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STATISTICAL LINEAR SPACES

Part I. Properties of ε, η-topology

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The definition of the statistical linear space in the Menger sense (*SLM*-space) is given in this paper. The ε, η-topology is introduced and the basic properties of *SLM*-spaces as linear topological spaces are investigated.

0. INTRODUCTION AND PRELIMINARIES

In this paper we shall deal with basic properties of statistical linear spaces in the Menger sense (*SLM*-space) which are a special case of statistical metric spaces in the Menger sense (*SMM*-space). *SMM*-spaces are a generalization of the usual notion of metric spaces in that sense that a metric is replaced by a collection of probability distribution functions. Similarly, *SLM*-spaces are a generalization of linear normed spaces where a norm is substituted by a suitable family of probability distribution functions.

This paper contains in Section 1 the definition of *SLM*-spaces and the main properties of them together with three examples.

The definition of the ε, η-topology and basic properties of *SLM*-spaces as linear topological spaces are in Section 2. Section 3 contains some properties of ε, η-neighbourhoods from a base for the ε, η-topology. In Section 4 properties of the mapping $\mathscr{J}$, which is defined on an *SLM*-space and takes its values in the Lévy space of probability distribution functions, are studied.

The notation of an *SMM*-space is studied in many details in [1]. A detailing discussion of the original Menger definition of the generalized triangular inequality is made there. Under these conclusions the authors suggested the following definition of an *SMM*-space.

Definition 1. By a statistical metric space in the sense of Menger we shall call a triple $(S, \mathscr{K}, T)$ where S is a nonempty set, $\mathscr{K}$ is a mapping $\mathscr{K} : S \times S \to \mathscr{F}$, where $\mathscr{F}$ is the set of all one-dimensional probability distribution functions, satisfying $(\mathscr{H}(x, y) = F_{xy}(\cdot))$

1. $(F_{xy}(u) = 1$ for $u > 0) \Leftrightarrow x = y$
2. $F_{xy}(0) = 0$ for every pair $x, y \in S$
3. $F_{xy}(u) = F_{yx}(u)$ for every $u \in \mathbb{R}$ and every pair $x, y \in S$ ($\mathbb{R}$ is the set of reals)
4. $F_{xz}(u + v) \geqq T(F_{xy}(u), F_{yz}(v))$ for every $x, y, z \in S$ and every $u, v \in \mathbb{R}$ where T is a t-norm defined on $\langle 0, 1\rangle \times \langle 0, 1\rangle$ with values in $\langle 0, 1\rangle$ and satisfying properties:
   (a) $T(a, b) = T(b, a); T(a, 1) = a$ for $a > 0$
   (b) $T(a, b) \leqq T(c, d)$ for $a \leqq c$, $b \leqq d$
   $T(T(a, b), c) = T(a, T(b, c))$
   (d) $T(0, 0) = 0$.

Definition 1 yields immediately that every t-norm T satisfies $T(a, b) \leqq \min(a, b)$ where min is a t-norm too. Further important examples of t-norms are $T(a, b) = ab$, $T(a, b) = \max(a + b - 1, 0)$. It is worth quoting [10] where one can see a close relation between t-norms and 2-dimensional copulas.

Further, in [1] the ε, η-topology is introduced by the neighbourhoods of the form

$$N_x(\varepsilon, \eta) = \{y \in S : F_{xy}(\eta) > 1 - \varepsilon\}, \quad x \in S, \quad \eta > 0, \quad 0 < \varepsilon \leqq 1$$

and under the continuity of the t-norm T it is proved that these neighbourhoods form a base for a Hausdorff topology in S. This topology is called the ε, η-topology. The paper [2] studies the question under which conditions the ε, η-topology is metrizable. If $\sup_{a<1} T(a, a) = 1$ then the system $\mathscr{N} = \{U(\varepsilon, \eta)\}$ where $U(\varepsilon, \eta) = \{(x, y) \in S \times S : F_{xy}(\eta) > 1 - \varepsilon\}$ $(\eta > 0, \varepsilon \in (0, 1\rangle)$ is a base of a Hausdorff uniformity in $S \times S$.

The mapping $\mathscr{H} : S \times S \to \mathscr{F}$ where $\mathscr{F}$ is the Lévy space of probability distribution functions is studied in [3]. If $\lim_{v \uparrow 1} T(a, v) = a$ uniformly in $\langle 0, 1\rangle$, then $\mathscr{H}$ is uniformly continuous with respect to the ε, η-topology in $S \times S$.

The problem of a completion of SMM-spaces is solved in [4]. It is proved (under certain conditions on the t-norm T) that every SMM-space can be (up to an isomorphism) completed by the maintaince of the t-norm in the unique way.

In [5] it is suggested one of the possible generalizations of the triangular inequality. The demand 4 in Definition 1 is replaced by 4': $(F_{xy}(u) = 1$ and $F_{yz}(v) = 1) \Rightarrow F_{xz}(u + v) = 1$, which is of course weaker than 4 in Definition 1. Further, in this paper a relation between the mapping $\mathscr{H}$ (mentioned above) and a certain class of semimetrics on S is studied and it is proved, in the case of the t-norm $T = \min(a, b)$ the existence of a probability space $(D, \mathscr{B}, \mu)$ where D contains some semimetrics on S, all sets of the form $\{d \in D : d(x, y) > u\}$ $x, y \in S$, $u \in \mathbb{R}$ belong to $\mathscr{B}$ and

$$\mu\{d \in D : d(x, y) > u\} = F_{xy}(u).$$

At the beginning the theory of SMM-spaces belonged rather to the functional analysis than to the probability theory; e.g. many articles are devoted to problems of fixed points of mappings defined on SMM-spaces. Recently, some papers occurred where the connection with the probability theory is quite evident, see, e.g. [7], [8], [9].

1. DEFINITION OF SML-SPACE, BASIC PROPERTIES, EXAMPLES

In this paper a special case of statistical metric spaces is considered. The definition of SMM-spaces is based on that fact that although the distance of two points is a fixed nonnegative number, an observer can measure this distance with certain errors. His measurements are affected by errors and from this point of view a distance is a random variable with its distribution function. Similarly, we can consider the case of a normed linear space, where a norm is the distance measured from the zero element. Properties of a norm and Definition 1 of the SMM-space lead us to the following definition of the linear statistical space.

Definition 2. Let S be a real linear space, let $\mathscr{F}$ be the set of all probability distribution functions defined on the real line $\mathbb{R}$. Let $\mathscr{J} : S \to \mathscr{F}$ be a given mapping. For every $x \in S$ let us denote $\mathscr{J}(x) = F_x \in \mathscr{F}$ and we demand that $\mathscr{J}$ satisfies:

1. $x = 0 \Leftrightarrow F_x = H$ where $H(u) = 0\ u \leqq 0$; $H(u) = 1\ u > 0$
2. $F_{\lambda x}(u) = F_x(u/|\lambda|)$ for every $x \in S$ and every $\lambda \neq 0$.
3. $F_x(u) = 0$ for every $u \leqq 0$ and every $x \in S$.
4. $T(F_x(u), F_y(v)) \leqq F_{x+y}(u + v)$ for every u, $v \in \mathbb{R}$ and every pair x, $y \in S$ where T is a t-norm satisfying (a), (b), (c), (d) in Definition 1.

Under these conditions the triple $(S, \mathscr{J}, T)$ is called a linear statistical space in the Menger sense (SLM-space).

Example 1. Let $S = \mathbb{R}$, let G be a distribution function with $G(0) = 0$ and $G \neq H$. If $x \in S$ let us define

$$\mathscr{J}(x) = F_x(\cdot) = G\left(\frac{\cdot}{|x|}\right) \quad \text{for} \quad x \neq 0$$

$$\mathscr{J}(0) = H(\cdot) \quad \text{and} \quad T(a, b) = \min(a, b);$$

then $(\mathbb{R}, \mathscr{J}, \min)$ is an SLM-space. As we assume $G \neq H$ then $x = 0$ if and only if $F_x = H$. Further, $F_x(0) = 0$ for every $x \in \mathbb{R}$ thanks to the assumption $G(0) = 0$. Thus, we have

$$F_{\lambda x}(u) = G\left(\frac{u}{|\lambda x|}\right) = G\left(\frac{u}{|\lambda|\,|x|}\right) = G\left(\frac{u}{|\lambda|}\frac{1}{|x|}\right) = F_x\left(\frac{u}{|\lambda|}\right)$$

for every $\lambda \in \mathbb{R}$, $\lambda \neq 0$ and every $x \in \mathbb{R}$. The main problem is to prove the triangular inequality in the form

$$F_{x+y}(u + v) \geqq \min(F_x(u), F_y(v)), \quad \text{i.e.}$$

$$(*) \qquad G\left(\frac{u + v}{|x + y|}\right) \geqq \min\left(G\left(\frac{u}{|x|}\right), G\left(\frac{v}{|y|}\right)\right).$$

If $u \leqq 0$ or $v \leqq 0$ then the inequality $(*)$ is true because $G(0) = 0$. In the case $u > 0$ and $v > 0$, $x = 0$ or $y = 0$ or $x + y = 0$ the generalized triangular inequality is trivial. As the function G is nondecreasing, the inequality $(*)$ for $u > 0$, $v > 0$, $|x + y| > 0$, $|x| > 0$, $|y| > 0$ follows from the inequality

$$\frac{u + v}{|x + y|} \geqq \min\left(\frac{u}{|x|}, \frac{v}{|y|}\right).$$

Indeed, let us assume $u > 0$, $v > 0$, $|x + y| > 0$ and $(u + v)/|x + y| > \min(u/|x|, v/|y|)$. It implies that simultaneously $(u + v)/|x + y| > u/|x|$ and $(u + v)/|x + y| > v/|y|$, thus $(u + v)|x| > u|x + y|$ and $(u + v)|y| > v|x + y|$, what gives $|x| + |y| > |x + y|$ and that is a contradiction. This completes the proof of that fact that $(\mathbb{R}, \mathscr{F}, \min)$ is an SLM-space.

Example 2. Let S be the set of all real sequences, i.e. $S = \{x : x = (x_1, x_2, x_3, \ldots, x_n, \ldots)\}$, where the operations of addition and scalar multiplication are defined coordinatewisely. Let $\{a_n\}_{n=1}^{\infty}$ be a sequence of positive numbers such that $\sum_{n=1}^{\infty} a_n = 1$. Let us define the mapping $\mathscr{F} : S \to \mathscr{F}$ in the following way:

if $x = (x_1, x_2, x_3, \ldots, x_n, \ldots)$ then we put

$$\begin{array}{lll}
F_x(u) = 0 & \text{for} & u \leqq |x_1| \\
F_x(u) = a_1 & \text{for} & |x_1| < u \leqq |x_1| + |x_2| \\
F_x(u) = a_1 + a_2 & \text{for} & |x_1| + |x_2| < u \leqq |x_1| + |x_2| + |x_3| \\
\vdots & \vdots & \vdots \\
F_x(u) = \sum_{i=1}^{n} a_i & \text{for} & \sum_{i=1}^{n} |x_i| < u \leqq \sum_{i=1}^{n+1} |x_i| \\
\vdots & \vdots & \vdots
\end{array}$$

In the case if $\sum_{i=1}^{\infty} |x_i| < \infty$ we must consider two possibilities:

a) $\sum_{i=1}^{\infty} |x_i|$ contains infinitely many non-zero elements, then $F_x(u) = 1$ for $u \geqq \sum_{i=1}^{\infty} |x_i|$

b) $\sum_{i=1}^{\infty} |x_i|$ contains finitely many non-zero elements only, then $F_x(u) = 1$ for $u > \sum_{i=1}^{\infty} |x_i|$.

We do not eliminate the case of an empty interval.

As a t-norm we choose again the function $\min(a, b)$. Then the triple $(S, \mathscr{F}, \min)$ is an SLM-space. Surely, $F_x = H$ if and only if $x = 0$ because for every $x \neq 0$ at least one coordinate x_i differs from zero. Further, $F_{\lambda x}(u) = F_x(u/|\lambda|)$ for every $x \in S$, $\lambda \neq 0$, $u \in \mathbb{R}$ because if $\lambda \neq 0$, $u > 0$, $x = 0$ then $\lambda x = 0$ and $F_{\lambda x}(u) = 1$. If $u \leqq 0$ then for every $x \in S$ it is $F_x(u) = 0$ hence $F_{\lambda x}(u) = 0$ also for every $\lambda \in \mathbb{R}$. Now, in the last case $\lambda \neq 0$, $u > 0$, $x \neq 0$ we have

$$F_x\left(\frac{u}{|\lambda|}\right) = \sum_{i=1}^{n} a_i \quad \text{if and only if} \quad \sum_{i=1}^{n} |x_i| < \frac{u}{|\lambda|} \leqq \sum_{i=1}^{n+1} |x_i|\,,$$

what is

$$\sum_{i=1}^{n} |\lambda x_i| < u \leqq \sum_{i=1}^{n+1} |\lambda x_i|\,.$$

The previous inequality expresses the value of $F_{\lambda x}$ at the point u, i.e.

$$F_{\lambda x}(u) = \sum_{i=1}^{n} a_i \quad \text{if and only if} \quad \sum_{i=1}^{n} |\lambda x_i| < u \leqq \sum_{i=1}^{n+1} |\lambda x_i|\,.$$

At the end we must verify the generalized triangular inequality with the t-norm min. If $u + v \in (\sum_{i=1}^{n} |x_i + y_i|, \sum_{i=1}^{n+1} |x_i + y_i|\rangle$ then either $u \leqq \sum_{i=1}^{n+1} |x_i|$ or $v \leqq \sum_{i=1}^{n+1} |y_i|$, hence either $F_x(u) \leqq \sum_{i=1}^{n} a_i$ or $F_y(v) \leqq \sum_{i=1}^{n} a_i$, but in every case the inequality $\min(F_x(u), F_y(v)) \leqq F_{x+y}(u + v)$ holds. The case $F_x(u) = 1$ is investigated in a similar way.

Example 3. Let $(\Omega, \mathscr{A}, \mathsf{P})$ be a probability space. Two random variables ξ, η on Ω with $\mathsf{P}\{\omega : \xi(\omega) = \eta(\omega)\} = 1$ shall belong to the same class of equivalence. Let S denote these classes of equivalence on Ω. Evidently, S is a linear space. Let us define a mapping $\mathscr{F}$ in the following way:

$$\mathscr{F}(\xi)[u] = \mathsf{P}\{\omega : |\xi(\omega)| < u\} = F_\xi(u), \xi \in S, \ u \in \mathbb{R}\,,$$

As a t-norm we choose $m(a, b) = \max(a + b + b - 1, 0)$. Then the triple $(S, \mathscr{F}, m)$ is an SLM-space.

It is clear that for every $\lambda \neq 0$ and $\xi \in S$ it holds

$$\mathsf{P}\{\omega : |\lambda\xi(\omega)| < u\} = \mathsf{P}\left\{\omega : |\xi(\omega)| < \frac{u}{|\lambda|}\right\}$$

and hence $F_{\lambda\xi}(u) = F_\xi(u/|\lambda|)$. Similarly, $\mathsf{P}\{\omega : |\xi(\omega)| < u\} = 0$ for $u \leqq 0$ gives $F_\xi(u) = 0$ for every $u \leqq 0$. Surely, $F_\xi(u) = H(u)$ for every $u \in \mathbb{R}$ if and only if $\xi = 0$. The validity of the generalized triangular inequality is based on the results in [10]. It holds that the joint distribution function $G_{\xi,\eta}(\cdot, \cdot)$ of $\xi, \eta \in S$ can be expressed as a function of their marginal distribution functions $g_\xi(\cdot)$, $g_\eta(\cdot)$ $G_{\xi,\eta}(u, v) = C(g_\xi(u), g_\eta(v))$ where C is a 2-dimensional copula generally depending on a couple ξ, η. This copula C is a function defined on $\langle 0, 1\rangle \times \langle 0, 1\rangle$ satisfying the following inequality

$$\min(a, b) \geqq C(a, b) \geqq m(a, b)\,.$$

The inclusions $\{\omega : |\xi(\omega) + \eta(\omega)| < u + v\} \supset \{\omega : |\xi(\omega)| + |\eta(\omega)| < u + v\} \supset$ $\supset \{\omega : |\xi(\omega)| < u, |\eta(\omega)| < v\}$ give

$$F_{\xi+\eta}(u + v) = \mathsf{P}\{\omega : |\xi(\omega) + \eta(\omega)| < u + v\} \geqq$$
$$\geqq \mathsf{P}\{\omega : |\xi(\omega)| < u, |\eta(\omega)| < v\} = C(F_\xi(u), F_\eta(v)) \geqq m(F_\xi(u), F_\eta(v)) .$$

It proves the validity of the generalized triangular inequality with the t-norm m.

Theorem 1. Every SLM-space is an SMM-space with the same t-norm.

Proof. Let $(S, \mathscr{J}, T)$ be an SLM-space. Let us define the mapping $\mathscr{K}(x, y) = \mathscr{J}(x - y)$, $\mathscr{K} : S \times S \to \mathscr{F}$. Then the triple $(S, \mathscr{K}, T)$ is an SMM-space. $\mathscr{J}(x) = H$ if and only if $x = 0$. The mapping $\mathscr{K}$ is surely symmetric, because $\mathscr{J}(x - y) = \mathscr{J}(y-x)$. If we denote $\mathscr{K}(x, y) = F_{xy}$, $\mathscr{J}(x) = F_x$, then the generalized triangular inequality holds, because

$$T(F_{xy}(u), F_{yz}(v)) = T(F_{x-y}(u), F_{y-z}(v)) \leqq F_{x-z}(u + v) = F_{xz}(u + v) . \qquad \square$$

Remark. Let S be an n-dimensional real linear space. Then the triple $(S, \mathscr{J}, T)$ is an SLM-space if and only if to every n-tuple of real numbers $(\lambda_1, \lambda_2, \ldots, \lambda_n)$ a probability distribution function $F_{(\lambda_1,\lambda_2,\ldots,\lambda_n)}$ corresponds such that

1. $F_{(\lambda_1,\lambda_2,\ldots,\lambda_n)} = H$ if and only if $\lambda_1 = \lambda_2 = \ldots = \lambda_n = 0$
2. $F_{(\mu\lambda_1,\mu\lambda_2,\ldots,\mu\lambda_n)}(u) = F_{(\lambda_1,\lambda_2,\ldots,\lambda_n)}(u/|\mu|)$ for every $\mu \neq 0$, $u \in \mathbb{R}$ and every n-tuple $(\lambda_1, \lambda_2, \ldots, \lambda_n)$
3. $F_{(\lambda_1,\lambda_2,\ldots,\lambda_n)}(0) = 0$ for every n-tuple $(\lambda_1, \lambda_2, \ldots, \lambda_n)$
4. $T(F_{(\lambda_1,\lambda_2,\ldots,\lambda_n)}(u), F_{(\mu_1,\mu_2,\ldots,\mu_n)}(v)) \leqq F_{(\lambda_1+\mu_1,\lambda_2+\mu_2,\ldots,\lambda_n+\mu_n)}(u + v)$ for every n-tuple $(\lambda_1, \lambda_2, \ldots, \lambda_n)$ and $(\mu_1, \mu_2, \ldots, \mu_n)$ and every $u, v \in \mathbb{R}$ (T is a t-norm).

2. TOPOLOGY IN SLM-SPACES

We shall use usual notions in the topology and in the theory of linear topological spaces; see, e.g. [11]. Only the notions important for us shall be defined explicitly.

Definition 4. Let $(S, \mathscr{J}, T)$ be a statistical linear space in the sense of Menger, let $x \in S$, $0 < \varepsilon \leqq 1$, $\eta > 0$. Then the subset of S

$$O(x, \varepsilon, \eta) = \{z \in S : F_{x-z}(\eta) > 1 - \varepsilon\}$$

is called the ε, η-neighbourhood of the point x.

As the space S is linear, it is sufficient to introduce neighbourhoods of the zero element only, i.e. the neighbourhoods of the form $O(\varepsilon, \eta) = \{z : F_z(\eta) > 1 - \varepsilon\}$. We shall assume the continuity of the t-norm T on $\langle 0, 1\rangle \times \langle 0, 1\rangle$. Under this assumption it is possible to prove that the collection of ε, η-neighbourhoods forms a base of a topology in the space $(S, \mathscr{J}, T)$. It is clear that $0 \in O(\varepsilon, \eta)$ for every $0 < \varepsilon \leqq 1$, $\eta > 0$, because $F_0(\eta) = H(\eta) = 1 > 1 - \varepsilon$. Further, if two ε, η-neighbourhoods $O(\varepsilon, \eta)$, $O(\varepsilon', \eta')$ are given, then there exists a neighbourhood $O(\varepsilon^*, \eta^*)$ such that

$$O(\varepsilon^*, \eta^*) \subset O(\varepsilon, \eta) \cap O(\varepsilon', \eta') .$$

It is sufficient to put $\varepsilon^* = \min(\varepsilon, \varepsilon')$, $\eta^* = \min(\eta, \eta')$ because

$$O(\varepsilon, \eta) \cap O(\varepsilon', \eta') = \{z \in S : F_z(\eta) > 1 - \varepsilon, F_z(\eta') > $$
$$> 1 - \varepsilon'\} \supset \{z : F_z(\min(\eta, \eta')) > 1 - \min(\varepsilon, \varepsilon')\} = O(\varepsilon^*, \eta^*) .$$

Similarly, if $\varepsilon \leqq \varepsilon'$, $\eta \leqq \eta'$ then

$$O(\varepsilon, \eta) \subset O(\varepsilon', \eta') .$$

The last property which is necessary for a base of neighbourhoods in a topology is that for every ε, η-neighbourhood $O(\varepsilon, \eta)$ and every $y \in O(\varepsilon, \eta)$ there exists such an ε, η-neighbourhood that $O(y, \varepsilon^*, \eta^*) \subset O(\varepsilon, \eta)$. Let $O(\varepsilon, \eta)$ and y be given. As the function F_y being a probability distribution function is left continuous at η, there exist $\eta_0 < \eta$, $\varepsilon_0 < \varepsilon$ that $F_y(\eta_0) > 1 - \varepsilon_0 > 1 - \varepsilon$. Now, we choose η^* such that $0 < \eta^* < \eta - \eta_0$ and ε^* such that $T(1 - \varepsilon_0, 1 - \varepsilon^*) > 1 - \varepsilon$ (such an ε^* exists because the t-norm T is assumed continuous and $T(a, 1) = a$). Let $s \in O(y, \varepsilon^*, \eta^*)$ then $F_s(\eta) \geqq T(F_y(\eta_0), F_{y-s}(\eta - \eta_0)) \geqq T(F_y(\eta_0), F_{y-s}(\eta^*)) \geqq T(1 - \varepsilon_0, 1 - \varepsilon^*) > 1 - \varepsilon$ and $s \in O(\varepsilon, \eta)$.

Definition 5. The topology generated under the continuity of the t-norm T by the base $\mathscr{U} = \{O(\varepsilon, \eta) : 0 < \varepsilon \leqq 1, \eta > 0\}$ of the neighbourhoods of the zero element in $(S, \mathscr{J}, T)$ will be called the ε, η-topology.

Definition 6. A sequence $\{x_n\}_{n=1}^{\infty} \subset (S, \mathscr{J}, T)$ will be called F-convergent at $x \in S$, if

$$\lim_{n \to \infty} F_{x_n - x}(u) = H(u)$$

for every $u \in \mathbb{R}$ (in symbols $x_n \xrightarrow{F} x$).

Lemma 1. A sequence $\{x_n\}_{n=1}^{\infty} \subset (S, \mathscr{J}, T)$ is F-convergent at $x \in S$ if and only if

$$(\forall \varepsilon \in (0, 1\rangle \, \forall \eta > 0 \, \exists n_0 \, \forall n \geqq n_0) \Rightarrow (x_n \in O(\varepsilon, \eta)) .$$

Proof. If $\lim_{n \to \infty} F_{x_n}(u) = H(u)$, $u > 0$, it is $\lim_{n \to \infty} F_{x_n}(u) = 1$, then

$$(\forall u > 0 \, \forall \varepsilon \in (0, 1\rangle \, \exists n_0 \, \forall n \geqq n_0) \Rightarrow F_{x_n}(u) > 1 - \varepsilon \Leftrightarrow x_n \in O(\varepsilon, u) .$$

Conversely, if $(\forall \varepsilon \in (0, 1\rangle \, \forall \eta > 0 \, \exists n_0 \, \forall n \geqq n_0) \Rightarrow x_n \in O(\varepsilon, \eta) \Leftrightarrow F_{x_n}(\eta) > 1 - \varepsilon$, it is precisely that $\lim_{n \to \infty} F_{x_n}(\eta) = 1$ for every $\eta > 0$. If $u \leqq 0$ we have $F_{x_n}(u) = 0$ for every n. □

Theorem 2. Every SLM-space $(S, \mathscr{J}, T)$ with a continuous t-norm is with respect to the ε, η-topology a Hausdorff linear topological space with a countable base of neighbourhoods of the zero element and hence it is metrizable.

Proof. If we choose any sequences $\{\varepsilon_n\}_1^\infty$, $\{\eta_n\}_1^\infty$ such that $\varepsilon_n \downarrow 0$, $\eta_n \downarrow 0$ then $\{O(\varepsilon_n, \eta_n)\}_1^\infty$ is a base of neighbourhoods of the origin for the ε, η-topology, because for every $O(\varepsilon, \eta)$ we can find a pair ε_{n_0}, η_{n_0} such that $\varepsilon_{n_0} \leqq \varepsilon$, $\eta_{n_0} \leqq \eta$ and hence $O(\varepsilon_{n_0}, \eta_{n_0}) \subset O(\varepsilon, \eta)$.
This space will be a Hausdorff space if and only if $\bigcap_{U \in \mathscr{B}(0)} U = \{0\}$ where $\mathscr{B}(0)$ is a base of neighbourhoods of the origin for the ε, η-topology. In our case it is necessary to prove that $\bigcap_{0<\varepsilon\leqq 1, \eta>0} O(\varepsilon, \eta) = \{0\}$. Let us suppose that $x \in \bigcap_{\varepsilon,\eta} O(\varepsilon, \eta)$. Then for every $\eta > 0$ and every $\varepsilon \in (0, 1)$ $F_x(\eta) > 1 - \varepsilon$, in other words $F_x(\eta) = 1$ for every $\eta > 0$. It implies that $x = 0$ in S. We have proved that a countable base of the origin for the ε, η-topology exists and hence the ε, η-topology is metrizable.

Using Lemma 1 and the existence of a countable base for the ε, η-topology at the origin we can easily prove that linear operations and the ε, η-topology are consistent. Let $\lambda_n \to \lambda$ in reals, let $x_n \to x$ in S in the ε, η-topology. Then $\lambda_n x_n - \lambda x = \lambda_n(x_n - x) + (\lambda_n - \lambda) x$ and the generalized triangular inequality proves immediately continuity of scalar multiplication in the product topology. In a similar way, using the generalized triangular inequality again, one can prove continuity of addition in S in the product topology. □

Theorem 3. Let $(S, \mathscr{J}, T)$ be a statistical linear space with the t-norm T satisfying $\lim_{a\uparrow 1, b\uparrow 1} T(a, b) = 1$. Then $(S, \mathscr{J}, T)$ with the topology defined by the F-convergence is a linear topological space.

Proof. When $x_n \xrightarrow{F} x$ then evidently for every subsequence $\{x_{n_k}\}_1^\infty \subset \{x_n\}_1^\infty$ $x_{n_k} \xrightarrow{F} x$ also. Further, for every stationary sequence $\{x_n\}_1^\infty$, i.e. $x_n = x$ for every $n \geqq n_0$, it holds that $x_n \xrightarrow{F} x$.

If $x_n \not\xrightarrow{F} x$, i.e. there exists at least one $u_0 > 0$ that $F_{x_n - x}(u_0) \not\to 1$, then an $\varepsilon_0 > 0$ and subsequence $\{x_{n_k}\}_1^\infty \subset \{x_n\}$ must exist such that for every subsequence $\{x_k^*\}_1^\infty \subset \{x_{n_k}\}_1^\infty$ $F_{x^*_k - x}(u_0) \leqq 1 - \varepsilon_0$, in other words $x_k^* \not\xrightarrow{F} x$.

In this way we have verified all demands put on the topological convergence and we must prove further that this convergence and linear operations defined on S are in accordance. When $x_n \xrightarrow{F} x$, $y_n \xrightarrow{F} y$ then using the generalized triangular inequality we obtain

$$F_{x_n + y_n}(2\eta) \geqq T(F_{x_n}(\eta), F_{y_n}(\eta)) \geqq T(1 - \varepsilon, 1 - \varepsilon)$$

for a suitable large n and the left continuity at $[1, 1]$ of the t-norm implies that $T(1 - \varepsilon, 1 - \varepsilon) \to 1$ if $\varepsilon \to 0$. Similarly, as it was done in the proof of Theorem 3 we can prove that $x_n \xrightarrow{F} x$, $\lambda_n \to \lambda$ imply that $\lambda_n x_n \xrightarrow{F} \lambda x$, too. It follows from the left continuity at $[1, 1]$ of the t-norm T that every F-convergent sequence has a unique limit point, because

$$F_{x_0-y_0}(2\eta) \geqq T(F_{x_n-x_0}(\eta), F_{x_n-y_0}(\eta)) > T(1-\varepsilon, 1-\varepsilon)$$

for a suitable large natural n and every $\eta > 0$. □

Remark. If the t-norm T is continuous then as we proved in Lemma 1 and Theorem 3, the ε, η-topology and the F-convergence are equivalent. Generally, this equivalence need not hold without the assumption of the continuity of the t-norm T, because ε, η-neighbourhoods need not form a base of neighbourhoods of the origin in S for the topology generated by the F-convergence.

In further considerations we shall deal with continuous t-norms only. In this case every statistical linear space $(S, \mathscr{J}, T)$ has the metrizable ε, η-topology and the question of its normability is interesting for us.

Definition 7. A subset $A \subset S$ where (S, τ) is a linear topological space with a topology τ is called bounded in topology τ if for every τ-neighbourhood U of the origin in S there exists $\lambda > 0$ that

$$A \subset \lambda U .$$

In our case of an SLM-space $(S, \mathscr{J}, T)$ a subset $A \subset S$ is ε, η-bounded if and only if for every $O(\varepsilon, \eta)$ there exists $\lambda(\varepsilon, \eta) > 0$ that

$$A \subset \lambda(\varepsilon, \eta) \,.\, O(\varepsilon, \eta) = O(\varepsilon, \lambda(\varepsilon, \eta) \,.\, \eta)) .$$

In other words, the ε, η-boundedness of A can be expressed as follows: a subset A is ε, η-bounded if and only if for every sequence $\{x_n\}_1^\infty \subset A$ and every sequence $\{\lambda_n\}_1^\infty$, $\lambda_n \to 0$ of reals $\lambda_n x_n \xrightarrow{F} 0$ also in S.

Now, we use very important criterion of normability of linear topological spaces due to Kolmogorov, see [11]. A Hausdorff linear topological space is normable if and only if there exists a bounded convex neighbourhood of the origin in it. If U is such a neighbourhood then the norm in question can be expressed as

$$\|x\| = \inf\{\lambda > 0 : x \in \lambda U\} , \quad x \in S .$$

In the case of an SLM-space $(S, \mathscr{J}, T)$ if such a neighbourhood $O(\varepsilon_0, \eta_0)$ exists, then a possible norm $\|\cdot\|$ has the form

$$\begin{aligned} \|x\| &= \inf\{\lambda > 0 : x \in \lambda\, O(\varepsilon_0, \eta_0)\} = \\ &= \inf\{\lambda > 0 : x \in O(\varepsilon_0, \lambda\eta_0) = \\ &= \inf\{\lambda > 0 : F_x(\lambda\eta_0) > 1 - \varepsilon_0\} . \end{aligned}$$

With this question of normability an important property is connected as the following Theorem 4 states.

In the next Theorem 4 we shall need the following notation:

$\overline{\text{conv}}\, A$ is the absolutely convex hull of A, conv A is the convex hull of A.

Theorem 4. Let an SLM-space $(S, \mathcal{J}, T)$ be finite-dimensional. Then the ε, η-topology is normable and is equivalent to the usual Euclidean topology.

Proof. We suppose that the space $(S, \mathcal{J}, T)$ is finite-dimensional and hence every $x \in S$ can be expressed in the form

$$x = \sum_{i=1}^{n} \lambda_i e_i \, ;$$

$(e_1, e_2, \ldots, e_n)$ is any linear base in S. As the number of the elements in a base is finite, we can find an ε, η-neighbourhood $O(\varepsilon, \eta)$ which contains all elements of the base. Further, every $x \in \overline{\text{conv}}\,(e_1, e_2, \ldots, e_n)$ can be expressed as an absolutely convex combination of $e_1, e_2, \ldots, e_n$, i.e. $x = \sum_{i=1}^{n} \mu_i e_i$, $\sum_{i=1}^{n} |\mu_i| \leqq 1$, and because conv $O(\varepsilon, \eta)$ is also absolutely convex in S then $\overline{\text{conv}}\,(e_1, e_2, \ldots, e_n) \subset \text{conv}\, O(\varepsilon, \eta)$.

Now, it is necessary to prove that conv $(e_1, e_2, \ldots, e_n)$ is at the same time a neighbourhood of the zero element in the ε, η-topology; for this fact it is sufficient to find $O(\varepsilon^*, \eta^*)$ such that

$$O(\varepsilon^*, \eta^*) \subset \overline{\text{conv}}\,(e_1, e_2, \ldots, e_n) \, .$$

Let us suppose, that such a neighbourhood does not exist, i.e. for every $O(\varepsilon, \eta)$ there exists at least one point $x_0 \in O(\varepsilon, \eta)$ so that $x_0 \notin \overline{\text{conv}}\,(e_1, e_2, \ldots, e_n)$. Taking $\varepsilon_n \downarrow 0$, $\eta_n \downarrow 0$ we can construct a sequence $\{x_m\}_1^\infty$ which has the zero element as its limit point, let us say $x_m = \sum_{i=1}^{n} \lambda_i^m e_i$, but $x_m \notin \overline{\text{conv}}\,(e_1, e_2, \ldots, e_n)$, i.e. $\sum_{i=1}^{n} |\lambda_i^m| > 1$. First, we can suppose that $M \geqq \sum_{i=1}^{n} |\lambda_i^m| > 1$ for all m, where $M < +\infty$. Then there exists a subsequence $\{\lambda_1^{m_k}; \lambda_2^{m_k}, \ldots, \lambda_n^{m_k}\}$ that is convergent and hence

$$x_{m_k} = \sum_{i=1}^{n} \lambda_i^{m_k} e_i \xrightarrow{F} x_0 \quad \text{but} \quad x_0 \neq 0 \quad \text{because}$$

$$x_0 = \sum_{i=1}^{n} \lambda_i^0 e_i \, , \quad \lambda_i^0 = \lim_k \lambda_i^{m_k} \quad \text{and} \quad \sum_{i=1}^{n} |\lambda_i^0| \geqq 1 \, .$$

If there exists a subsequence $\sum_{i=1}^{n} |\lambda_i^{m_k}|$ unbounded from above, i.e.

$$\lim_k \sum_{i=1}^{n} |\lambda_i^{m_k}| = +\infty \, ,$$

then we can consider the sequence

$$x_{m_k}^* = \sum_{i=1}^{n} \frac{\lambda_i^{m_k}}{\sum_{j=1}^{n} |\lambda_j^{m_k}|} e_i = \frac{1}{\sum_{j=1}^{n} |\lambda_j^{m_k}|} x_{m_k} \, ,$$

instead of the original $\{x_m\}_m$. However, at the same time, we have $x^*_{m_k} = \sum_{i=1}^{n} \mu_i^{m_k} e_i$ with $\sum_{i=1}^{n} |\mu_i^{m_k}| = 1$ and this case can be transformed to the previous one. This fact proves that $\overline{\text{conv}}\,(e_1, e_2, \ldots, e_n)$ must be a neighbourhood of the zero element in the ε, η-topology. The boundedness of $\overline{\text{conv}}\,(e_1, e_2, \ldots, e_n)$ is clear, because if $\{x_m\}_1^\infty$ is any sequence from $\overline{\text{conv}}\,(e_1, e_2, \ldots, e_n)$, $\lim_m \varrho_m = 0$, $\varrho_m \in R$ then

$$\varrho_m x_m = \varrho_m \sum_{i=1}^{n} \lambda_i^m e_i, \sum_{i=1}^{n} |\lambda_i^m| \leqq 1 \quad \text{and}$$

$$F_{\varrho_m x_m}(u) \geqq T^{(n)} \left(\left(F_{e_1} \left(\frac{u}{|\varrho_m \lambda_1^m|} \right), \ldots, F_{e_n} \left(\frac{u}{|\varrho_m \lambda_n^m|} \right) \right) \right);$$

$$(T^{(n)}(a_1, a_2, \ldots, a_n) = T(a_1, T(a_2, \ldots, T(a_{n-1}, a_n) \ldots)),$$

with $|\lambda_i^m| \leqq 1$ and this fact implies that $\varrho_n x_n \xrightarrow{F} 0$. We proved that in the case of a finite dimensional *SLM*-space $(S, \mathcal{J}, T)$ the ε, η-topology is equivalent to the topology generated by the coordinate convergence and the ε, η-topology is normable. □

Lemma 2. Every *SLM*-space $(S, \mathcal{J}, T)$ where $T(a, b) = \min(a, b)$ is a locally convex linear topological space.

Proof. The proof is very simple. Let us consider any ε, η-neighbourhood $O(\varepsilon, \eta)$ in $(S, \mathcal{J}, T)$ and let $x, y \in O(\varepsilon, \eta)$, $\alpha \in \langle 0, 1 \rangle$, then $F_x(\eta) > 1 - \varepsilon$, $F_y(\eta) > 1 - \varepsilon$ and hence

$$F_{\alpha x + (1-\alpha) y}(\eta) \geqq \min(F_{\alpha x}(\alpha\eta), F_{(1-\alpha)y}((1-\alpha)\eta)) = \min(F_x(\eta), F_y(\eta)) > 1 - \varepsilon. \ \square$$

3. PROPERTIES OF ε, η-NEIGHBOURHOODS

Lemma 3. Let $O(\varepsilon, \eta)$ be an ε, η-neighbourhood of the zero element in an *SLM*-space $(S, \mathcal{J}, T)$. Then for every $|\lambda| \leqq 1$, $\lambda \in R$ and every $x \in O(\varepsilon, \eta)$

$$\lambda x \in O(\varepsilon, \eta).$$

Proof. Let $x \in O(\varepsilon, \eta)$, i.e. $F_x(\eta) > 1 - \varepsilon$ then $F_{\lambda x}(\eta) = F_x(\eta/|\lambda|) \geqq F_x(\eta) > 1 - \varepsilon$ and hence $\lambda x \in O(\varepsilon, \eta)$. □

Lemma 4. Every ε, η-neighbourhood $O(\varepsilon, \eta)$ is a symmetric set.

Proof. If $x \in O(\varepsilon, \eta)$ then $F_{-x}(\eta) = F_x(\eta) > 1 - \varepsilon$ also, what implies that $-x \in O(\varepsilon, \eta)$. □

Lemma 5. Let an ε, η-neighbourhood $O(\varepsilon, \eta)$ be given. Then for every $x \in (S, \mathcal{J}, T)$ there exists a $\lambda > 0$ such that $x \in \mu\, O(\varepsilon, \eta)$ for every μ, $|\mu| \geqq \lambda$. This property is called the absorbing property of ε, η-neighbourhoods.

Proof. Since for every $x \in (S, \mathcal{J}, T) \lim_{n \to \infty} F_x(u) = 1$, i.e. for every $\varepsilon > 0$ there exists $u_x(\varepsilon) > 0$ such that for every $u \geqq u_x(\varepsilon)$ we have $F_x(u) > 1 - \varepsilon$, it is evident to put $\lambda = u_x(\varepsilon)/\eta$. If μ is an arbitrary real number with $|\mu| \geqq \lambda$ then $F_x(|\mu|\ \eta) \geqq$ $\geqq F_x(u_x(\varepsilon)) > 1 - \varepsilon$ and hence $x \in O(\varepsilon, |\mu|\ \lambda)$. As every $O(\varepsilon, \eta)$ is a symmetric set, then $O(\varepsilon, |\mu|\ \eta) = \mu\ .\ O(\varepsilon, \eta)$. □

Lemma 6. If an ε, η-neighbourhood $O(\varepsilon, \eta)$ is a convex set, then it is an absolutely convex set in $(S, \mathcal{J}, T)$.

Proof. It follows immediately from Lemma 3 and Lemma 4.

Lemma 7. For every ε, η-neighbourhood of the zero element in $(S, \mathcal{J}, T)$

$$S = \bigcup_{n=1}^{\infty} O(\varepsilon, n\ .\ \eta)\ .$$

Proof. Let $x \in (S, \mathcal{J}, T)$ and let $O(\varepsilon, \eta)$ be an arbitrary ε, η-neighbourhood of the zero element in S. As Lemma 5 states for the chosen $\varepsilon > 0$ there exists $u(\varepsilon) > 0$ such that $F_x(u(\varepsilon)) > 1 - \varepsilon$. Now, it is sufficient to choose a natural n in such a way that $n\ .\ \eta \geqq u(\varepsilon)$, at this moment $x \in O(\varepsilon, n\eta) = n\ .\ O(\varepsilon, \eta)$. This proves that $S =$ $= \bigcup_{n=1}^{\infty} n\ .\ O(\varepsilon, \eta)\ .$ □

Lemma 8. Let x_0 be a cluster point of an ε, η-neighbourhood $O(\varepsilon, \eta)$ in an *SLM*-space $(S, \mathcal{J}, T)$. Then

$$\lim_{u \to \eta+} F_{x_0}(u) \geqq 1 - \varepsilon\ .$$

Proof. Let $\{x_n\} \subset O(\varepsilon, \eta)$, $x_n \xrightarrow{F} x_0$, let $\lambda > 1$. Then, according to the generalized triangular inequality

$$F_{x_0}(\lambda\eta) \geqq T(F_{x_n - x_0}((\lambda - 1)\ \eta), F_{x_n}(\eta)) \geqq T(F_{x_n - x_0}((\lambda - 1)\ \eta), 1 - \varepsilon)$$

for every natural n because $x_n \in O(\varepsilon, \eta)$. But $x_n \xrightarrow{F} x_0$, i.e. $F_{x_n - x_0}((\lambda - 1)\ \eta) > 1 - \varepsilon'$ for a suitable large n and hence $F_{x_0}(\lambda\eta) \geqq T(1 - \varepsilon', 1 - \varepsilon)$. As ε' is quite arbitrary, the t-norm T is continuous and $T(a, 1) = a$ for $a > 0$, this implies $F_{x_0}(\lambda\eta) \geqq 1 - \varepsilon$ for every $\lambda > 1$. $F_{x_0}(\cdot)$ is a probability distribution function, therefore the limit $\lim_{u \to \eta+} F_{x_0}(u)$ must exist and in this case $\lim_{u \to \eta+} F_{x_0}(u) \geqq 1 - \varepsilon$. □

Lemma 9. If $O(\varepsilon, \eta)$ is a convex set in an *SLM*-space $(S, \mathcal{J}, T)$ then its closure $\overline{O(\varepsilon, \eta)}$ in the ε, η-topology can be described as

$$\overline{O(\varepsilon, \eta)} = \{x \in S : \inf\{\lambda > 0 : F_x(\lambda\eta) > 1 - \varepsilon\} \leqq 1\}\ .$$

Proof. If $O(\varepsilon, \eta)$ is a convex set in $(S, \mathcal{J}, T)$ then it is at the same time absolutely convex and absorbing. Let us define a functional (Minkowski functional)

$$\begin{aligned} p_{\varepsilon\eta}(x) &= \inf\{\lambda > 0 : x \in O(\varepsilon, \lambda\eta)\} = \\ &= \inf\{\lambda > 0 : F_x(\lambda\eta) > 1 - \varepsilon\}\ . \end{aligned}$$

From the properties of the ε, η-neighbourhood $O(\varepsilon, \eta)$ mentioned above it follows that $p_{\varepsilon\eta}(\cdot)$ is a seminorm defined on S. As $O(\varepsilon, \eta)$ is a neighbourhood in the ε, η-topology this seminorm $p_{\varepsilon\eta}(\cdot)$ is continuous in the ε, η-topology, and the closure $\overline{O(\varepsilon, \eta)}$ can be expressed as

$$\overline{O(\varepsilon, \eta)} = \{x \in S : \inf\{\lambda > 0 : F_x(\lambda\eta) > 1 - \varepsilon\} \leqq 1\} = \{x : n_{1-\varepsilon}(x) \leqq \eta\}$$

where $n_{1-\varepsilon}(x) = \inf\{\lambda > 0 : F_x(\lambda) > 1 - \varepsilon\}$. □

4. PROPERTIES OF MAPPING $\mathscr{J}$

Let an SLM-space $(S, \mathscr{J}, T)$ be given. The mapping $\mathscr{J}$ is defined on the linear space S with values in the set $\mathscr{F}$ of all probability distribution functions defined on real numbers. In $\mathscr{F}$ we can introduce a metric L defined by

$$L(F, G) = \inf\{h > 0 : F(u - h) - h \leqq G(u) \leqq F(u + h) + h \quad \text{for every} \quad u \in \mathbb{R}\};$$

this metric is called Lévy's metric and the pair $(\mathscr{F}, L)$ is a complete metric space.

Definition 9. Let $(S, \mathscr{J}, T)$ and $(S, \mathscr{J}', T')$ be two SLM-spaces defined on the same linear space S. We shall say that $(S, \mathscr{J}, T)$ and $(S, \mathscr{J}', T')$ are topologically equivalent if the mappings $\mathscr{J}, \mathscr{J}'$ define equivalent ε, η-topologies.

Theorem 5. SLM-spaces $(S, \mathscr{J}, T)$, $(S, \mathscr{J}', T')$ are topologically equivalent if and only if the mapping $L(\mathscr{J}(\cdot), \mathscr{J}'(\cdot))$ defined on S is continuous at 0 in both the ε, η-topologies.

Proof. If the ε, η-topologies are equivalent, i.e. if $x_n \xrightarrow{F} 0$ in $(S, \mathscr{J}, T)$ then $x_n \xrightarrow{F} 0$ in $(S, \mathscr{J}', T')$ and vice versa, then $\mathscr{J}(x_n)(u) = F_{x_n}(u) \to H(u)$, $\mathscr{J}'(x_n)(u) = F'_{x_n}(u) \to H(u)$ for every $u \in \mathbb{R}$ what can be expressed also in the form $L(\mathscr{J}(x_n), H)) \underset{n\to\infty}{\to} 0$, $L(\mathscr{J}'(x_n), H)) \to 0$. From the triangular inequality in the metric space $(\mathscr{F}, L)$

$$L(\mathscr{J}(x_n), \mathscr{J}'(x_n)) \leqq L(\mathscr{J}(x_n), H)) + L(\mathscr{J}'(x_n), H))$$

it immediately follows that

$$\lim_{n\to\infty} L(\mathscr{J}(x_n), \mathscr{J}'(x_n)) = 0 .$$

Conversely, if $x_n \xrightarrow{F} 0$ in $(S, \mathscr{J}, T)$, i.e. $L(F_{x_n}, H) \to 0$ and we assume that $L(\mathscr{J}(x_n), \mathscr{J}'(x_n)) \to 0$ also, then $L(\mathscr{J}'(x_n), H) \leqq L(\mathscr{J}(x_n), H) + L(\mathscr{J}(x_n), \mathscr{J}'(x_n))$ for every n and hence $\lim_{n\to\infty} L(\mathscr{J}'(x_n), H) = 0$. This fact says that $x_n \xrightarrow{F} 0$ in $(S, \mathscr{J}', T')$ and the ε, η-topology in $(S, \mathscr{J}, T)$ is stronger than the ε, η-topology in $(S, \mathscr{J}', T')$. In a similar way we can prove the opposite implication what completes the proof of Theorem 5. □

Theorem 6. Let an SLM-space $(S, \mathscr{J}, T)$ be given. Then the mapping $\mathscr{J} : S \to (\mathscr{F}, L)$ is uniformly continuous in the ε, η-topology.

Proof. The t-norm T is continuous on $\langle 0, 1\rangle \times \langle 0, 1\rangle$ and therefore T is uniformly continuous on $\langle 0, 1\rangle \times \langle 0, 1\rangle$ and $\lim_{x \uparrow 1} T(a, x) = a$ uniform in a. It means that $(\forall \eta > 0\ \exists \varepsilon \in (0, 1)\ \forall a \in \langle 0, 1\rangle) \Rightarrow T(a, 1 - \varepsilon) > a - \eta$. Let $x_n \to x_0$ in the ε, η-topology, we can find a natural number n_0 such that for every $n \geqq n_0$

$$x_n \in O(x_0, \varepsilon, \eta) \Leftrightarrow F_{x_n - x_0}(\eta) > 1 - \varepsilon .$$

Let $u \in \mathbb{R}$ be arbitrary, then

$$F_{x_0}(u + \eta) \geqq T(F_{x_0 - x_n}(\eta), F_{x_n}(u)) \geqq T(F_{x_n}(u), 1 - \varepsilon) > F_{x_n}(u) - \eta .$$

From this inequality we obtain that $F_{x_0}(u + \eta) + \eta > F_{x_n}(u)$. In a similar way we can prove the opposite inequality $F_{x_n}(u) > F_{x_0}(u - \eta) - \eta$. Both the obtained inequalities express together that $L(F_{x_n}, F_{x_0}) < \eta$. The continuity of the mapping $\mathscr{J}$ in the ε, η-topology is proved. It is necessary to note that a choice of ε and η does not depend on x_n, x_0 and the continuity of $\mathscr{J}$ can be expressed in a stronger form as follows $(\forall \eta > 0\ \forall \varepsilon \in (0, 1)\ \forall x, y \in S,\ x - y \in O(\varepsilon, \eta)) \Rightarrow L(F_x, F_y) < \eta$. This implication means, of course, the uniform continuity of the mapping $\mathscr{J}$ in the ε, η-topology. □

Theorem 7. A set $K \subset (S, \mathscr{J}, T)$ is bounded in the ε, η-topology if and only if the image $\mathscr{J}(K)$ in $(\mathscr{F}, L)$ is compact.

Proof. Let K be a bounded subset in $(S, \mathscr{J}, T)$. It means that for every ε, η-neighbourhood $O(\varepsilon, \eta)$ there exists an $\alpha = \alpha(\varepsilon, \eta) \in \mathbb{R}$ such that for every real λ, $|\lambda| \geqq \alpha$

$$K \subset \lambda\, O(\varepsilon, \eta) = O(\varepsilon, |\lambda|\, \eta) .$$

Let $\mathscr{J}(K) = \{F_x : x \in K\}$. If we choose the neighbourhood $O(\varepsilon, 1)$ then for every λ, $|\lambda| \geqq \alpha(\varepsilon, 1)$ $K \subset O(\varepsilon, |\lambda|)$. It implies that $\mathscr{J}(K) \subset \mathscr{J}(O(\varepsilon. |\lambda|))$ what means for every $|\lambda| \geqq \alpha(\varepsilon, 1)$ and every $x \in K$ $F_x(|\lambda|) > 1 - \varepsilon$. We have proved that for every $F \in \mathscr{J}(K)$ and every $u \geqq \alpha(\varepsilon, 1)$

$$F(u) > 1 - \varepsilon .$$

This fact can be expressed in the form $\lim_{u \to \infty} F_x(u) = 1$ uniformly in $x \in K$. As we know that the subset $\mathscr{J}(K)$ is compact in $(\mathscr{F}, L)$ if and only if

$$\lim_{u \to \infty} F(u) = 1 , \quad \lim_{u \to -\infty} F(u) = 0 \quad \text{uniformly in } \mathscr{J}(K)$$

the necessary part of the proof is finished. Let us suppose that $\mathscr{J}(K)$ is compact in $(\mathscr{F}, L)$, $K \subset (S, \mathscr{J}, T)$. Then $\lim_{u \to \infty} F_x(u) = 1$ uniformly in $x \in K$, i.e.

$$(\forall \varepsilon \in (0, 1\rangle\ \exists \alpha = \alpha(\varepsilon)\ \forall u \geqq \alpha\ \forall x \in K) \Rightarrow F_x(u) > 1 - \varepsilon .$$

Let $\{x_n\}_1^\infty$ be an arbitrary sequence in K and let $\lambda_n \to 0$ in reals. Then

$$F_{\lambda_n x_n}(u) = F_{x_n}\left(\frac{u}{|\lambda_n|}\right) > 1 - \varepsilon \quad \text{for} \quad u \geqq \alpha |\lambda_n| .$$

As $\lambda_n \to 0$, then for every $u > 0$ there exists such a natural n_0 that $u \geqq \alpha|\lambda_n|$ for every $n \geqq n_0$. So, for $u \geqq u_0$ we have $\lambda_n x_n \in O(\varepsilon, u)$. The convergence $\lambda_n x_n \xrightarrow{F} 0$ is proved and hence the subset K is bounded in the ε, η-topology. □

Theorem 8. An *SLM*-space $(S, \mathscr{J}, T)$ with the t-norm $T = \min$ is normable if and only if there exists such an ε, η-neighbourhood $O(\varepsilon, \eta)$ of the zero element that its image $\mathscr{J}(O(\varepsilon, \eta))$ is compact in $(\mathscr{F}, L)$.

Proof. This statement immediately follows from Theorem 7 and Criterion of normability. □

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