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A POLYNOMIAL SOLUTION TO REGULATION AND TRACKING

Part II. Stochastic Problem

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Recent results on polynomial techniques in solving the discrete-time linear-quadratic regulation and/or tracking problems are presented. Both deterministic and stochastic problems are considered in order to let appear their formal similarity and to contrast the inherent differences. The analysis is based on external polynomial models and the construction of the optimal controller or control sequence is reduced to the solution of linear polynomial equations, combined with spectral factorization. The existence of admissible controls that yield a finite performance index is studied and all such controls are specified in a parametric form. The optimal control then corresponds to the zero parameter and is shown to be recurrent, i.e. realizable by a linear finite dimensional system.

The paper is divided into two parts. Part I is concerned with the deterministic problem, i.e. with the existence of open-loop control strategies and their realization by various feedback schemes. Part II investigates the stochastic problem, i.e. the existence of closed-loop control strategies including the constraints of causality and stability.

3. STOCHASTIC REGULATION AND TRACKING

3.1. Formulation

Consider the *stochastic plant*

(3.1a)
$$x_{t+1} = Ax_t + Bu_t + En_{\sigma t}$$
$$y_t = Cx_t + Du_t + Kn_{\sigma t}$$

and the *reference qenerator*

(3.2a)
$$x_{Rt+1} = Fx_{Rt} + Gn_{\psi t}$$
$$y_{Rt} = Hx_{Rt} + Ln_{\psi t}$$

for the discrete times $t = 0, 1, \ldots$. Here $u_t \in \mathbb{R}$ is the control input, $x_t \in \mathbb{R}^n$ is the plant state, $n_{\sigma t} \in \mathbb{R}$ is the background noise, $y_t \in \mathbb{R}$ is the output and $x_{Rt} \in \mathbb{R}^m$ is the generator state, $n_{\psi t} \in \mathbb{R}$ is the noise driving the generator, and $y_{Rt} \in \mathbb{R}$ is the reference.

Let the available output be corrupted by an additive observation noise $n_{\varrho t} \in \mathbb{R}$ and let the available reference be corrupted by an additive observation noise $n_{\varphi t} \in \mathbb{R}$.

All four random sources n_ϱ, n_σ and n_φ, n_ψ are pairwise independent zero-mean covariance-stationary white random processes with intensities ϱ, σ and φ, ψ respectively. These intensities are allowed to assume any nonnegative real values.

We shall consider *causal linear controllers* which operate on the available data, i.e. on $y_t + n_{\varrho t}$ and $y_{Rt} + n_{\varphi t}$ to generate the plant input u_t. Such systems are described by

$$\begin{aligned} z_{t+1} &= \boldsymbol{P}z_t + \boldsymbol{Q}(y_t + n_{\varrho t}) + \boldsymbol{R}(y_{Rt} + n_{\varphi t}) \\ u_t &= \boldsymbol{S}z_t + \boldsymbol{T}(y_t + n_{\varrho t}) + \boldsymbol{U}(y_{Rt} + n_{\varphi t}) . \end{aligned} \tag{3.3a}$$

The controller, as a purely discrete-time system, can generate u_t only after $y_t + n_{\varrho t}$ and $y_{Rt} + n_{\varphi t}$ have been processed. Thus it is in fact strictly causal; i.e. $\boldsymbol{T} = \boldsymbol{U} = 0$ in (3.3a). It is convenient to view the plant as that part of the system which is given and the controller as that part of the system which is completely unknown and to be found. To be consistent with this philosophy, we must incorporate in the plant the one-step delay that is known to be present in the controller. This leaves the controller unconstrained and simplifies the synthesis procedure. Thus we *assume* hereafter that $\boldsymbol{D} = 0$ in (3.1a) in exchange for nonzero $\boldsymbol{T}$ and $\boldsymbol{U}$ in (3.3a).

As the plant, reference generator and controller are linear systems, each signal in the resulting interconnection of (3.1), (3.2) and (3.3) can be thought of as the sum of two components: the free motion starting from (nonzero) initial states and the response to the random sources. Thus the reference y_{Rt} can be written as

$$y_{Rt} = y_{RDt} + y_{RSt} \tag{3.4}$$

where y_{RDt} is the deterministic component due to generator's initial state x_{R0} and y_{RSt} is the stochastic component due to the noises $n_{\varphi t}$ and $n_{\psi t}$. Similarly, the plant input u_t and output y_t can be written as

$$\begin{aligned} u_t &= u_{Dt} + u_{St} \\ y_t &= y_{Dt} + y_{St} \end{aligned} \tag{3.5}$$

where u_{Dt} and y_{Dt} are the deterministic components due to the initial states x_0, x_{R0} and z_0 while u_{St} and y_{St} are the stochastic components due to the noises $n_{\varrho t}$. $n_{\sigma t}$, $n_{\varphi t}$ and $n_{\psi t}$.

Once all the signals are looked at as composed of two different components the following two different problems arise naturaly: first we want the system to respond to the initial conditions in a suitable way (assuming the noise is absent); second we want the system to behave suitably under the action of noises (assuming that it is initially at rest).

In practice, we are usually to deal with both problems at the same time. In our analysis, however, we shall first treat each problem separately. Their simultaneous solution will be discussed in the sections to follow.

Asymptotic trackinq: given the plant and the reference generator, the problem is to find a controller such that

$$\lim_{t\to\infty} \{\lambda u_{Dt}^2 + \mu(y_{RDt} - y_{Dt})^2\} = 0 \tag{3.6}$$

for all x_0, x_{R0} and z_0. Here $\lambda \geqq 0$ and $\mu \geqq 0$ are real constants, not both zero.

This rather standard formulation can be interpreted as follows: If $\lambda \neq 0$ the plant input is to tend asymptotically to zero ($\lim_{t\to\infty} u_{Dt} = 0$); if $\mu \neq 0$ the output is to follow asymptotically the reference ($\lim_{t\to\infty} (y_{RDt} - y_{Dt}) = 0$). Moreover, the resulting system should behave this way for all possible initial states of its subsystems: of the plant, of the reference generator and of the controller.

Optimal ırackinq: given the plant and the reference generator, the problem is to find a controller such that the ensemble average

$$J = \lim_{t\to\infty} \mathsf{E}\{\lambda u_{St}^2 + \mu(y_{RSt} - y_{St})^2\} \tag{3.7a}$$

ıs finite and attains its minimum. Here $\lambda \geqq 0$ and $\mu \geqq 0$ are real constants, not both zero.

This is the standard formulation of the infinite-horizon linear-quadratic stochastic optimal tracking problem. The cost defined in (3.7a) represents certain variances in the system as $t \to \infty$, and its interpretation depends on the actual values of λ and μ: If $\lambda = 0$ the output is to follow the reference as closely as possible (in the sense of minimizing the error variance in steady state); if $\mu = 0$ the control effort is to be minimized in steady state; if $\lambda\mu > 0$ a compromise of the two is to be found with λ and μ weighting the relative importance of both components.

The special case of $y_R = 0$ is called (either asymptotic or optimal) *requlation* problem.

In addition to the internal models (3.1a) and (3.2a) it is convenient to introduce the input-output model of the plant

$$Ay = Bu + Cn_\sigma + D \tag{3.1b}$$

where A, B, C and D are polynomials in d having no common factor independent of x_0 defined by

$$\begin{aligned} \frac{B(d)}{A(d)} &= \boldsymbol{C}(\boldsymbol{I}_n - \boldsymbol{A}d)^{-1}\, \boldsymbol{B}d \\ \frac{C(d)}{A(d)} &= \boldsymbol{K} + \boldsymbol{C}(\boldsymbol{I}_n - \boldsymbol{A}d)^{-1}\, \boldsymbol{E}d \\ \frac{D(d)}{A(d)} &= \boldsymbol{C}(\boldsymbol{I}_n - \boldsymbol{A}d)^{-1}\, x_0 \end{aligned} \tag{3.8}$$

and the input-output model of the reference generator

$$F y_{\mathrm{R}} = G n_{\psi} + E \tag{3.2b}$$

where F, G and E are polynomials in d having no common factor independent of x_{R0} defined by

$$\begin{aligned} \frac{G(d)}{F(d)} &= L + H(I_m - Fd)^{-1} Gd \\ \frac{E(d)}{F(d)} &= H(I_m - Fd)^{-1} x_{\mathrm{R0}} \,. \end{aligned} \tag{3.9}$$

Evidently, both A and F are causal polynomials and B is strictly causal, i.e. $\langle B \rangle = 0$. To avoid trivialities we assume that $B \neq 0$ and $G \neq 0$.

We shall look for the desired controller in the form of the input-output model

$$pu = -q(y + n_{\varrho}) + r(y_{\mathrm{R}} + n_{\varrho}) + t \tag{3.3b}$$

where p, q, r and t are causal recurrent sequences defined by

$$\begin{aligned} \frac{q(d)}{p(d)} &= T + S(I - Pd)^{-1} Qd \\ \frac{r(d)}{p(d)} &= U + S(I - Pd)^{-1} Rd \\ \frac{t(d)}{p(d)} &= S(I - Pd)^{-1} z_0 \,. \end{aligned} \tag{3.10}$$

Evidentely, p is bicausal.

We shall refer to q/p and r/p as the feedback and feedforward parts of the controller, respectively.

The composite system consisting of the plant (3.1) and of the controller (3.3) is a causal system if and only if $Ap + Bq$ is a bicausal sequence (see Kučera [5]). In our situation this is always the case as we have A causal polynomial, q causal sequence, p bicausal sequence and $\langle B \rangle = 0$. Therefore, to describe all controllers (3.3), we can simply consider only such p, g and r from (3.10) for which

$$Ap + Bq = 1 \,, \tag{3.11}$$

a typical representative of bicausal sequences.

To solve the tracking problems it is convenient to define a causal sequence s by the relation

$$Fs + Br = 1. \tag{3.12}$$

Note that s is well defined since F and r are causal and $\langle B \rangle = 0$.

Finally denote by $\hat{D}$ a greatest common divisor of A and F so that

(3.13) $$A = A_0\hat{D}\,, \quad F = F_0\hat{D}$$

with A_0 and F_0 relatively prime. Using this notation, we have

(3.14a) $$u_D = -Dq + At + \frac{A_0E}{F_0}\,r$$

(3.14b) $$y_{RD} - y_D = -Dp - Bt + Es$$

(3.15) $$u_S = -Aqn_\varrho - Cqn_\sigma + Arn_\varphi + \frac{A_0G}{F_0}\,rn_\psi$$
$$y_{RS} - y_S = Bqn_\sigma - Cpn_\sigma - Brn_\varphi + Gsn_\psi$$

where use has been made of (3.1)–(3.5) and (3.11)–(3.12). The correlation functions of the input u_S and the tracking error $y_{RS} - y_S$ for $t \to \infty$, if they exist, are given by

(3.16a) $$\mathsf{C}_u = q_*A_*\varrho Aq + q_*C_*\sigma Cq + r_*A_*\varphi Ar + r_*\,\frac{A_{0*}G_*\psi GA_0}{F_{0*}F_0}\,r$$

(3.16b) $$\mathsf{C}_{y_R-y} = q_*B_*\varrho Bq + p_*C_*\sigma Cp + r_*B_*\varphi Br + s_*G_*\psi Gs$$

respectively, so that the cost (3.7a) can be expressed in the form

(3.7b) $$J = \langle \lambda\mathsf{C}_u + \mu\mathsf{C}_{y_R-y}\rangle\,.$$

3.2. Asymptotic tracking

Conforming to the preceding sections denote by D' a greatest common divisor of A and B so that

(3.18) $$A = A'D' \quad B = B'D'$$

where A' and B' are relatively prime.

Now we can solve the asymptotic tracking problem:

Theorem 3.

a) There exists a causal controller satisfying (3.6) if and only if $\lambda\mu/D'F_0$ is a stable sequence.

b) The set of feedback parts of all causal controllers satisfying (3.6) is given by

(3.19) $$p = v_p$$

(3.20) $$q = v_q$$

where v_p and v_q are any causal recurrent sequences satisfying

$$Av_p + Bv_q = 1 \tag{3.21}$$

such that μv_p and λv_q are stable sequences.

The set of feedforward parts of all causal controllers satisfying (3.6) is given by

$$r = \begin{cases} w_r & \text{if} \quad \mu \neq 0 \\ \dfrac{F_0}{A_0} w & \text{if} \quad \mu = 0 \end{cases} \tag{3.22}$$

where w is any causal stable recurrent sequence and w_r along with w_s are any causal recurrent sequences satisfying

$$Fw_s + Bw_r = 1 \tag{3.23}$$

such that μw_s and λw_r are stable.

Proof. Using (3.14) we have

$$\lambda u_{\mathrm{D}} = -\lambda Dq + \lambda At + \lambda \frac{A_0 E}{F_0} r \tag{3.24a}$$

$$\mu(y_{\mathrm{RD}} - y_{\mathrm{D}}) = -\mu Dp - \mu Bt + \mu Es \tag{3.24b}$$

so that (3.6) holds for any x_0, x_{R0} and z_0 iff all the terms in (3.24) are stable sequences for any D, E and t.

The first terms are evidently stable for any D iff (3.19)–(3.21) holds. Such p and q can be found iff $\lambda\mu/D'$ is a stable sequence.

The second terms can not be directly affected be the choice of a controller. It can be shown, however, that they are stable whenever the corresponding first terms are so.

Finally we have to tackle the third terms. For $\lambda\mu = 0$ the situation is trivial. Hence let $\lambda\mu > 0$. Clearly, s must be stable. Taking (3.12) into account this implies that Br is stable as well. Now the third term of (3.24a) is stable only if $A_0 r$ is stable. As a result, r itself must be stable. This immediately yields (3.22) and (3.23). Note that (3.23) has a stable solution if D' and F_0 are stable. If now r is stable then the stability of the third term in (3.24a) hinges on F_0. Note that (3.12) prevents the cancellation of unstable factors between F_0 and r whenever s and r are stable. □

Theorem 3 covers the "regular" case of $\lambda\mu > 0$ as well as the "singular" cases of $\lambda = 0$ or $\mu = 0$. The regular problem is solvable if and only if the plant is stabilizable (i.e., D' is stable) and the unstable modes of the refernece generator are contained in those of the plant (i.e., F_0 is stable). The singular problems have always solutions.

In the singular case of $\lambda = 0$ ($\mu = 0$) the set of all controllers to achieve (3.6) is parametrized explicitly: just take any bicausal (causal) stable v_p (v_q) – the first free parameter – and calculate the corresponding $v_q(v_p)$ via (3.21) to get the feedback and simply take any causal stable w_r (w) – the second free parameter – to get the feedforward. In contrast, this parametrization is only implicit in the regular case: equations (3.21) and (3.23) must be solved for stable sequences to get the controller.

When all the sequence v_p, v_q and w_r or w are taken to be recurrent, the resulting controller is a finite-dimensional system (3.3). The family of controllers that achieve (3.6) is much broader, however. Also non-recurrent sequences lead to controllers that provide asymptotic tracking for any initial state of the plant and of the generator but – as such controllers are no longer finite-dimensional – not for all own initial states. Asymptotic tracking is achieved only for those initial states of the controller for which the associated t results in stable sequences λAt and μBt.

To illustrate the results consider a simple example. Given the plant (3.1) with

$$\boldsymbol{A} = \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix} \quad \boldsymbol{B} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$\boldsymbol{C} = \begin{bmatrix} 1 & 2 \end{bmatrix} \quad \boldsymbol{D} = \begin{bmatrix} 0 \end{bmatrix}$$

and the reference generator (3.2) with

$$\boldsymbol{F} = \begin{bmatrix} \frac{1}{2} & 1 \\ \frac{1}{2} & 0 \end{bmatrix} \quad \boldsymbol{H} = \begin{bmatrix} \frac{1}{2} & 0 \end{bmatrix}.$$

We assume that the systems are not affected by noise; hence the matrices E, K and G, L are immaterial. We are to find all controllers which make it possible to satisfy (3.6).

The plant and the reference generator give rise to the polynomials

$$A = 1 - d\,, \quad B = d(1 + d)$$

and

$$F = (1 - \mathrm{d})(2 + d)\,.$$

Here clearly $D' = 1$ and $F_0 = 2 + d$ so that the condition is satisfied even for $\lambda\mu > 0$. The equations (3.21) and (3.23) have now the form

$$(1 - d)\, v_p + d(1 + d)\, v_q = 1$$

$$(1 - d)(2 + d)\, v_s + d(1 + d)\, v_r = 1\,.$$

The set of the feedback parts of all suitable controllers is given by (3.19)–(3.20) as follows.

When $\lambda\mu > 0$,

$$p = \frac{2 + d}{2} + d(1 + d)\, \bar{u}$$

$$q = \tfrac{1}{2} - (1 - d)\, \bar{u}$$

where $\bar{u}$ is any causal stable sequence;

when $\lambda = 0$,

$$p = 1 + \mathrm{d}\bar{u}$$

$$q = \frac{1}{1+d} - \frac{1+d}{1-d}\bar{u}$$

where $\bar{u}$ is any causal stable sequence.
And when $\mu = 0$,

$$p = \frac{1}{1-d} + \frac{d(1+d)}{1-d}\bar{u}$$

$$q = \bar{u}$$

where $\bar{u}$ is any causal stable sequence.

The set of the feedforward parts of all suitable controllers is given by (3.22) as follows.
When $\lambda\mu > 0$,

$$r = \tfrac{1}{2} - (1-d)(2+d)\bar{v};$$

when $\lambda = 0$,

$$r = \frac{1}{2} - \frac{(1-d)(2+d)}{1+d}\bar{v};$$

and when $\mu = 0$,

$$r = \bar{v}$$

where $\bar{v}$ is any causal stable sequence.

3.3. Optimal tracking

Now let us go into the optimal tracking. For further reference let $\bar{B}$ be the greatest *causal factor* of B, i.e.

(3.25) $$B' = d^k \bar{B}$$

for some $k > 0$, let $\bar{A}$ and $\bar{F}$ be the greatest causal factors of A' and F, i.e.

(3.26) $$A' = \bar{A}, \quad F = \bar{F}$$

as A' and F themselves are causal, and let $\bar{C}$ and $\bar{G}$ be the greatest causal factors of C and G, i.e.

(3.27) $$C = d^l \bar{C}, \quad G = d^m \bar{G}$$

for some $l \geqq 0$, $m \geqq 0$. Also let $\bar{H}$, $\bar{M}$ and $\bar{N}$ be causal Hurwitz polynomials such that

(3.28) $$A'_* \lambda A' + B'_* \mu B' = \bar{H}_* \bar{H}$$

(3.29) $$A_* \varrho A + C_* \sigma C = \bar{M}_* \bar{M}$$

(3.30) $$F_* \varphi F + G_* \psi G = \bar{N}_* \bar{N}.$$

Such $\bar{H}$, $\bar{M}$ and $\bar{N}$ are called the *spectral factors*; when $\lambda + \mu > 0$, $\varrho + \sigma > 0$ and $\varphi + \psi > 0$, respectively, they exist and are unique up to the signs.

Stochastic components of signals as defined are not affected by initial states. We may therefore asume in this section, without loss of generality, that A, B, C as well as F, G are relatively prime.

Now we are ready to solve the optimal tracking problem.

Theorem 4. Define

$$H = \begin{cases} \sqrt{(\mu)}\,\bar{B} & \text{if } \lambda = 0 \\ \sqrt{(\lambda)}\,\bar{A} & \text{if } \mu = 0 \\ \bar{H} & \text{if } \lambda\mu > 0 \end{cases} \tag{3.31}$$

$$M = \begin{cases} \sqrt{(\sigma)}\,\bar{C} & \text{if } \varrho = 0 \\ \sqrt{(\varrho)}\,\bar{A} & \text{if } \sigma = 0 \\ \bar{M} & \text{if } \varrho\sigma > 0 \end{cases} \tag{3.32}$$

$$N = \begin{cases} \sqrt{(\psi)}\,\bar{G} & \text{if } \varphi = 0 \\ \sqrt{(\varphi)}\,\bar{F} & \text{if } \psi = 0 \\ \bar{N} & \text{if } \varphi\psi > 0\,. \end{cases} \tag{3.33}$$

Let P, Q, T and R, S, V be the solutions of the equations

$$H_*P - T_*B = A'_*\lambda M \tag{3.34a}$$

$$H_*Q + T_*A = B'_*\mu M \tag{3.34b}$$

and

$$H_*S - U_*BF_0 = A'_*\lambda N A_0 \tag{3.35a}$$

$$H_*R + U_*F = B'_*\mu N \tag{3.35b}$$

that satisfy $\langle T \rangle = \langle U \rangle = 0$. Then

a) there exists a causal controller which makes J finite if and only if

$$\lambda\varrho \frac{AQ}{D'HM}, \quad \lambda\sigma \frac{CQ}{D'HM}$$
$$\mu\varrho \frac{BQ}{D'HM}, \quad \mu\sigma \frac{CP}{D'HM} \tag{3.36}$$

and

$$\lambda\varphi \frac{AR}{D'HN}, \quad \lambda\psi \frac{A_0GR}{F_0D'HN}$$
$$\mu\varphi \frac{BR}{D'HN}, \quad \mu\psi \frac{GS}{F_0D'HN} \tag{3.37}$$

are all causal stable sequences;

b) the set of feedback parts of all controllers that yield finite J is generated by the following formulas:

— if $\varrho + \sigma = 0$ then

$$p = \frac{1 - Bv}{A} \tag{3.38a}$$

$$q = v \tag{3.38b}$$

where v is any causal recurrent sequence;
— if $\varrho + \sigma > 0$ then

$$p = \frac{P - Bv}{D'HM} \tag{3.39a}$$

$$q = \frac{Q + Av}{D'HM} \tag{3.39b}$$

where v is any causal recurrent l_2-sequence.

The set of feedforward parts of all controllers which yield finite J is generated by the following formulas:
— if $\varphi + \psi = 0$ then

$$r = w \tag{3.40}$$

where w is any causal recurrent sequence;
— if $\varphi + \psi > 0$ then

$$r = \frac{R + Fw}{D'HN} \tag{3.41}$$

where w is any causal recurrent l_2-sequence;

c) The feedback part $\hat{p}$, $\hat{q}$ of the controllers which minimizes J is obtained as follows:
— if $\varrho + \sigma = 0$ then

$$\hat{p} = \frac{1 - Bv}{A} \qquad \hat{q} = v \tag{3.42}$$

where v is any causal recurrent sequence;
— if $\varrho + \sigma > 0$ then

$$\hat{p} = \frac{P}{D'HM} \tag{3.43a}$$

$$\hat{q} = \frac{Q}{D'HM} \tag{3.43b}$$

i.e. its transfer function is unique and corresponds to $v = 0$ in (3.39).

The feedforward part $\hat{r}$ of the controller which minimizes J is obtained as follows:

– if $\varphi + \psi = 0$ then

(3.44) $$\hat{r} = w$$

where w is any causal recurrent sequence;

– if $\varphi + \psi > 0$ then

(3.45) $$\hat{r} = \frac{R}{D'HN}$$

i.e. its transfer function is unique and corresponds to $w = 0$ in (3.41).

The associated minimal cost $\hat{J}$ is given by

(3.46) $$\hat{J} = \hat{J}_{\text{REG}} + \hat{J}_{\text{TR}}$$

where

(3.47a) $$\hat{J}_{\text{REG}} = 0$$

for $\sigma = 0$ and

(3.47b) $$\hat{J}_{\text{REG}} = \left\langle \mu\lambda \frac{M_* M}{D'_* H_* H D'} + \frac{T_* T}{H_* H} - \mu\varrho \right\rangle$$

for $\sigma \neq 0$ while

(3.48a) $$\hat{J}_{\text{TR}} = 0$$

for $\psi = 0$ and

(3.48b) $$\hat{J}_{\text{TR}} = \left\langle \mu\lambda \frac{N_* N}{D'_* H_* H D'} \frac{A_{0*} A_0}{F_{0*} F_0} + \frac{U_* U}{H_* H} - \mu\varphi \right\rangle$$

for $\psi \neq 0$.

Proof. Using (3.16) and (3.11), (3.12) the cost (3.7) reads

(3.49) $$J = J_{\text{REG}} + J_{\text{TR}}$$

where

(3.50) $$J_{\text{REG}} = \lambda\langle q_* A_* \varrho A q + q_* C_* \sigma C q \rangle + \mu \left\langle q_* B_* \varrho B q + (1 - Bq)_* \frac{C_* \sigma C}{A_* A} (1 - Bq) \right\rangle$$

(3.51) $$J_{\text{TR}} = \lambda \left\langle r_* A_* \varphi A r + r_* G_* \frac{A_* \psi A}{F_* F} G r \right\rangle + \mu \left\langle r_* B_* \varphi B r + (1 - Br)_* \frac{G_* \psi G}{F_* F} (1 - Br) \right\rangle .$$

Notice that J_{REG}, which is related to regulation, does not depend on r while J_{TR}, which is related to tracking, does not depend on q. We can therefore minimize J_{REG} and J_{TR} independently by a suitable choice of q and r, respectively. The associated p will then be obtained via (3.11).

Let us start with J_{REG}. First of all observe that if $\varrho + \sigma = 0$ then $J_{REG} = \hat{J}_{REG} = 0$ for all p and q, i.e. the cost is finite and even minimal (zero) for all feedbacks. All the sequences (3.36) are clearly zero and hence stable.

If $\varrho \neq 0$ and $\sigma = 0$ then, employing (3.50),

$$J_{REG} = \varrho\langle v_* v\rangle \tag{3.52}$$

where

$$v = D'Hq \tag{3.53}$$

so that the cost is finite for any l_2-sequence v and minimal ($\hat{J}_{REG} = 0$) for $v = 0$.

Notice that equations (3.24) have now the form

$$H_*P - T_*B = A'_*\lambda A\sqrt{\varrho}$$

$$H_*Q + T_*A = B'_*\mu A\sqrt{\varrho}$$

and as such possess the solution $P = D'H\sqrt{\varrho}$, $Q = 0$ and $T = B'\mu\sqrt{\varrho}$ so that all the sequences (3.36) are again zero and hence stable.

Finally if $\sigma \neq 0$ then, using (3.31) and (3.32), taking into account that $\langle B\rangle = 0$ and completing the squares, J_{REG} is expressed in the desirable form

$$J_{REG} = \langle v_{1*}v_1\rangle + \langle v_2\rangle - \mu\varrho \tag{3.54}$$

where

$$v_1 = \frac{D'HM}{A}q - \mu\frac{B'_*M}{H_*A} \tag{3.55}$$

$$v_2 = \mu\lambda\frac{M_*M}{D'_*H_*HD'}\,. \tag{3.56}$$

Using the equation (3.34b) decompose the second term of v_1 as follows

$$\mu\frac{B'_*M}{H_*A} = \frac{T_*}{H_*} + \frac{Q}{A}\,. \tag{3.57}$$

Then

$$J_{REG} = \hat{J}_{REG} - 2\left\langle\frac{T}{H}v\right\rangle + \langle v_*v\rangle \tag{3.58}$$

where

$$\hat{J}_{REG} = \langle v_2\rangle + \left\langle\frac{T_*T}{H_*H}\right\rangle - \mu\varrho \tag{3.59}$$

and

$$v = \frac{D'HM}{A}q - \frac{Q}{A}\,. \tag{3.60}$$

For causal A and q the sequence v is also causal. Then $\langle T \rangle = 0$ entails

$$\left\langle \frac{T}{H} v \right\rangle = 0$$

since H is causal by definition and we finally get

$$J_{\text{REG}} = \hat{J}_{\text{REG}} + \langle v_* v \rangle \,. \tag{3.61}$$

To prove (3.36) first note that when $\lambda\mu = 0$ equations (3.34) are always solvable. On the other hand, if $\lambda\mu > 0$ and $\sigma > 0$ then J_{REG} is finite iff the first two sequences in (3.15) are l_2. Since $\sigma > 0$ this entails that D' is Hurwitz so that the equations (3.34) are again solvable. As a consequence the first two sequences from (3.15a) and (3.15b) premultiplied by λ and μ respectively can be expressed be means of (3.60) as

$$\begin{aligned}
\lambda\varrho A q &= \lambda\varrho \frac{AQ}{D'HM} + \frac{\lambda A'}{H} \frac{\varrho A}{M} v \\
\lambda\sigma C q &= \lambda\sigma \frac{CQ}{D'HM} + \frac{\lambda A'}{H} \frac{\sigma C}{M} v \\
\mu\varrho B q &= \mu\varrho \frac{BQ}{D'HM} + \frac{\mu B'}{H} \frac{\varrho A}{M} v \\
\mu\sigma C p &= \mu\sigma \frac{PC}{D'HM} - \frac{\mu B'}{H} \frac{\sigma C}{M} v \,.
\end{aligned} \tag{3.62}$$

Now the second terms in (3.62) are all l_2 for any l_2-sequence v since the unstable part of H divides both $\lambda A'$ and $\mu B'$ by (3.31) and the unstable part of M divides both ϱA and σC by (3.32). As a result, the solvability hinges on the stability of (3.36).

Multiplying (3.34a) by A', (3.34b) by B', and adding them up gives

$$A'P + B'Q = HM \,. \tag{3.63}$$

Now (3.39a) follows from (3.11), (3.60) and (3.63). Here P is evidently causal by (3.63) so that p is bicausal and the claim a) is proved.

Claim (3.39) follows from (3.60), (3.11), and (3.63). Note that v is to be causal (so that p and q may be causal) and l_2 (so that J_{REG} may be finite).

As far as (3.43) is concerned, observe that $\hat{J}_{\text{REG}}$ is independent of the controller. The best we can do to minimize J_{REG} is to put $v = 0$ whence (3.43) and (3.57b) follow.

Now we focus our attention on J_{TR}. First notice that for $\varphi = \psi = 0$ we have $J_{\text{TR}} = \hat{J}_{\text{TR}} = 0$ for all r, i.e. it is finite and even minimal for all feedforwards. Moreover, all the sequences (3.37) are zero and hence stable.

If, on the other hand, $\varphi \neq 0$ and $\psi = 0$ then (3.51) gives

$$J_{\text{TR}} = \varphi \langle w_* w \rangle \tag{3.64}$$

where

$$w = D'Hr\,.$$

Hence J_{TR} is finite for any l_2-sequence w and minimal ($\hat{J}_{TR} = 0$) for $w = 0$. Notice that the equations (3.35) have now the form

$$H_*S - U_*BF_0 = A'_*\lambda\,\sqrt{(\varphi)}\,\bar{F}A_0$$

$$H_*R + U_*F = B'_*\mu\,\sqrt{(\varphi)}\,\bar{F}$$

and as such possess the solutions $S = \sqrt{(\varphi)}\,D'HF_0$, $R = 0$ and $U = \mu\sqrt{(\varphi)}\,B'$. Therefore all the sequences (3.37) are again zero and hence stable. Moreover, both (3.41) and (3.45) evidentely cover this case.

Finally if $\psi \neq 0$ we use (3.31) and (3.33), take into account that $\langle B\rangle = 0$ and complete the squares to express J_{TR} in the desirable form

(3.65)
$$J_{TR} = \langle w_{1*}w_1\rangle + \langle w_2\rangle - \mu\varphi$$

where

(3.66)
$$w_1 = \frac{D'HN}{F}\,r - \mu\,\frac{B'_*N}{H_*F}$$

(3.67)
$$w_2 = \mu\lambda\,\frac{N_*N}{D'_*H_*HD'}\,\frac{A_{0*}A_0}{F_{0*}F_0}\,.$$

Using the equation (3.35b) decompose the second term of w_1 as follows

(3.68)
$$\mu\,\frac{B'_*N}{H_*F} = \frac{U_*}{H_*} + \frac{R}{F}\,.$$

Then

(3.69)
$$J_{TR} = \hat{J}_{TR} - 2\left\langle\frac{U}{H}\,w\right\rangle + \langle w_*w\rangle$$

where

(3.70)
$$\hat{J}_{TR} = \langle w_2\rangle + \left\langle\frac{U_*U}{H_*H}\right\rangle - \mu\varphi$$

and

(3.71)
$$w = \frac{D'NH}{F}\,r - \frac{R}{F}\,.$$

For causal F and r the sequence w is also causal. Then $\langle U\rangle = 0$ entails

$$\left\langle\frac{U}{H}\,w\right\rangle = 0$$

since H is causal by definition and we finally get

(3.72)
$$J_{TR} = \hat{J}_{TR} + \langle w_*w\rangle\,.$$

To prove (3.37) note that when $\lambda\mu = 0$ equations (3.35) are always solvable. On the other hand, if $\lambda\mu > 0$ then J_{TR} is finite iff the last two sequences from (3.16) are l_2. Since $\psi > 0$ this entails that F and B have no unstable factor in common and F_0 is Hurwitz so that the equations (3.35) are again solvable. As a consequence, the last two sequences in (3.15a) and (3.15b), premultiplied respectively by λ and μ, can be expressed by means of (3.71) as

$$\begin{aligned} \lambda\varphi A r &= \lambda\varphi \frac{AR}{D'HN} + \frac{\lambda A'}{H}\frac{\varphi F}{N} w \\ \lambda\psi \frac{A}{F} G r &= \lambda\psi \frac{A_0 G R}{F_0 D' H N} + \frac{\lambda A'}{H}\frac{\psi G}{N} w \\ \mu\varphi B r &= \mu\varphi \frac{BR}{D'HN} + \frac{\mu B'}{H}\frac{\varphi F}{N} w \\ \mu\psi G s &= \mu\psi \frac{GS}{F_0 D' H N} - \frac{\mu B'}{H}\frac{\psi G}{N} w . \end{aligned} \tag{3.73}$$

Now the second terms in (3.73) are all l_2 for any l_2-sequence w since the unstable part of H divides both $\lambda A'$ and $\mu B'$ by (3.31) and the unstable part of N divides both φF and ψG by (3.33). As a result, the solvability hinges on the stability of (3.37).

Relation (3.41) follows from (3.71). Note that w is to be causal (so that r may be causal) and l_2 (so that J_{TR} may be finite).

Finally, to prove (3.45) observe that $\hat{J}_{TR}$ is independent of the controller. The best we can do to minimize J_{TR} is clearly to put $w = 0$ whence (3.45) and (3.48b) follow.

□

The proof is based on the following simple idea: To separate the cost into two parts, one of them depending on the feedback and the other one depending on the feedward, and then further separate each of them into two parts of which only one depends on the controller. These parts are then set to zero in order to obtain the optimal controllers; the sum of the remaining parts identifies the minimal cost. This is accomplished by completing the squares (by means of H, M and N) in several stages. The first completion results in (3.54) and (3.65). The temptation to minimize J by setting $v_1 = w_1 = 0$ would, however, yield a non-causal controller. Therefore we isolate the non-causal parts of v_1 and w_2 by means of the decompositions (3.57) and (3.68). The requirements $\langle T \rangle = \langle U \rangle = 0$ are crucial in obtaining the final complete squares (3.61) and (3.72). Here $v = w = 0$ already yields a causal controller. Thus J can be reduced below $\hat{J}_{REG} + \hat{J}_{TR}$ by non-causal controllers only, the minimum attainable by causal controllers is $\hat{J}_{REG} + \hat{J}_{TR}$.

Theorem 4 covers again the "regular" case of $\lambda\mu > 0$ as well as the "singular" cases of $\lambda = 0$ or $\mu = 0$. This is made possible through the way H is defined. In the regular case we just take H to be $\bar{H}$ of (3.28). When $\lambda = 0$ the synthesis of optimal controllers simplifies. This stems from the fact that the control u need no longer be stable since its steady state variance C_u is no longer included in the cost. Indeed, in view of (3.25) and (3.31) the couples of equations (3.34) and (3.35) reduce to the two single equations

(3.74a) $$d^k Q' + T'A = M$$

(3.75) $$d^k R' + U'F = N$$

where deg $T' < k$, deg $U' < k$ and

(3.74b) $$P' = D'\bar{B}T' .$$

The original P, Q, T and R, U are then given by the relationships

(3.76) $$P = \sqrt{(\mu)}\, P' , \quad Q = \sqrt{(\mu)}\, Q' , \quad T_* = \mu B'_* T'$$

and

(3.77) $$R = \sqrt{(\mu)}\, R' \quad U_* = \mu B'_* U' .$$

When $\mu = 0$ the problem becomes trivial as the tracking erorr no longer appears in the cost. Now (3.25) and (3.31) result in

(3.78) $$P = \sqrt{(\lambda)}\, M , \quad Q = 0 , \quad T_* = 0$$

and

(3.79) $$R = 0 , \quad U_* = 0 .$$

Indeed, the best we can now do is not to control at all, i.e. take $q = r = 0$ which results in $u_t = 0$ for all t.

It is important to note that (like in the deterministic tracking treated in Section 2) the singular cases are *not* obtained as limits for $\lambda \to 0$ or $\mu \to 0$ of the regular case. the difference stems from the fact that μ need not be a stable sequence when $\lambda = 0$ and similarly for $y_R - y$ when $\mu = 0$. For positive λ and μ, no matter how small both u and $y_R - y$ must be stable sequences. This discontinuity is embodied in the definition (3.31) of H. The limit cases would correspond to taking $H = \bar{H}$ for and λ and μ.

Besides, every case discussed above splits naturally into three subcases:

The subcase of $\psi = 0$ is referred to as *optimal requlation*. The controller should just optimally eliminate the influence of a disturbance in the plant. As no reference signal is to be tracked, the optimal feedforward either can be arbitrary (when $\varphi = 0$) or it must be zero (when $\varphi > 0$). While the former is described separately by (3.40), the latter is included in (3.41) and (3.45): when $\varphi > 0$ and $\psi = 0$ equations (3.35) take the form

(3.80a) $$H_* S - U_* B' D' F_0 = A'_* \lambda A' \sqrt{(\varphi)}\, D' F_0$$

(3.80b) $$H_* R + U_* F = B'_* \mu \sqrt{(\varphi)}\, F$$

and possess the solution

$$S = \sqrt{(\varphi)}\, HD'F_0\,, \quad R = 0\,, \quad U_* = \sqrt{(\varphi)}\, B'_*\mu\,. \tag{3.81}$$

On the other hand, the subcase of $\sigma = 0$ is referred to as *pure optimal tracking*. The resulting system should just optimally track the reference. Since there is no disturbance in the plant, the optimal feedback either can be arbitrary (when $\varrho = 0$) or it must be zero (when $\varrho > 0$). Again the former is described separately by (3.38) while the latter is included in (3.39) and (3.43); when $\varrho > 0$ and $\sigma = 0$ equations (3.34) take the form

$$H_*P - T_*B = A'_*\lambda A\sqrt{\varrho} \tag{3.82a}$$

$$H_*Q + T_*A = B'_*\mu A\sqrt{\varrho} \tag{3.82b}$$

and possess the solution

$$P = \sqrt{(\varrho)}\, HD'\,, \quad Q = 0\,, \quad T_* = \sqrt{(\varrho)}\, B'_*\mu\,. \tag{3.83}$$

In the third subcase, $\sigma\psi > 0$, the simultaneous regulation and tracking is to be achieved. Therefore the optimal controller possesses both feedback and feedforward parts with uniquely determined transfer functions.

It is important to note that again the cases of $\varrho\sigma = 0$ and $\varphi\psi = 0$ are *not* obtained as limits for $\varrho\sigma \to 0$ and $\varphi\psi \to 0$ of the cases with $\varrho\sigma > 0$ and $\varphi\psi > 0$, respectively. This difference stems from the fact that one requires just minimality of the cost (rather than simultaneous internal stability or ability to track asymptotically). This discontinuity is embodied in the definitions (3.32) and (3.33) of M and N, respectively. The limit cases would again correspond to taking $M = \overline{M}$ and $N = \overline{N}$ for any $\varrho\sigma$ and $\varphi\psi$.

One more remark to complete the discussion. In any case a finite cost can be achieved by controllers characterized by recurrent and/or non-recurrent sequences p, q and r. The optimal controller, however, has always the p, q and r recurrent. As a consequence, the optimal controller is always a finite-dimensional system.

Let us now illustrate the results by simple examples. First consider the plant (3.1) with

$$\boldsymbol{A} = \begin{bmatrix} 0 & 1 \\ 0 & 2 \end{bmatrix} \quad \boldsymbol{B} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \quad \boldsymbol{E} = \begin{bmatrix} 2/7 \\ 4/7 \end{bmatrix}$$

$$\boldsymbol{C} = \begin{bmatrix} 1 & 3 \end{bmatrix} \quad \boldsymbol{D} = \begin{bmatrix} 0 \end{bmatrix} \quad \boldsymbol{K} = \begin{bmatrix} 1 \end{bmatrix}$$

the reference generator (3.2) with

$$\boldsymbol{F} = [1] \quad \boldsymbol{G} = [1]$$

$$\boldsymbol{H} = [1] \quad \boldsymbol{L} = [1]$$

and the noise intensities

$$\varrho = 1\,, \quad \sigma = 0$$

$$\varphi = 0\,, \quad \psi = 1$$

Let the cost (3.7) be specified by

$$\lambda = 0\,, \quad \mu = 1\,.$$

The plant and the reference generator give rise to the polynomials

$$A = 1 - 2d\,, \quad B = 3d + d^2\,, \quad C = 1$$

and

$$F = 1 - d\,, \quad G = 1\,.$$

We calculate

$$D' = 1\,, \quad H = \bar{B} = 3 + d\,, \quad M = 1 - 2d\,, \quad N = 1\,.$$

As $\lambda = 0$, we can use directly equations (3.74)–(3.75) which have the form

$$dQ' + T'(1 - 2d) = 1 - 2d$$

$$dR' + U'(1 - d) = 1$$

$$P' = (3 + d)\,T'$$

and we get

$$P = 3 + d$$

$$Q = 0$$

$$R = 1\,.$$

The set of all controllers that yield a finite cost is characterized by (3.39) and (3.41) as

$$p = \frac{1 - dv}{1 - 2d}$$

$$q = \frac{v}{3 + d}$$

$$r = \frac{1 + (1 - d)\,w}{3 + d}$$

for any causal l_2-sequences v and w. The minimum cost is than achieved by the controller (3.43)–(3.45) with

$$\hat{p} = \frac{1}{1 - 2d}$$

$$\hat{q} = 0$$

$$\hat{r} = \frac{1}{3 + d}\,.$$

As another example consider the plant (3.1) with

$$\boldsymbol{A} = [1] \quad \boldsymbol{B} = [1] \quad \boldsymbol{E} = [1]$$

$$\boldsymbol{C} = [1] \quad \boldsymbol{D} = [0] \quad \boldsymbol{K} = [2]$$

the reference generator (3.2) with

$$F = [1] \quad G = [1]$$
$$H = [1] \quad L = [1]$$

and the intensities

$$\varrho = 0 \quad \sigma = 1$$
$$\varphi = 3 \quad \psi = 4\,.$$

The cost (3.7) is specified by

$$\lambda = 2 \quad \mu = 1\,.$$

The plant and the reference generator give rise to the polynomials

$$A = 1 - d\,, \quad B = d\,, \quad C = 2 - d$$

and

$$F = 1 - d\,, \quad G = 1\,.$$

First we calculate the spectral factors from (3.28)–(3.29). They are

$$\bar{H} = 2 - d$$
$$\bar{M} = 2 - d$$
$$\bar{N} = 3 - d\,.$$

Since $D = 1$, $F_0 = 1$ and also $H = \bar{H}$, $M = \bar{M}$, $N = \bar{N}$ are all stable polynomials, the sequences (3.36)–(3.37) are stable and the problem is therefore solvable.

To obtain the solution, solve the couple of equations (4.34)

$$(2 - d^{-1})\,P - T_*d \qquad = -4d^{-1} + 6 - 2d$$
$$(2 - d^{-1})\,Q + T_*(1 - d) = 2d^{-1} - 1$$

to get

$$P = 4 - d\,, \quad Q = 1\,, \quad T = 3d$$

and the couple of equations (3.35)

$$(2 - d^{-1})\,S - U_*d \qquad = -6d^{-1} + 8 - 2d$$
$$(2 - d^{-1})\,R + U_*(1 - d) = 3d^{-1} - 1$$

to get

$$S = 6 - d\,, \quad R = 2\,, \quad U = 5d\,.$$

Now all causal controllers which yield finite J are described by (3.39) and (3.41) as

$$p = \frac{4 - d - dv}{(2 - d)^2}$$
$$q = \frac{1 + (1 - d)\,v}{(2 - d)^2}$$
$$r = \frac{2 + (1 - d)\,w}{(2 - d)(3 - d)}$$

where v and w are any causal l_2-sequences. The minimum cost is assured by the optimal controller (3.43) and (3.45)

$$\hat{p} = \frac{4 - d}{(2 - d)^2} \quad \hat{q} = \frac{1}{(2 - d)^2} \quad \hat{r} = \frac{2}{(2 - d)(3 - d)}.$$

3.4. Simultaneous asymptotic and optimal tracking

To obtain a deeper analysis we have studied asymptotic and optimal tracking problems separately. In practice, however, one usually encounters both problems at the same time. That is why we are interested in their simultaneous solution.

Theorem 5.

a) Every finite dimensional controller providing asymptotic tracking renders J finite.

b) If $\varrho + \sigma = 0$ then the feedback part of any controller which provides asymptotic tracking (i.e. given by (3.19)–(3.21)) can be used to form the optimal controller. If $\varphi + \psi = 0$ then the feedforward part of any controller which provides asymptotic tracking (i.e. given by (3.22–3.23) can be used to form the optimal controller. If $\varrho + \sigma > 0$ and $\varphi + \psi > 0$ then the optimal controller (3.43) and (3.45) provides asymptotic tracking if and only if

$$\frac{\mu P}{D'HM}, \quad \frac{\lambda Q}{D'HM}$$

$$\frac{\mu S}{F_0 D'HN}, \quad \frac{\lambda A_0 R}{F_0 D'HN}$$

are causal stable sequences.

Proof. To prove a) we just need to compare (3.14) and (3.15) and to take into account that stability of $\lambda(A_0/F_0)\, r$ yields stability of Ar, stability of μs implies stability of μBq as well as stability of μs implies stability of μBr.

Recall that all the p, q, r and s are l_2-sequences iff they are stable whenever the controller is finite dimensional.

First two claims in b) are trivial. To prove the third one just observe that an optimal controller provides asymptotic tracking iff all the sequences $\mu\hat{p}$, $\lambda\hat{q}$, $\lambda(A_0/F_0)\,\hat{r}$, and $\mu\hat{s} = \mu(1 - B\hat{r})/F$ are stable. □

Hence any asymptotically tracking finite-dimensional system gives a finite cost. The converse, however, is by no means true: There may exist systems that are *not* able to track asymptotically even though they yield finite J. To see this just have a look at the first example of Section 3.3: The optimal controller possesses $\hat{p} = 1/(1 - 2d)$

whereby $\mu\hat{p} = \hat{p}$ is not a stable sequence. This controller therefore does not accomplish asymptotic tracking. Why? Because the two requirements, optimality and asymptotic tracking, are antagnoistic in this case. As there is no noise in the plant ($\sigma = 0$) but there is an output noise ($\varrho \neq 0$), the best we can do to minimize J is to do nothing, i.e. not to apply any feedback at all. This way we obtain $J_{\text{REG}} = 0$. However, the given plant is unstable. Unless it is not stabilized by feedback, it is not able to track asymptotically the given reference for some initial states. This stabilization, however, results in injecting the noise n_ϱ into the system. Then J_{REG} is no longer zero and the system is not optimal.

In the antagonistic cases described above it may be of interest to identify, among the controllers that provide asymptotic tracking, the one which minimizes the cost. Such a suboptimal controller exists if and only if the sequences

$$\frac{\mu P}{D'\bar{H}\bar{M}}, \quad \frac{\lambda Q}{D'\bar{H}\bar{M}} \tag{3.84}$$
$$\frac{\mu S}{F_0 D'\bar{H}\bar{N}}, \quad \frac{\lambda A_0 R}{F_0 D'\bar{H}\bar{N}}$$

are all stable. We can get it via (3.43)–(3.45) when taking $H = \bar{H}$, $M = \bar{M}$ and $N = \bar{N}$ no matter what λ, μ, ϱ, σ and φ, ψ are.

Let us illustrate this idea on the first example of Section 3.3. When taking $M = \bar{M} = 2 - d$ (instead of $M = A = 1 - 2d$), (3.74) takes the form

$$dQ' + T'(1 - 2d) = 2 - d$$
$$P' = (3 + d)\,T'$$

and we get

$$P = 6 + 2d$$
$$Q = 3$$
$$T = 2d(3 + d)\,.$$

So we have found that in the class of controllers providing asymptotic tracking, the minimum cost is assured by the suboptimal controller with the feedback part

$$p = \frac{2}{2 - d}$$
$$q = \frac{3}{2 - d}$$

and with an arbitrary feedforward part. This controller, however, yields the cost

$$J_{\text{REG}} = 3 > \hat{J}_{\text{REG}} = 0\,.$$

3.5. Stability

Motivated by the desire to obtain the most general results, we did not require stability of the control system in the preceding analysis. In practice we are of course primarily interested in control system that are internally stable. The following questions are therefore quite natural: Does asymptotic tracking imply stability? Is the optimal system stable? As expected, these are not always the cases.

First one should realize that the reference generator is fixed and not necessarily stable. In fact, it is the unstable generators that are used to model signals of practical interest like steps, ramps, or random walks. Thus the question is to stabilize just the control system consisting of the plant and of the controller.

Theorem 6. Let both plant (3.1) and controller (3.3) be realized without unstable hidden modes. Then the composite system consisting of (3.1) and (3.3) is internally stable if and only if p , q and r are stable sequences.

Proof. Take into account that controller (3.3) is a dynamical system having two inputs and one output. Then apply the general theorem on stability of feedback systems given in Kučera [5]. □

The relationship between internal stability and asymptotic tracking clearly depends on actual values of λ and μ. Recall that internal stability of the control systems means the stability of p, q and r while the ability to track asymptotically means the stability of λq, μp, $\lambda(A_0/F_0)\, r$ and μs. Therefore, if $\lambda\mu > 0$ then asymptotic tracking implies internal stability while if $\mu = 0$ then internal stability implies asymptotic tracking. If, however, $\lambda = 0$ then asymptotic tracking does not yield internal stability and vice versa.

If $\mu\psi = 0$ or if F is stable then every internally stable control system gives finite cost. On the other hand, if $\mu\psi \neq 0$ and F is unstable then an internally stable control system gives finite J only if the corresponding s is stable as well.

Finally, the optimal control system is internally stable if and only if $\hat{p}$, $\hat{q}$ and $\hat{r}$ given by (3.43) and (3.45) are stable sequences.

To illustrate the issue of stability let us consider simple examples. Take the plant (3.1) with

$$\boldsymbol{A} = [1] \quad \boldsymbol{B} = [1] \quad \boldsymbol{E} = [2]$$

$$\boldsymbol{C} = [1] \quad \boldsymbol{D} = [0] \quad \boldsymbol{K} = [1]$$

and

$$\varrho = 0 \,, \quad \sigma = 1$$

and solve the optimal regulation problem specified by $\lambda = 0$ and $\mu = 1$. The plant gives rise to the polynomials

$$A = 1 - d \,, \quad B = d \,, \quad C = 1 + d \,.$$

Since

$$H = 1\,, \quad M = 1 + d$$

equation (3.74a) reads

$$dQ' + (1 - d)\,T' = 1 + d$$

and gives

$$P = 1$$

$$Q = 2\,.$$

The set of all controllers that yield finite J is given by

$$p = \frac{1 - dv}{1 + d}$$

(3.85) $$q = \frac{2 + (1 - d)\,v}{1 + d}$$

$$r = w$$

for any causal l_2-sequence v and any causal sequence w. Even if some of these controllers makes the resulting control system stable (e.g. when $v = -1$), the optimal one, which is specified by $v = 0$, does not.

On the other hand, consider the optimal regulation problem specified by $\lambda = 0$ and $\mu = 1$ for the plant (3.1) with

$$\boldsymbol{A} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \quad \boldsymbol{B} = \begin{bmatrix} 0 \\ 2 \\ -1 \end{bmatrix} \quad \boldsymbol{E} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$$

$$\boldsymbol{C} = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} \quad \boldsymbol{D} = \begin{bmatrix} 0 \end{bmatrix} \quad \boldsymbol{K} = \begin{bmatrix} 1 \end{bmatrix}$$

and

$$\varrho = 0\,, \quad \sigma = 1\,.$$

The plant yields the polynomials

$$A = 1 - d^2\,, \quad B = 2d^2 - d^3\,, \quad c = 1 + d\,.$$

Since we have

$$H = 1 - d\,, \quad M = 1 + d$$

equation (3.74) reads

$$d^2 Q' + (1 - d^2)\,T' = 1 + d$$

so that

$$P = (1 + d)(2 - d)$$

$$Q = (1 + d)$$

$$T' = (1 + d)\,.$$

In spite of the fact that MH is again unstable, the optimal controller is

$$\hat{p} = 1$$

$$\hat{q} = \frac{1}{2 - d}$$

$$\hat{r} = w$$

and the resulting optimal control system is internally stable whenever w is chosen stable.

In the cases when the optimal controller does not make the control system internally stable, one may wish to find a stabilizing controller that minimizes the cost. Such a suboptimal controller exists if and only if

$$\frac{P}{D'\bar{H}\bar{M}}, \quad \frac{Q}{D'\bar{H}\bar{M}}, \quad \frac{R}{D'\bar{H}\bar{N}} \tag{3.86}$$

are stable sequences. It is given by (3.43)–(3.45) when taking $H = \bar{H}$, $M = \bar{M}$ and $N = \bar{N}$ no matter what λ, μ, ϱ, σ and φ, ψ are.

In the first example, however, this is not the case. Although there are stabilizing controller which make J finite in (3.85), none of them is the best as $M = \bar{M}$ remains unstable.

3.6. Conclusion

A deep analysis of the discrete-time linear-quadratic regulation and/or tracking problems for single-input single-output systems and infinite control horizon has been presented in a compact and unified way. The underlying idea has been to make use of input-output polynomial models and reduce the synthesis of the control to the solution of linear polynomial equations, possibly in conjuction with the spectral factorization. The analysis results in necessary and sufficient conditions for the existence of admissible controls that make the given performance criterion finite, and all such controls an specified in a parametric form. The optimal control is then obtained by setting the parameter to zero.

The paper consists of two parts: deterministic and stochastic problem. The solution of the *deterministic problem* which is analysed in Section 2.2., is an open-loop one. The optimal control strategy is obtained in the form of a sequence that depends on the given data including the initial states of the plant and the reference generator. When these states are known there is no need for feedback control. The optimal control sequence can nevertheless be realized via state feedback. This feedback law is then independent of the initial states. The resulting system, however, is not practicable unless it is stable in some sense. This is discussed in Section 2.3. If the initial states are not known, we have to resort to output feedback. However, the optimal feedback control law does depend on initial states and hence it cannot be found. A reasonable practical solution is to use an observer-based control law even though it is by no means optimal (Section 2.4).

When one wishes to track for all initial states, the *asymptotic tracking system* treated in Section 3.2 will do the job. If the system is desturbed by random signals, there arises another problem of minimizing the effect of noises in steady state, provided all systems were initially at rest, the so called *stochastic optimal tracking* (Section 3.3.). As stochastic systems exhibit the free motion starting from nonzero initial states in addition to the response to random sources, one usually encounters in practice both problems at the same time. This is a difficult task, however. These two requirements may be in conflict and it is of interest to find a compromise. This is the topic discussed in Section 3.4. The solution presented therein allows for any combination of deterministic and stochastic exogenous inputs, a feature of practical significance that is often ignored in the literature. It is highly desirable in practice that the control system be stable. Therefore Section 3.5 delivers the conditions under which both asymptotic and optimal tracking systems may be stable. The investigation of stability has been purposely separated from that of optimality in order to obtain a deep insight in the problem. All these requirements have been finally merged together to provide a meaningful design.

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