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A SUPPLEMENT TO GOTTWALD'S NOTE ON FUZZY CARDINALS

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We supplement the review of fuzzy cardinality definitions placed in [3]. To be exact, we present approaches in which cardinality of a finite fuzzy subset is expressed by a fuzzy natural number and indicate the most appropriate one.

S. Gottwald placed in [3] a comparative review of approaches to the problem how to define fuzzy cardinality, i.e. how to count elements of a universe which are in its fuzzy subset. In accordance with the concepts presented in [3], cardinality of a fuzzy subset was defined either as a non-negative real number or as a family of usual cardinals. In this note we shall present and compare such approaches in which cardinality of finite fuzzy subset is expressed by means of a fuzzy number. To this end, we must introduce some notation and terminology.

Throughout this note, by a fuzzy subset A of some fixed universal set U we shall mean a function $A : U \to I$, where $I := [0, 1]$ with $:=$ standing for "equals by the definition". Membership grade of an element $x \in U$ in A will be denoted by $A(x)$. The classical subset $\{x : A(x) > 0\}$ will be called support of A and denoted supp (A). If support of a fuzzy subset is finite, then that subset is called finite, too. Throughout the paper we shall assume that A is finite and card (supp (A)) $= n$, where card (M) denotes the usual cardinality of a classical subset M of U. The subset $A_t :=$ $:= \{x : A(x) \geqq t\}$, where $t \in I_0$ and $I_0 := (0, 1]$, is called t-level set of A. The sequence

$$a_0 \geqq a_1 \geqq a_2 \geqq \ldots \geqq a_n > a_{n+1} = a_{n+2} = a_{n+3} = \ldots$$

is defined in the following way: $a_0 := 1$, a_i $(1 \leqq i \leqq n)$ denotes the ith element in descending sequence consisting of positive membership grades in A, $a_i := 0$ for $i > n$.

Let $N := \{0, 1, 2, \ldots\}$. If $F : N \to I$ (i.e. $U := N$), the F will be called fuzzy natural number (in short, fn-number). F is said to be convex iff $F(j) \geqq \min(F(i),$

$F(k))$ for each triplet $i \leqq j \leqq k$ (cf. [4]). Let $\oplus$ denote addition of fn-numbers. Then the fn-number $F \oplus G$ is defined by membership grades

$$(F \oplus G)(k) := \sup_{i+j=k} \min (F(i), G(j)) .$$

As a chronologically first fuzzy approach to cardinality of finite fuzzy subsets, we shall consider the fn-number $\mathsf{FGCount}_A^0$ (see [1, 7]) with membership grades

$$\mathsf{FGCount}_A^0(k) := \begin{cases} \max \{t \in I_0 : \operatorname{card}(A_t) = k\} , \\ 0 \text{ if } \operatorname{card}(A_t) \neq k \text{ for each } t . \end{cases}$$

The values $\mathsf{FGCount}_A^0(k)$ may be considered degrees to which cardinality of A equals k. One can easy notice (see [1]) that $\mathsf{FGCount}_A^0$

(a) is always normalized, i.e. there exists a natural number h such that $\mathsf{FGCount}_A^0(h) = 1$,

(b) is strictly decreasing on its support,

(c) is a non-convex fn-number,

(d) does not fulfil the additivity property

$$\mathsf{FGCount}_A^0 \oplus \mathsf{FGCount}_B^0 = \mathsf{FGCount}_{A \cap B}^0 \oplus \mathsf{FGCount}_{A \cup B}^0 ,$$

where $A \cap B$ and $A \cup B$ denote (resp.) intersection and union of A and B, i.e. $(A \cap B)(x) := \min (A(x), B(x))$, $(A \cup B)(x) := \max (A(x), B(x))$.

In order to avoid the lack of convexity, an important modification of the definition of $\mathsf{FGCount}_A^0$ was proposed in [2] and [8]. As a consequence, we get then a new fn-number defining fuzzy cardinality, namely the $\mathsf{FGCount}_A$ where

$$\mathsf{FGCount}_A(k) := \begin{cases} \max \{t \in I_0 : \operatorname{card}(A_t) \geqq k\} , \\ 0 \text{ if } \operatorname{card}(A_t) < k \text{ for each } t. \end{cases}$$

Let T be a finite fn-number such that $T(0) = g_0$, $T(1) = g_1, \ldots, T(s) = g_s$ and $T(r) = 0$ for $r = s+1, s+2, \ldots$. In such a case we shall use the following "vectorial" notation $T = (g_0, g_1, \ldots, g_s)$.

It is easy to prove (see e.g. [2], [6], [8]) that the following propositions are valid:

(a) $\mathsf{FGCount}_A(k) = \max_{j \geqq k} \mathsf{FGCount}_A^0(j)$.

(b) $\mathsf{FGCount}_A = (a_0, a_1, \ldots, a_n)$. Hence $\mathsf{FGCount}_A$ is convex.

(c) If $A \subset B$, then $\mathsf{FGCount}_A \subset \mathsf{FGCount}_B$ (monotonicity).

Remark. $Y \subset Z := (Y(x) \leqq Z(x)$ for all $x \in U)$.

(d) $\mathsf{FGCount}_A \oplus \mathsf{FGCount}_B = \mathsf{FGCount}_{A \cap B} \oplus \mathsf{FGCount}_{A \cup B}$ (additivity).

Let D denote a classical n-element subset of U. Then, contrary to expectation, we get $\mathsf{FGCount}_D = (1, 1, \ldots, 1)$ with support of $\mathsf{FGCount}_D$ consisting of $n + 1$ elements. This result is sensible provided that $\mathsf{FGCount}_A(k)$ defines degree to which A has at least rather than exactly k elements. Thus FGCount, as definition of fuzzy cardinality, is unsatisfactory. Namely, for classical subsets it does not collapse to usual cardinal number. That is why a next definition of fuzzy cardinality was introduced in [2]. To be exact, the new definition is again a simple modification of the previous one.

Let $\mathscr{L}_k(A)$ denote the family of k-element classical subsets (of U) containing A_1. Then fuzzy cardinality of A will be defined by the finite fn-number Crd_A with membership grades

$$\mathsf{Crd}_A(k) := \begin{cases} \max\limits_{Z \in \mathscr{L}_k(A)} \min\limits_{x \in Z} A(x), \\ 0 \text{ if } \mathscr{L}_k(A) \text{ is empty} \end{cases}$$

(if A_1 is empty, what implies $\mathscr{L}_0(A) = \{\emptyset\}$, we additionally put $\min\limits_{x \in \emptyset} A(x) := 1$).

One can consider $\mathsf{Crd}_A(k)$ to be degree to which cardinality of A equals k. It is easy to verify that (cf. [2], [6])

(a) $\mathsf{Crd}_A = (0, 0, \ldots, 0, 1, a_{m+1}, a_{m+2}, \ldots, a_n)$, where $m := \operatorname{card}(A_1)$ and the constant sequence composed of zeros is m-element one. Thus Crd_A is always convex.
(b) $\mathsf{Crd}_D = (0, \ldots, 0, 1)$ with the figure one placed at the $(n + 1)$th position and D as previously.
(c) $\mathsf{Crd}_A \oplus \mathsf{Crd}_B = \mathsf{Crd}_{A \cap B} \oplus \mathsf{Crd}_{A \cup B}$.
(d) $\mathsf{Crd}_A = \mathsf{FGCount}_A$ iff $\operatorname{card}(A_1) = 0$.

Unfortunately, the monotonicity does not hold for Crd-cardinality. But it is quite obvious that property (b) excludes, in principle, monotonicity. On the other hand, property (b) is, from the practical as well set-theoretical points of view, more important than monotonicity.

This is well-known that the theory of fuzzy subsets is closely connected with the Łukasiewicz many-valued logic (see e.g. [5]). Indeed, it suffices to interpret each membership grade $A(x)$ as representing the truth-value of the statement "x is in A". Therefore, the next approach is based on that logic.

Let $\mathscr{P}_k(A)$ denote the family of all the k-element classical subsets of $\operatorname{supp}(A)$. Moreover, let $p \to q := \min(1, 1 - p + q)$ (Łukasiewicz implication operator) and $p \leftrightarrow q := \min(p \to q, q \to p)$ for $p, q \in I$. Then $\deg(R, S) := \inf\limits_{x \in U} (R(x) \leftrightarrow S(x))$ for arbitrary fuzzy subsets R and S of U. One can consider $\deg(R, S)$ to be degree to which R equals S. Let us define finite fn-number Cd_A by means of membership grades

$$\mathsf{Cd}_A(k) := \begin{cases} \max\{\deg(A, Y) : Y \in \mathscr{P}_k(A)\}, \\ 0 \text{ if } \mathscr{P}_k(A) \text{ is empty.} \end{cases}$$

Then $\mathsf{Cd}_A(k)$ will be considered degree to which A has exactly k elements. This is, in fact, a quality of the best (using the criterion $\deg(A, Y)$) approximation of A by elements from $\mathscr{P}_k(A)$. One can easy verify (see [6]) that

(a) $\mathsf{Cd}_A(k) = \min(a_k, 1 - a_{k+1})$ for $k = 0, 1, 2, \ldots$.
(b) For the classical n-element subset D of U we get $\mathsf{Cd}_D(n) = 1$ and $\mathsf{Cd}_D(j) = 0$ for $j \neq n$.

(c) $\mathrm{Cd}_A = (1 - a_1, 1 - a_2, \ldots, 1 - a_p, a_p, a_{p+1}, \ldots, a_n)$, where $p := \min\{l : a_l + a_{l+1} \leqq 1\}$. Hence Cd_A is always convex.

(d) At most one cardinal number is "favoured", i.e. there exists at most one natural number k_f such that $\mathrm{Cd}_A(k_f) > 0{\cdot}5$.

(e) $\mathrm{FGCount}_A = 2\mathrm{Cd}_{0.5A}$, where membership grades in $0{\cdot}5A$ and $2\mathrm{Cd}_{0.5A}$ are defined as follows: $(0{\cdot}5A)(x) := 0{\cdot}5A(x)$ and $(2\mathrm{Cd}_{0.5A})(k) := \min(1, 2\mathrm{Cd}_{0.5A}(k))$.

(f) $\mathrm{Cd}_A \oplus \mathrm{Cd}_B = \mathrm{Cd}_{A \cap B} \oplus \mathrm{Cd}_{A \cup B}$.

(g) Let A^c denote the complement of A, i.e. $A^c(x) := 1 - A(x)$. If U is finite and $\mathrm{card}(U) = m$, then $\mathrm{Cd}_{A^c}(j) = \mathrm{Cd}_A(m - j)$ for $j = 0, 1, \ldots, m$.

One can easy give counterexamples that both the important properties (d) and (g) do not hold for $\mathrm{FGCount}_A$ and Crd_A. Obviously (g) is a counterpart of the elementary law card $(D^c) = m -$ card (D), where D denotes now a classical subset of m-element universe.

To summarize the discussion, it seems to be more suitable to define cardinality of a finite fuzzy subset as a fuzzy natural rather than positive real number (or a family consisting of usual cardinals). Then the fn-number Cd_A is, from the set-theoretical point of view, defined in a most natural way and fulfills many natural postulates (see e.g. properties (b), (f), (g)) except the monotonicity (what is, however, explicable).

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