emptyset(\overline{\mu(a')} \cap \overline{\nu(F')} \neq \emptyset)$.

Furthermore we shall derive analogous results for the dual pair $[M_p^s]$ and $[M_d^s]$.

Lemma 2.9. Suppose

$$[B_1] \qquad \left.\begin{array}{l} P \cap Q_{y^*} \neq \emptyset \\ P \cap Q_y \;\; = \emptyset \quad \forall y \gtrsim y^* \end{array}\right\} \Rightarrow P^* \cap Q_{y^*}^* \neq \emptyset$$

then

$$M_p^s \subset M_d^s\,.$$

Proof. Let $y^* \in M_p^s$ then by Lemma 2.3 and condition $[B_1]$ we have $P^* \cap Q_{y^*}^* \neq \emptyset$. That means that $y^* \in \bigcup_{F \in P_0^*} \nu(F)$ and with regard to Corollary 1 of the weak duality principle we have $y^* \in M_d^s$. □

Lemma 2.10. Let the following conditions hold

$$[B_2] \qquad P \cap Q_y = \emptyset \Rightarrow P^* \cap Q_y^* \neq \emptyset$$

$[B_3]$ $$\bigcup_{a \in P_0} \mu(a) \quad \text{is closed}$$

then

$$M_d^s \subset M_p^s$$

Proof. Let $y^* \in M_d^s$, then

$$P^* \cap Q_y^* = \emptyset \quad \forall y \lesssim y^* . \tag{2.12}$$

If $P \cap Q_{y^*} = \emptyset$, then for the closedness of $\bigcup_{a \in P_0} \mu(a)$, there exists a $y < y^*$ such that $P \cap Q_y = \emptyset$. According to the condition $[B_2]$ $P^* \cap Q_y^* \neq \emptyset$ that contradicts (2.12) Thus we have $P \cap Q_{y^*} \neq \emptyset$ and it means $y^* \in \bigcup_{a \in P_0} \mu(a)$. Finally from Corollary 1 it follows $y^* \in M_p^s$. □

Summarizing Lemmas 2.9 and 2.10 we have

Theorem 2.4. (Minimum Strong Duality Principle.)

Suppose $P_0 \neq \emptyset$ or $P_0^* \neq \emptyset$ and conditions $[B_1]$, $[B_2]$ and $[B_3]$ hold, then

$$M_p^s = M_d^s .$$

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