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STATISTICAL LINEAR SPACES

Part II. Strongest t-norm

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This paper is devoted to one of the most important cases of SLM-spaces when the generalized triangular inequality is given in the strongest form, i.e. using the t-norm $T(a, b) = \min(a, b)$. Only in Theorem 11 a case of a weaker t-form is considered. Symbols and denotations are the same as in the first part of the paper.

1. SEMINORMS AND t-NORM MIN

In the first part of this paper (cf. [1]) the notion of a statistical linear space in the sense of Menger was introduced and the ε,η-topology was investigated. The main goal of the second part of the paper is a more detailed investigation of one of the most important cases of SLM-spaces. We shall deal with a statistical linear space $(S, \mathscr{J}, \min)$ where the generalized triangular inequality is expressed in the form

$$F_{x+y}(u + v) \geqq \min (F_x(u), F_y(v)) .$$

Let $a \in \langle 0, 1)$ and let us determine for every $x \in S$ the number $n_a(x)$ by the relation

$$n_a(x) = \inf \{\lambda > 0 : F_x(\lambda) > a\} .$$

Since every probability distribution function $F_x(\cdot)$ is nondecreasing and left continuous the following inequalities are equivalent

$$n_a(x) < \lambda \Leftrightarrow F_x(\lambda) > a .$$

On the contrary, we can express F_x using n_a as follows $F_x(u) = 0$ for $u \leqq 0$; $F_x(u) = \sup \{a \in \langle 0, 1) : n_a(x) < u\}$. Let us denote $N_S = \{n_a(\cdot) : a \in \langle 0, 1)\}$ for a given space $(S, \mathscr{J}, \min)$.

Theorem 1. The class N_S is a class of seminorms in $(S, \mathscr{J}, \min)$.

Proof. Let $n_a \in N_S$, then for every $x \in (S, \mathscr{J}, \min)$ $n_a(x) \geqq 0$ because $F_x(0) = 0$. Further, $n_a(0) = 0$, $F_0(u) = H(u)$ and hence $n_a(0) = 0$ for every $a \in \langle 0, 1)$.

If $\lambda \neq 0$ is a real number then the definition of $n_a(x)$ yields

$$n_a(\lambda x) = \inf\{u > 0 : F_{\lambda x}(u) > a\} = \inf\{u > 0 : F_x(u/|\lambda|) > a\} =$$
$$= \inf\{|\lambda| v > 0 : F_x(v) > a\} = |\lambda| \inf\{v > 0 : F_x(v) > a\} = |\lambda| n_a(x).$$

For $\lambda = 0$ $n_a(0) = \inf\{u > 0 : H(u) > a\} = 0$ for every a. We have proved that for every $a \in \langle 0, 1)$ and every real λ

$$n_a(\lambda x) = |\lambda| n_a(x).$$

Thus, we have yet to prove the triangular inequality

$$n_a(x + y) \leqq n_a(x) + n_a(y).$$

According to the definition $n_a(x + y) = \inf\{\lambda > 0 : F_{x+y}(\lambda) > a\} = \inf\{u + v > > 0 : F_{x+y}(u + v) > a\} \leqq \inf\{u + v > 0 : \min(F_x(u), F_y(v)) > a\} \leqq \inf\{u > > 0 : F_x(u) > a\} + \inf\{v > 0 : F_y(v) > a\} = n_a(x) + n_a(y)$. □

Theorem 2. The class $N_S - \{n_0\}$ is a class of norms in $(S, \mathscr{J}, \min)$ if and only if every function $F_x(\cdot)$ for $x \neq 0$ is continuous at 0.

Proof. In the previous Theorem 1 we proved that $n_a(0) = 0$ for every $a \in \langle 0,1)$. We can prove that the equality for every $a \in (0, 1)$ $n_a(x) = 0$ implies that $x = 0$, if every function F_x is continuous at 0, $x \neq 0$. Let us suppose that $n_{a_0}(x_0) = 0$ for some $a_0 \in (0, 1)$ and some $x_0 \in S$, $x_0 \neq 0$. However, at the same moment $F_{x_0}(\lambda) > a_0$ for every $\lambda > 0$, it is clear that the function F_{x_0} cannot be continuous at 0.

If $N_S - \{n_0\}$ is a class of norms in $(S, \mathscr{J}, \min)$, i.e. $n_a(x) = 0 \Rightarrow x = 0$ for every $a \in (0, 1)$, it must be then $n_a(x) > 0$ for every $x \neq 0$ and every $a \in (0, 1)$. If for some $x_0 \neq 0 \lim_{u \downarrow 0} F_{x_0}(u) = \varepsilon_0$ would be nonzero then we obtain immediately $0 = = \inf\{\lambda > 0 : F_{x_0}(\lambda) > \varepsilon_0/2\} = n_{\varepsilon_0/2}(x_0)$ and this conclusion is a contradiction to the assumption that $n_a(\cdot)$ is a norm for every $a \in (0, 1)$. □

It is well known that every locally convex topology in a linear topological space can be determined by a suitable collection of seminorms which are defined by all absolutely convex and absorbing neighbourhoods in the given topology. Similarly, in our case it is natural to ask about the relation between the ε,η-topology in $(S, \mathscr{J}, \min)$ and the class N_S of seminorms in S.

Theorem 3. Let an *SLM*-space $(S, \mathscr{J}, \min)$ be given. Then for $n \to \infty$ $x_n \to x_0$ in the ε,η-topology if and only if $n_a(x_n - x_0) \to 0$ for every $a \in \langle 0, 1)$.

Proof. The convergence $x_n \to x_0$ in the ε,η-topology means that $(\forall \varepsilon \in (0, 1\rangle$ $\forall \eta > 0\ \exists n_0\ \forall n \geqq n_0) \Rightarrow F_{x_n - x_0}(\eta) > 1 - \varepsilon$. It follows from this implication that for every $n \geqq n_0$ $\inf\{\lambda > 0 : F_{x_n - x_0}(\lambda) > 1 - \varepsilon\} < \eta$, i.e.

$$n_{1-\varepsilon}(x_n - x_0) < \eta \quad \text{for every} \quad n \geqq n_0.$$

In other words, if we choose $a \in \langle 0, 1)$ and $\eta > 0$ arbitrarily and if we put $\varepsilon = 1 - a$, then $\varepsilon \in (0, 1\rangle$ and there exists a natural n_0 such that for every $n \geqq n_0$

$$n_{1-\varepsilon}(x_n - x_0) = n_a(x_n - x_0) < \eta$$

and hence $n_a(x_n - x_0) \to 0$ for every $a \in \langle 0, 1)$. Now, let for every $a \in \langle 0, 1)$ $n_a(x_n - x_0) \to 0$, i.e.

$$(\forall a \in \langle 0, 1) \, \forall \eta > 0 \, \exists n_0 \, \forall n \geqq n_0) \Rightarrow n_a(x_n - x_0) < \eta \, .$$

The definition of the seminorm $n_a(\cdot)$ yields that $n_a(x_n - x_0) < \eta \Leftrightarrow F_{x_n - x_0}(\eta) > a$ and for every $n \geqq n_0$ $x_n - x_0 \in O(1 - a, \eta)$. This fact proves that $x_n \to x_0$ in the ε,η-topology. □

Remark 1. If an *SLM*-space $(S, \mathscr{J}, \min)$ is normable and hence the existence of a bounded convex neighbourhood in the ε,η-topology is ensured, see [1], then there exists a bounded ε,η-neighbourhood $O(\varepsilon_0, \eta_0)$ and a norm can be given by

$$\|x\| = \inf\{\lambda > 0 : F_x(\lambda\eta_0) > 1 - \varepsilon_0\} = \eta_0^{-1} n_{1-\varepsilon_0}(x) \, .$$

From this follows that $n_{1-\varepsilon_0}(\cdot)$ must be also a norm in $(S, \mathscr{J}, \min)$.

Theorem 4. Let $(S, \mathscr{J}, \min)$ be given. Then the ε,η-topology is equivalent to the topology induced by a countable collection of seminorms $\{n_{a_k}(\cdot)\}$, where $a_k \uparrow 1$.

Proof. In [1], the metrizability of the ε,η-topology is proved, hence we can consider sequences in S only. If $x_n \to 0$ in the ε,η-topology, then $n_a(x_n) \to 0$ for every $a \in \langle 0, 1)$ as Theorem 3 states.
Conversely, let us suppose that $x_n \to 0$ in the topology induced by the seminorms $\{n_{a_k}\}$ where $a_k \uparrow 1$, i.e. $n_{a_k}(x_n) \to 0$ for every k. Since $a_k \uparrow 1$, then for an arbitrary $\varepsilon \in (0, 1\rangle$ there exists q such that for every $k \geqq q$ $1 > a_k > 1 - \varepsilon$. If $\eta > 0$ is arbitrary, then there exists n_0 such that for $n \geqq n_0$ $n_{a_q}(x_n) < \eta$. This last inequality implies that $F_{x_n}(\eta) > a_q$, hence $1 - \varepsilon < F_{x_n}(\eta)$. In other words, it means that $x_n \in O(\varepsilon, \eta)$ for every $n \geqq n_0$. This proves the convergence of $\{x_n\}$ to the origin in the ε,η-topology. □

Remark 2. The collection of seminorms $\{n_{a_k}\}$ mentioned in Theorem 4 induces a metric

$$d(x, y) = \sum_{k=1}^{\infty} \frac{1}{2^k} \frac{n_{a_k}(x - y)}{1 + n_{a_k}(x - y)}$$

in S which is equivalent to the ε,η-topology.

Theorem 5. Let S be a linear space, let $N = \{n_a : a \in \langle 0, 1)\}$ be a class of seminorms in S. If for every $x \in S$ the function $n_a(x)$ is nondecreasing and right continuous in $\langle 0,1)$ and $n_a(x) = 0$ for every $a \in \langle 0, 1)$ implies that $x = 0$, then S can be assumed as an SLM-space $(S, \mathscr{J}, \min)$ and the ε,η-topology in S is identical with the topology induced by the class N of seminorms.

Proof. If N is any class of seminorms in S then it is possible to introduce a topology in S, which is locally convex and at the same time the weakest topology in which all seminorms in N are continuous. This topology is a Hausdorff topology if and only if $n(x) = 0$ for every $n \in N$ implies that $x = 0$ in S. A base of neighbourhoods of the origin in this weakest topology is formed by all sets

$$\{x \in S : \max_{1 \leqq i \leqq k} n_i(x) < \varepsilon\}$$

where $\varepsilon > 0$ and $n_i \in N$, $k = 1, 2, \ldots$. In our case this base is formed by all sets of the form

$$\{x \in S : n_a(x) < \varepsilon\}, \quad a \in \langle 0, 1)$$

because we assume that $n_a(x)$ is nondecreasing on $\langle 0, 1)$ in a. Using the seminorms n_a, $a \in \langle 0, 1)$ we can define a mapping $\mathscr{J} : S \to \mathscr{F}$. If $x \in S$ then we put

$$\mathscr{J}(x)[u] = F_x(u) = \sup\{a \in \langle 0, 1) : n_a(x) < u\}$$

in the case that $\{a \in \langle 0, 1) : n_a(x) < u\} \neq \emptyset$ and $\mathscr{J}(x)[u] = F_x(u) = 0$ if $\{a \in \langle 0, 1) : : n_a(x) < u\} = \emptyset$. First, we shall prove that $F_x(\cdot)$ is a probability distribution function. It is sufficient to consider the case $u > 0$ only. If $u_1 \leqq u_2$ then

$$F_x(u_1) = \sup\{a : n_a(x) < u_1\} \leqq \sup\{a : n_a(x) < u_2\} = F_x(u_2)$$

hence $F_x(\cdot)$ is a nondecreasing function. Further, we shall prove that $F_x(\cdot)$ is left continuous at every $u > 0$. Let $u_n \uparrow u$. Then $F_x(u_n) = \sup\{a : n_a(x) < u_n\} \leqq \leqq F_x(u_{n+1}) \leqq F_x(u)$ and $\lim_{n\to\infty} F_x(u_n) = \sup_n F_x(u_n)$ exists and $\sup_n F_x(u_n) \leqq F_x(u)$. But $F_x(u) = \sup\{a \in \langle 0, 1) : n_a(x) < u\}$ and hence for every $\varepsilon > 0$ there exists $a_\varepsilon \in \langle 0, 1)$ such that $F_x(u) - \varepsilon < a_\varepsilon$ where $n_{a_\varepsilon}(x) < u$. We assume that $n_a(x)$ is right continuous at a_ε; thus for suitably small $\delta > 0$ and $\varepsilon_0 > 0$ we have $n_{a_\varepsilon+\delta}(x) < < u - \varepsilon_0$.
If we denote $\varepsilon_n = u - u_n$ then $\varepsilon_n \downarrow 0$ and for large n $\varepsilon_n \leqq \varepsilon_0$ and hence $n_{a_\varepsilon+\delta}(x) < u - - \varepsilon_n = u_n$. But this inequality implies that $F_x(u_n) < F_x(u) + \delta - \varepsilon$ and therefore $\lim_{n\to\infty} F_x(u_n) = F_x(u)$.

Finally, it is necessary to prove that $\lim_{u\to\infty} F_x(u) = 1$. Let us suppose that $\lim_{u\to\infty} F_x(u) = = a_0 < 1$. Then for every $u > 0$ $F_x(u) \leqq a_0$. The definition of $F_x(\cdot)$ yields that for every $u > 0$ $a_0 \geqq \sup\{a > 0 : n_a(x) < u\}$. It implies that $n_{a_0+\delta}(x) = \infty$ if $a_0 + \delta < 1$, $\delta > 0$. This conclusion is in a contradiction to the assumption that $n_a(\cdot)$ for $a \in \langle 0, 1)$ are seminorms on S.

If $x = 0$ and $n_a(0) = 0$ for every $a \in \langle 0, 1)$, then for $u > 0$ $F_0(u) = \sup\{a \in \in \langle 0, 1) : 0 < u\} = 1$ and $F_0(u) = H(u)$ for every $u \in \mathbb{R}$. On the contrary, if $F_x(u) = = H(u)$ for every $u \in \mathbb{R}$, i.e. $F_x(u) = 1$ for every $u > 0$, then $1 = \sup\{a : n_a(x) < u\}$

for every $u > 0$. But this implies that $n_a(x) = 0$ for every $a \in \langle 0, 1)$ and by the assumptions of Theorem 5 we have $x = 0$. A further property, which must be verified, is the equality

$$F_{\lambda x}(u) = F_x(u/|\lambda|)$$

for every $x \in S$ and every real $\lambda \neq 0$.

$$F_{\lambda x}(u) = \sup\{a : n_a(\lambda x) < u\} = \sup\{a : |\lambda| \, n_a(x) < u\} =$$
$$= \sup\{a : n_a(x) < u/|\lambda|\} = F_x(u/|\lambda|) .$$

Finally, we must verify the generalized triangular inequality for the t-form $T(a, b) = \min(a, b)$. Since $\{n_a(\cdot)\}$ are seminorms in S then

$$F_{x+y}(u + v) = \sup\{a : n_a(x + y) < u + v\} \geqq \sup\{a : n_a(x) + n_a(y) < u + v\} \geqq$$
$$\geqq \sup\{\{a : n_a(x) < u\}, \{a : n_a(y) < v\}\} = \min(F_x(u), F_y(v))$$

because the functions $n_a(x)$, $n_a(y)$ are nondecreasing on $\langle 0, 1)$ in a.

A base for the topology generated by the class $N = \{n_a : a \in \langle 0, 1)\}$ of seminorms in S is formed by all sets

$$\{x : n_a(x) < \eta\} , \quad a \in \langle 0, 1) , \quad \eta > 0 .$$

The ε,η-topology has the base of ε,η-neighbourhoods $O(\varepsilon, \eta) = \{x \in S : F_x(\eta) > 1 - \varepsilon\}$. As $F_x(\eta) = \sup\{a : n_a(x) < \eta\}$ then $O(\varepsilon, \eta) = \{x : n_{1-\varepsilon}(x) < \eta\}$. This implies that both these topologies are identical. □

2. PROBABILITY AND t-NORM MIN

Theorem 6. Let an SLM-space $(S, \mathscr{J}, \min)$ be given. Then there exists a probability space $(N, \mathscr{A}, \mathsf{P})$ where

1) N is a suitable set of seminorms in S
2) for every $u \in \mathbb{R}$ and every $x \in S$ $\{n \in N : n(x) < u\} \in \mathscr{A}$
3) $\mathsf{P}(\{n \in N : n(x) < u\}) = F_x(u)$.

Proof. In Theorem 1 it was proved that for every $a \in \langle 0, 1)$ $n_a(x) = \inf\{\lambda > 0 : F_x(\lambda) > a\}$ is a seminorm in S. Let us denote $N = \{n_a : a \in \langle 0, 1)\}$ and let $\mathscr{B}_0 = \{\{n_a : u_1 \leqq n_a(x) < u_2\} : x \in S, u_1 \geqq 0, u_2 \geqq 0\}$. Since $n_a(x) < u_2$ if and only if $F_x(u_2) > a$ and similarly $n_a(x) \geqq u_1$ if and only if $F_x(u_1) \leqq a$, evidently $\{n_a : u_1 \leqq n_a(x) < u_2\} = \{n_a : F_x(u_1) \leqq a < F_x(u_2)\}$ and hence $\mathscr{B}_0 \subset \mathscr{A}_0 = \{\{n_a \in N : a_1 \leqq a < a_2\}, a_i \in \langle 0, 1\rangle, i = 1, 2\}$. Let us prove that A_0 is a semiring in N. If $A = \{n_a : a_1 \leqq a < a_2\}$, $B = \{n_a : b_1 \leqq a < b_2\}$ then $A \cap B = \{n_a : \max(a_1, b_1) \leqq a < \min(a_2, b_2)\}$ is also an element of $\mathscr{A}_0$. If $A \subset B$, i.e. $b_1 \leqq a_1$, $a_2 \leqq b_2$, then if we put $C_0 = A$, $C_1 = \{n_a : b_1 \leqq a < a_2\}$, $C_2 = B$ the sets C_0, C_1, C_2 belong to the system $\mathscr{A}_0$ and $C_1 - C_0 = \{n_a : b_1 \leqq a < a_1\}$, $C_2 - C_1 = \{n_a : a_2 \leqq$

$\leqq a < b_2\}$, and $C_1 - C_0$, $C_2 - C_1$ belong to $\mathscr{A}_0$ also. Since $N = \{n_a : 0 \leqq a < 1\}$ then $N \in \mathscr{A}_0$; we have proved that $\mathscr{A}_0$ is a semiring.

Since it can happen that $n_a(x) = n_b(x)$ for every $x \in S$ although $a \neq b$ then, in general, there is no one-to-one correspondence of all the sets $\langle a, b) \subset \langle 0, 1)$ and the elements of $\mathscr{A}_0$. Since it holds $n_a(x) \leqq n_b(x)$ for every $x \in S$ and for every pair $a \leqq b$, and $n_a(x)$ is right continuous for every $x \in S$, then every subset $\{n_a \in N : n_a = n_b\}$ has the minimal point, i.e. there exists such a number $a_0 \in \langle 0, 1)$ that if $n_a = n_{a_0}$ then $a \geqq a_0$. It is clear that every subset $\{n_a : n_a = n_b\}$ can be expressed in the unique way either as $\{n_a : a_0 \leqq a < b_0\}$ or as $\{n_a : a_0 \leqq a \leqq b_0\}$, where $b_0 = \sup\{b : n_b = n_{a_0}\}$. Let us consider the cases $a_0 < b_0$ only. There exists a countable number of such intervals at most. Let us denote by $\langle a_i, b_i)$, $i = 0, 1, 2, \ldots$ all exceptional subsets in $\langle 0, 1)$, i.e. for every $a \in \langle a_i, b_i)$ $n_a = n_{a_i}$ $i = 0, 1, 2, \ldots$. Then there exists a one-to-one correspondence between N and the set $\langle 0, 1) - \bigcup_i (a_i, b_i)$.

Let us define a probability measure μ on the measurable space $(\langle 0, 1), \mathscr{B})$ (where $\mathscr{B}$ is the σ-algebra of all Borel sets in $\langle 0, 1)$) in the following way:

if $\langle a_i, b_i)$ is an exceptional set in $\langle 0, 1)$ then we put $\mu(\langle a_i, b_i)) = b_i - a_i$ and $\mu((a_i, b_i)) = 0$, i.e. all the exceptional sets shall be atoms; if $\langle a, b) \cap \bigcup_i \langle a_i, b_i) = \emptyset$ then we put $(\langle a, b)) = b - a$, i.e. the ordinary Lebesgue measure will be considered outside all atoms.

Let us denote by $G(\cdot)$ the distribution function of this measure μ in $\langle 0, 1)$, i.e.

$$\mu(\langle a, b)) = G(b) - G(a), \quad G(0) = 0 .$$

Using this measure μ we shall define a probability measure P on the semiring $\mathscr{A}_0$. If $\{n_a \in N : a_0 \leqq a < b_0\}$ is an element of $\mathscr{A}_0$ then we put

$$\mathsf{P}(\{n_a \in N : a_0 \leqq a < b_0\}) = G(b_0) - G(a_0) .$$

It is no problem to verify that the set function P is a measure on the semiring $\mathscr{A}_0$ and hence P can be in the unique way enlarged onto the smallest σ-algebra $\mathscr{A}$ over $\mathscr{A}_0$. In this way the probability space $(N, \mathscr{A}, \mathsf{P})$ is constructed. Since for every $x \in S$

$$\mathscr{A}_0 \in \{n_a \in N : u_1 \leqq n_a(x) < u_2\} = \{n_a : F_x(u_1) \leqq a < F_x(u_2)\} ,$$

we can determine the probability P of this random event as follows

$$\mathsf{P}(\{n_a \in N : F_x(u_1) \leqq a < F_x(u_2)\}) = G(F_x(u_2)) - G(F_x(u_1)) .$$

If we express all the elements in $\mathscr{A}_0$ in the unambiguous form $\{n_a \in N : c \leqq a < d\}$ then there exists a one-to-one correspondence between $\mathscr{A}_0$ and some semiintervals in $\langle 0, 1)$. That unambiguous form can be obtained in this way:

if c and d are no exceptional points, i.e. $c, d \notin \bigcup_i \langle a_i, b_i)$, $\{n_a : c \leqq a < d\}$ remains without any changes;

if $c \in \langle a_i, b_i)$ then c is replaced by a_i;

if $d \in \langle a_j, b_j)$ then d is replaced by b_j.

This expression of all intervals $\{n_a \in N : c \leqq a < d\}$ will be called the maximal expression. It can be easily proved that all maximal intervals in $\mathscr{A}_0$ form a semiring and also $N = \{n_a : 0 \leqq a < 1\}$ belongs to this semiring. Let us note that the probability measure P on this semiring can be defined as

$$\mathsf{P}(\{n_a \in N : c \leqq a < d\}) = d - c = G(d) - G(c),$$

where c, d are boundary points of a maximal interval. Now, since $\{n_a : u_1 \leqq n_a(x) < u_2\} = \{n_a : F_x(u_1) \leqq a < F_x(u_2)\}$ then $\{n_a : u_1 \leqq n_a(x) < u_2\}$ belongs among the maximal intervals in every case. At the first sight it is clear that the interval $\{n_a : 0 \leqq a < d\}$ is not maximal if and only if for every $x \in S$ there exists no $u > 0$ with $F_x(u) = d$. From these reasons every interval $\{n_a : F_x(u_1) \leqq a < F_x(u_2)\}$ is maximal and hence $\mathsf{P}(\{n_a : 0 \leqq n_a(x) < \lambda\}) = F_x(\lambda)$ for every $\lambda > 0$. □

3. LOCAL CONVEXITY AND *SLM*-SPACES.

For every $x \in S$ the seminorm $n_a(x)$ as a function of the argument $a \in \langle 0, 1)$ is nondecreasing and hence Borel measurable. It is reasonable to consider the value $n(x) = \int_0^1 n_a(x)\,\mathrm{d}a$ (which of course can be infinite too).

Theorem 7. Let an *SLM*-space $(S, \mathscr{J}, \min)$ be given such that for every $x \in S$

$$n(x) = \int_0^1 n_a(x)\,\mathrm{d}a$$

is finite. Then $n(\cdot)$ is a norm in S and, in general, the topology induced by this norm $n(\cdot)$ is stronger than the ε,η-topology.

Proof. Since for every $a \in \langle 0, 1)$ $n_a(\lambda x) = |\lambda|\, n_a(x)$ and $n_a(x + y) \leqq n_a(x) + n_a(y)$ hold, then $n(\cdot)$ must be a seminorm in S also. If $n(x) = 0$ then $n_a(x) = 0$ a.s. with respect to the Lebesgue measure in $\langle 0, 1)$. As $n_a(x)$ is a nondecreasing function in the argument $a \in \langle 0, 1)$, then for every $a \in \langle 0, 1)$ it must hold $n_a(x) = 0$ what implies $x = 0$. Hence $n(\cdot)$ is a norm in S. Let us suppose that for some sequence $\{x_n\}_1^\infty \subset S$ $n(x_n) \to 0$ if $n \to \infty$, i.e.

$$\int_0^1 n_a(x_n)\,\mathrm{d}a \to 0,$$

but this is the convergence in the mean with respect to the Lebesgue measure l of $\{n_a(x_n)\}_1^\infty$ in $\langle 0, 1)$. This convergence in the mean implies the convergence in measure and hence

$$(\forall \varepsilon \in (0, 1\rangle\ \forall \eta > 0\ \exists n_0 \quad \forall n \geqq n_0) \Rightarrow l(\{a : n_a(x_n) < \eta\}) > 1 - \varepsilon.$$

Since $n_a(x) < \eta \Leftrightarrow F_x(\eta) > a$ then we have $l\{a : F_{x_n}(\eta) > a\} = F_{x_n}(\eta) > 1 - \varepsilon$ and this fact proves convergence of $\{x_n\}_1^\infty$ in the ε,η-topology. Since the convergence in the norm $n(\cdot)$ is equivalent with the convergence of $\{n_a(x_n)\}_1^\infty$ in the mean which is generally stronger than the convergence in measure of $\{n_a(x_n)\}_1^\infty$ then we proved that the ε,η-topology is weaker than the norm $n(\cdot)$ in $(S, \mathcal{J}, \min)$.

Indeed, the following example shows that the norm $n(\cdot)$ is stronger than the ε, η-topology. Let S be a linear space of all real sequences with finitely many non-zero members only. Let $x = (x_1, x_2, \ldots) \in S$. Let us define a sequence of seminorms in S $\{n_k\}_{k=1}^\infty$ as follows

$$n_k(x) = \sum_{i=1}^{k} |x_i| \,, \quad k = 1, 2, \ldots.$$

For every $a \in \langle 0, 1)$ define a seminorm n_a in this way

$$n_a = n_k \quad \text{for} \quad a \in \langle 1 - 2^{-k+1}, 1 - 2^{-k}) \,, \quad k = 1, 2, \ldots.$$

For every $x \in S$ $n_{(\cdot)}(x)$ is a nondecreasing, right continuous; $x = 0$ if and only if $n_a(x) = 0$ for every $a \in \langle 0, 1)$. A norm $n(x)$ is then given by

$$n(x) = \int_0^1 n_a(x) \,\mathrm{d}a = \sum_{i=1}^{\infty} 2^{-i+1} |x_i| \,.$$

Let $\{x^{(n)}\}$ be a sequence of elements in S defined by

$$x^{(n)} = 0 \quad \text{for} \quad i \neq n$$

$$x^{(n)} = 2^{n-1} \,.$$

Then $\lim_{n\to\infty} n_a(x^{(n)}) = 0$ for every $a \in \langle 0, 1)$, but $\lim_{n\to\infty} n(x^{(n)}) = \lim_{n\to\infty} 1 = 1$. □

Theorem 8. Let $(S, \mathcal{J}, \min)$ be finite-dimensional. Let $(e_1, e_2, \ldots, e_n)$ be an arbitrary base in S. If $M = \max_{1 \leq i \leq n} \{\int_0^1 n_a(e_i) \,\mathrm{d}a\}$ is finite, then the ε,η-topology is equivalent to the topology induced by the norm $n(\cdot) = \int_0^1 n_a(\cdot) \,\mathrm{d}a$.

Proof. Since every $x \in S$ can be expressed as $x = \sum_{i=1}^{n} \lambda_i e_i$, then $n_a(x) \leq \sum_{i=1}^{n} |\lambda_i| \, n_a(e_i)$ and $\int_0^1 n_a(x) \,\mathrm{d}a$ is finite for every $x \in S$. Let $x_k \to 0$ according to the ε,η-topology. As S is finite-dimensional then the ε,η-topology is equivalent to the usual Euclidean topology, see [1]; hence $\|x_k\| = \sum_{i=1}^{n} |\lambda_i^k| \to 0$ too, (recall that $x_k = \sum_{i=1}^{n} \lambda_i^k e_i$). The following inequality

$$\int_0^1 n_a(x_k) \,\mathrm{d}a \leq \sum_{i=1}^{n} |\lambda_i^k| \int_0^1 n_a(e_i) \,\mathrm{d}a \leq M \|x_k\|$$

completes the proof. □

Theorem 9. Let an *SLM*-space $(S, \mathcal{J}, \min)$ be given. Then

$$\varrho(x, y) = \int_0^1 \frac{n_a(x - y)}{1 + n_a(x - y)} \,\mathrm{d}a$$

is a metric in S and the topology induced by ϱ is equivalent to the ε,η-topology in S.

Proof. Surely, it holds for every pair $x, y \in S$ $\varrho(x, y) \geqq 0$ because for every $a \in \langle 0, 1)$ $n_a(x - y) \geqq 0$; similarly, $\varrho(x, y) = \varrho(y, x)$ and $\varrho(x, x) = 0$. If $\varrho(x, y) = 0$, i.e.

$$\int_0^1 \frac{n_a(x - y)}{1 + n_a(x - y)}\, \mathrm{d}a = 0\,,$$

it implies that $n_a(x - y) = 0$ a.s. in $\langle 0, 1)$ with respect to the Lebesgue measure. But, for every $z \in S$ $n_a(z)$ is a nondecreasing function in $\langle 0, 1)$ and this fact implies that $n_a(x - y) = 0$ for every $a \in \langle 0, 1)$ and therefore $x = y$. The triangular inequality $\varrho(x, y) \leqq \varrho(x, z) + \varrho(y, z)$ for every triple x, y, z in S follows from the fact that for every $a \in \langle 0, 1)$ $n_a(\cdot)$ is a seminorm. We have proved that ϱ is a metric in S.

Now, let us suppose that $\varrho(x_n, x) \to 0$ if $n \to \infty$ i.e.

$$\int_0^1 \frac{n_a(x_n - x)}{1 + n_a(x_n - x)}\, \mathrm{d}a \to 0 \quad \text{if} \quad n \to \infty\,.$$

But this convergence describes the convergence in measure of the sequence $\{n_a(x_n - x)\}_1^\infty$ with respect to the Lebesgue measure in $\langle 0, 1)$ and we know from the proof of the previous Theorem 7 that this convergence is equivalent to the convergence induced by the ε,η-topology. □

Theorem 10. Every metrizable locally convex linear topological space (S, τ) is an *SLM*-space $(S, \mathscr{I}, \min)$.

Proof. Let a linear topological space (S, τ) be given and let the topology τ be locally convex and metrizable. Thanks to these properties of the topology τ there exists a countable base $\mathscr{B}_0$ of neighbourhoods of the zero element in S

$$\mathscr{B}_0 = \{U_n\}_1^\infty$$

where U_n is an absolutely convex subset in S and we can suppose that $U_n \supset U_{n+1}$ for every n. To every neighbourhood U_n it is possible to construct a seminorm $p_n(\cdot)$ in S defined by the relation

$$p_n(x) = \inf\{\lambda > 0 : x \in \lambda U_n\}\,.$$

Since $U_n \supset U_{n+1}$ then $p_n(x) \leqq p_{n+1}(x)$ for every $x \in S$. Let $\{a_n\}_1^\infty$ be a sequence of positive numbers, $\sum_{n=1}^{\infty} a_n = 1$. For every $x \in S$ we assign by the mapping $\mathscr{I}$ the distribution function $\mathscr{I}(x) = F_x$

$$\begin{array}{ll} F_x(u) = 0 & u \in (-\infty,\ p_1(x)\rangle \\ F_x(u) = a_1 & u \in (p_1(x),\ p_2(x)\rangle \\ F_x(u) = a_1 + a_2 & u \in (p_2(x),\ p_3(x)\rangle \\ \quad\vdots & \quad\vdots \\ F_x(u) = \sum_1^n a_i & u \in (p_n(x),\ p_{n+1}(x)\rangle \\ \quad\vdots & \quad\vdots \end{array}$$

If $\lim p_n(x)$ is finite then in the case $p_n(x) = p_{n+1}(x) = \ldots$ for every $n \geqq n_0$ we put $F_x(u) = 1$ for every $u \in (\lim p_n(x), \infty)$, otherwise $F_x(u) = 1$ for every $u \in \langle \lim p_n(x), \infty)$. We do not eliminate the case of empty intervals.

If $x = 0$ then $p_n(x) = 0$ for every n and hence $F_0 = H$. Similarly, if $F_x = H$ then it implies that $p_n(x) = 0$ for every n and therefore $x = 0$. Since every $p_n(\cdot)$ is a seminorm in S then for every real λ $p_n(\lambda x) = |\lambda|\, p_n(x)$ and as follows

$$F_{\lambda x}(u) = F_x(u/|\lambda|)\,, \quad u \in \mathbb{R}\,,$$

for every $\lambda \neq 0$ and every $x \in S$.

Further, we shall prove the generalized triangular inequality

$$F_{x+y}(u + v) \geqq \min\left(F_x(u), F_y(v)\right).$$

The cases $u + v \leqq 0$ and $u + v \geqq \lim_{n\to\infty} p_n(x + y)$ are trivial. If $u + v \in (p_n(x + y), p_{n+1}(x + y)\rangle$ then $F_{x+y}(u + v) = \sum_{i=1}^{n} a_i$ and it implies that $u + v \leqq p_{n+1}(x) + p_{n+1}(y)$. However, it means that either $u \leqq p_{n+1}(x)$ or $v \leqq p_{n+1}(y)$ which implies that either

$$F_x(u) \leqq \sum_{i=1}^{n} a_i \quad \text{or} \quad F_y(v) \leqq \sum_{i=1}^{n} a_i$$

and thus the generalized triangular inequality holds.

By the above procedure we have proved that (S, τ) can be understood as an *SLM*-space $(S, \mathscr{F}, \min)$. The topology τ is formed by the class $\{p_n(\cdot)\}_1^\infty$ of seminorms in S, in other words it means that $x_n \to 0$ in topology τ if and only if $p_k(x_n) \to 0$ for every k.

Thus, for every $\varepsilon > 0$ and every k there exists an n_0 such that for every $n \geqq n_0$ $p_k(x_n) < \varepsilon$.

From the construction of the mapping $\mathscr{F}$ we conclude that

$$F_{x_n}(\varepsilon) > \sum_{i=1}^{k-1} a_i\,.$$

Since k is arbitrary, we have $F_{x_n}(\varepsilon) \to 1$ and thus we have proved that $x_n \to 0$ in ε,η-topology induced by the mapping $\mathscr{F}$.

Let $x_n \to 0$ in the ε,η-topology, i.e. $F_{x_n}(u) \to 1$ if $n \to \infty$ for every $u > 0$. Let a natural number k and $\eta > 0$ be given. Let us set $\varepsilon = 1 - \sum_{i=1}^{k-1} a_i$. Then there exists a natural number n_0 such that for every $n \geqq n_0$ $F_{x_n}(\eta) > 1 - \varepsilon = \sum_{i=1}^{k-1} a_i$. According to the definition of F_x it follows that $\eta > p_k(x_n)$ for every $n \geqq n_0$ and hence $p_k(x_n) \underset{n}{\to} 0$. In this way we have proved that the ε,η-topology implies the original topology τ in S. □

We have shown that an *SLM*-space $(S, \mathscr{F}, \min)$ can be characterized by a special class $N_S = \{n_a(\cdot), a \in \langle 0, 1)\}$ of seminorms in S. Now, we shall generalize this result for a weaker t-norm than min. Let us consider such t-norms which are strictly increasing functions on the diagonal in $\langle 0, 1\rangle \times \langle 0, 1\rangle$. If $a \in \langle 0, 1)$ and if we consider similarly $n_a(x) = \inf\{\lambda > 0 : F_x(\lambda) > a\}$ we obtain for every $a \in \langle 0, 1)$ a positively homogeneous nonnegative functional in S.

Since the t-norm T is weaker than min, it need not hold $n_a(x + y) \leqq n_a(x) + n_a(y)$ in S. But we can prove an analogous inequality

$$n_{T(a \cdot a)}(x + y) \leqq n_a(x) + n_a(y) .$$

From the relation between $F_x(\cdot)$ and $n_a(\cdot)$ it follows that $n_a(x) < u$ if and only if $F_x(u) > a$, $n_a(y) < v$ if $F_y(v) > a$ and $n_{T(a \cdot a)}(x + y) < u + v$ if and only if $F_{x+y}(u + v) > T(a, a)$. Since

$$n_{T(a \cdot a)}(x + y) = \inf\{u + v > 0 : F_{x+y}(u + v) > T(a, a)\} \leqq$$

$$\leqq \inf\{u + v > 0 : T(F_x(u), F_y(v)) > T(a, a)\} \leqq$$

$$\leqq \inf\{u > 0 : F_x(u) > a\} + \inf\{v > 0 : F_y(v) > a\} = n_a(x) + n_a(y) ,$$

t-norm $T(a, a)$ is strictly increasing on the diagonal in $\langle 0, 1\rangle \times \langle 0, 1\rangle$, then we obtained a weaker form of the triangular inequality

$$n_{T(a \cdot a)}(x + y) \leqq n_a(x) + n_a(y)$$

determined by the t-norm T in S. The class $N_S^T = \{n_a : a \in \langle 0, 1)\}$ of functionals in S will be called the class of the generalized seminorms in S. Every generalized seminorm $n_a(\cdot)$ satisfies in S:

1) $n_a(0) = 0$; if $n_a(x) = 0$ for every $a \in \langle 0, 1)$ then $x = 0$.
2) $n_a(\lambda x) = |\lambda|\, n_a(x)$ for every $\lambda \in R$ and every $x \in S$.
3) $n_{T(a \cdot a)}(x + y) \leqq n_a(x) + n_a(y)$ for every $x, y \in S$ and every $a \in \langle 0, 1)$.

Analogously, as we constructed a probability space to the SLM-space $(S, \mathscr{J}, \min)$ we can construct a similar probability space $(N_S^T, \mathscr{A}, \mathrm{P})$ to the space $(S, \mathscr{J}, T)$ with quite analogous properties.

At the end of this we introduce two theorems which describe bounded and totally bounded sets in the ε,η-topology.

Theorem 11. A subset $K \subset (S, \mathscr{J}, T)$ is bounded in the ε,η-topology if and only if

$$(\forall a \in \langle 0, 1)\ \exists k_a > 0\ \forall x \in K) \Rightarrow F_x(k_a) > a .$$

Proof. If K is ε,η-bounded, then for every ε,η-neighbourhood $O(\varepsilon, \eta)$ there exists such a $\lambda > 0$ that

$$K \subset \lambda\, O(\varepsilon, \eta) = O(\varepsilon, \lambda\eta)\,, \quad \text{i.e. for every} \quad x \in K \quad F_x(\lambda\eta) > 1 - \varepsilon .$$

If we put $1 - \varepsilon = a$, $k_a = \lambda\eta$ then $F_x(k_a) > a$.

On the contrary, if $\{x_n\}_1^\infty$ is a sequence in K and $\{\lambda_n\}_1^\infty$ is a sequence of reals with $\lambda_n \to 0$ then for arbitrary $u > 0$ $F_{\lambda_n x_n}(u) = F_{x_n}(u/|\lambda_n|)$ and because $(u/|\lambda_n|) \to \infty$

if $n \to \infty$ then there exists such an n_0 that for every $n \geqq n_0$ $u \geqq k_a|\lambda_n|$ and hence $F_{\lambda_n x_n}(u) > a$ which proves that $\lambda_n x_n \to 0$ in the ε,η-topology. The set K is ε,η-bounded. $\square$

Remark 3. Especially, for an SLM-space $(S, \mathscr{J}, \min)$ Theorem 11 can be formulated in a more obvious sense. If K is ε,η-bounded, then

$$(\forall a \in \langle 0, 1) \, \exists k_a > 0 \, \forall x \in K) \Rightarrow F_x(k_a) > a$$

and hence $n_a(x) = \inf\{\lambda > 0 : F_x(\lambda) > a\} < k_a$ for every $x \in K$. Thus, a subset $K \subset S$ is ε,η-bounded if an only if K is bounded in every topology τ_a induced in S by the seminorm $n_a(\cdot)$.

Quite analogously, it is possible to characterize totally bounded sets in an SLM-space $(S, \mathscr{J}, \min)$.

Theorem 12. A set $K \subset (S, \mathscr{J}, \min)$ is totally bounded in the ε,η-topology if and only if it is totally bounded in every topology τ_a.

Proof. Let K be totally bounded in every topology τ_a. It means that for $a \in \langle 0, 1)$ and every $\eta > 0$ there exists a finite η-net in the topology τ_a, i.e. there exist elements $x_1^a, x_2^a, \ldots, x_{s_a}^a$ in S such that for every $x \in K$ $n_a(x - x_i^a) < \eta$ at least for one $i \in \{1, 2, \ldots, s_a\}$. It follows that $F_{x - x_i^a}(\eta) > a$, i.e. $x \in O(x_i^a, 1 - a, \eta)$ and hence

$$K \subset \bigcup_{i=1}^{s_a} O(x_i^a, 1 - a, \eta) = \bigcup_{i=1}^{s_a} \{x_i^a + O(1 - a, \eta)\}.$$

We have proved that for every neighbourhood $O(\varepsilon, \eta)$ we can find a finite set $\{x_i\}_1^{n(\varepsilon,\eta)}$ of elements in S such that

$$K \subset \bigcup_{i=1}^{n(\varepsilon,\eta)} \{x_i + O(\varepsilon, \eta)\}$$

and hence K is totally bounded in the ε,η-topology. The opposite inclusion is quite clear because the ε,η-topology in S is stronger than every topology τ_a. $\square$

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A more detailed list of references is given in [1].

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