The present paper, closely connected with [2], investigates the possibility of expressing P. Martin-Löf's complexity theory of strings in terms of Kolmogorov's complexity of strings which uses algorithms $\varphi$. We find for every recursive P. Martin-Löf test $V$ an algorithm $\varphi$ which in turn gives a P. Martin-Löf test $V(\varphi)$ such that $V \subseteq V(\varphi)$. The equality $V = V(\varphi)$ holds for some particular P. Martin-Löf tests called representable.

In this paper we continue our efforts to approach Kolmogorov's complexity theory of strings which uses algorithms $\varphi$ (see [3]) and P. Martin-Löf's complexity theory of strings which uses $M$—$L$ tests $V$ (see [5]). A very good up-to-date survey paper is [7]. The present authors have already done some attempts in this direction in [2]. We work within the general framework of a not necessarily binary alphabet (see [1]).

It has already been noticed that these theories are not equivalent (see [2]).

In this paper we find for every recursive $M$—$L$ test $V$ an algorithm $\varphi$ which in turn gives a $M$—$L$ test $V(\varphi)$ such that $V \subseteq V(\varphi)$ (see Theorem 2). The equality $V = V(\varphi)$ holds for some particular $M$—$L$ tests $V$ which we call representable (see Theorem 3). Such an equality $V = V(\varphi)$ would be a good interpretation of the somewhat unprecise term "equivalence" between Kolmogorov's and P. Martin-Löf's theories. In this respect see also Theorem 4.

The last section of our paper contains remarks and open problems.

1. BASIC NOTIONS

Throughout the paper $N$ will be the set of all natural numbers, i.e. $N = \{0, 1, 2, \ldots\}$. If $A$ is a finite set, $\text{card } A$ will be the number of elements in $A$.

For every non-empty sets $A$ and $B$ and for every function $f : A' \rightarrow B$ (where $A' \subset A$) we shall write $f : A \rightarrow B$. We shall say that $f$ is a partial function from $A$ to $B$. We consider that $f(x) = \infty$ in case $f$ is not defined in the point $x$. 

526
If \( f : A \to B \) is a partial function, then the domain of \( f \) is the set \( \text{dom}(f) = \{ x \in A \mid f(x) \neq \infty \} \); range \( f \) is \( \{ f(x) \mid x \in \text{dom}(f) \} \); graph \( f \) is \( \{ (x, f(x)) : x \in \text{dom}(f) \} \).

Let \( X = \{ a_1, a_2, \ldots, a_p \} \), \( p \geq 2 \) be a finite alphabet. Denote by \( X^* \) the free monoid generated by \( X \) under concatenation, i.e. \( X^* \) consists of all strings \( x = x_1x_2\ldots x_n \), where the \( x_i \) belong to \( X \); also the null string \( \lambda \) belongs to \( X^* \). For every \( a \) in \( X \) and every natural \( n > 0 \), \( a^n = aa \ldots a \) (\( n \) copies of \( a \)). We shall consider that \( a^0 = \lambda \).

For every \( x \) in \( X^* \), \( l(x) \) is the length of \( x \), i.e. \( l(x) = m \) in case \( x = x_1x_2\ldots x_m \) and \( l(\lambda) = 0 \). For Recursive Function Theory see [4] and [6]. We shall consider partial recursive functions (p.r. functions in the sequel) \( \varphi : X^* \times N \to X^* \) or \( g : N \to \{ 0 \} \cup X^* \times N \).

For every p.r. function \( \varphi : X^* \times N \to X^* \), the Kolmogorov complexity induced by \( \varphi \) is a function \( K_\varphi : X^* \times N \cup \{ \infty \} \to \mathbb{N} \), defined by \( K_\varphi(x \mid m) = \min \{ l(y) \mid y \in X^*, \varphi(y, m) = x \} \) in case such \( m \) exists, and \( K_\varphi(x \mid m) = \infty \), otherwise.

For every \( W \subseteq X^* \times (N \cup \{ 0 \}) \) and for every natural \( m \geq 1 \) we shall write \( W_m = \{ x \in X^* \mid (x, m) \in W \} \). We define the critical level induced by \( W \) to be the function \( m_W : X^* \to N \cup \{ \infty \} \) given by \( m_W(x) = \sup \{ m \in N \mid m \geq 1, \ x \in W_m \} \) in case such \( m \) exists, and \( m_W(x) = 0 \), in the opposite case.

A non-empty recursively enumerable set \( V \subseteq X^* \times (N \cup \{ 0 \}) \) will be called Martin-Löf test (M-L test) if it possesses the following two properties:

1) For every natural \( m \geq 1 \), \( V_{m+1} \subseteq V_m \).

2) For every natural numbers \( m, n, m \geq 1 \),

\[
\text{card} \{ x \in X^* \mid l(x) = n, x \in V_m \} < p^{n-m}(p - 1).
\]

We agree upon the fact that the empty set is a \( M-L \) test.

The second condition enables us to say that \( m_W \) takes only finite values for every \( M-L \) test \( V \), because in case \( (x, m) \in V \), then \( m \leq l(x) - 1 \) (directly from the definition).

For every p.r. function \( \varphi : X^* \times N \to X^* \) we can obtain the particular \( M-L \) test \( V(\varphi) = \{ (x, m) \in X^* \times (N \cup \{ 0 \}) \mid K_\varphi(x \mid l(x)) < l(x) - m \} \), see [1]. We shall call a \( M-L \) test \( V \) representable in case there exists a p.r. function \( \varphi : X^* \times N \to X^* \) such that \( V = V(\varphi) \), see [2].

The lexicographical order on \( X^* \) induced by \( a_1 < a_2 < \ldots < a_p \) is given by \( \lambda < a_1 < a_2 < \ldots < a_p < a_1a_2 < \ldots < a_1a_2a_3 < \ldots \). The enumeration of \( X^* \) in this order will be \( y(1) = \lambda, y(2) = a_1, y(3) = a_2, \ldots, y(p + 1) = a_p, y(p + 2) = a_1a_2, \ldots \). It follows that \( a_p^p = \lambda(s(m)), \) where \( s(m) = 1 + p + p^2 + \ldots + p^n = (p^{n+1} - 1)/(p - 1) \). This enumeration of \( X^* \) is recursive.

In the sequel, the lexicographical order will mean this lexicographical order on \( X^* \).
2. RESULTS

It is easily seen that there exist \( M - L \) tests which are not recursive. For instance, take \( A \subseteq \{a_1, a_2, a_3, \ldots \} \) which is recursively enumerable and not recursive. Then \( V = (A - \{a_1\}) \times \{1\} \) is a non-recursive \( M - L \) test.

The following theorem gives necessary and sufficient conditions under which a \( M - L \) test is recursive.

**Theorem 1.** A \( M - L \) test \( V \) is recursive iff the function \( m_V \) is recursive.

**Proof.** If \( m_V \) is recursive we can compute \( m_V(x) \) for every \( x \in X^* \). Let \( (x, m) \in X^* \times (N - \{0\}) \). If \( m_V(x) \geq m \), then \( (x, m) \in V \); if \( m_V(x) < m \), then \( (x, m) \notin V \).

Thus \( V \) is recursive.

Now suppose \( V \) is recursive which means that \( \chi_V \) is a recursive function (\( \chi_V \) is the characteristic function of \( V \)). It is easy to see that for every \( x \in X^* \) we have \( m_V(x) = \max \{m \in N : \chi_V(x, m) = 1\} \), in case \( (x, 1) \in V \), and \( m_V(x) = 0 \), in case \( (x, 1) \notin V \).

This shows that \( m_V \) is recursive. \( \square \)

Actually, the object of our paper will be the study of some properties of recursive \( M - L \) tests.

**Theorem 2.** Let \( V \subseteq X^* \times X^* \) be a recursive \( M - L \) test. Then there exists a p.r. function \( \phi : X^* \times N \rightarrow X^* \) such that \( V = V(\phi) \).

The p.r. function \( \phi \) can be taken to possess the following properties:

(a) The function \( \phi \) is injective.

(b) The graph of \( \phi \) is recursive.

(c) For every \( x \in X^* \), we have the equivalence: \( (x, 1) \in V \) iff \( (x, 1) \in V(\phi) \).

**Proof.** The set \( A = \{(x, m_V(x)) \mid x \in V_1\} \) is obviously recursive. We distinguish two cases: i) \( V \) is infinite and in this case there exists an injective recursive function \( g : N - \{0\} \rightarrow X^* \times N \) such that \( g(N - \{0\}) = A \); ii) \( V \) is finite and \( A \) has \( q \) elements, and in this case there exists an injective (p.r.) function \( g : \{1, 2, \ldots, q\} \rightarrow X^* \times N \) such that \( g([1, 2, \ldots, q]) = A \). In all cases, if \( i \) is in the domain of \( g \), we put \( g(i) = (x_i, m_V(x_i)) \). Moreover, due to the recursiveness of \( V \), we can suppose that \( g \) has the following "lexicographical" property: for all natural \( 1 \leq i < j : \)

u) \( l(x_i) \leq l(x_j) \),

v) if \( l(x_i) = l(x_j) \), then \( m_V(x_i) \geq m_V(x_j) \).

We can define the procedure for \( \phi \).

For \( i = 1 \), \( g(1) = (x_i, m_V(x_i)) \) and we put \( \varphi(x_i, l(x_i)) = x_i \), where \( z_i = y(s(l(x_i) - m_V(x_i) - 1)) \). See the definition of \( s \) in Section 1.

Next, let \( i = 2 \), so \( g(2) = (x_2, m_V(x_2)) \). In case \( l(x_1) \neq l(x_2) \), we put \( \varphi(x_2, l(x_2)) = x_2 \), where \( z_2 = y(s(l(x_2) - m_V(x_2) - 1)) \). In case \( l(x_1) = l(x_2) \), we consider the greatest element (according to the lexicographical order) of the set \( \{y(1), y(2), \ldots \} \).
\[ \ldots, \mathcal{y}(l(x_2) - m_\nu(x_2) - 1)) = \{z_1\}, \text{ and we shall call this element } z_2. \text{ Put } \mathcal{y}(z_2, l(x_2)) = x_2. \]

Continuing the procedure we reach the step \( i > 1 \). There are two cases. In the first case let \( l(x_i) = l(x_j) \), for all \( j < i \). In this case we put \( \mathcal{y}(z_i, l(x_i)) = x_i \), where \( z_i = \mathcal{y}(s(l(x_i) - m_\nu(x_i) - 1)) \). In the second (opposite) case let \( j(1) < j(2) < \ldots \) \( < j(k) < i \) be all indices \( j < i \) such that \( l(x_j) = l(x_i) \). In fact, \( j(2) = j(1) + 1 \), \( j(3) = j(2) + 1, \ldots \), due to the properties of the enumeration function \( g \). We define \( z_i \) to be the greatest element (in lexicographical order) of the set \( \{j(1), j(2), \ldots \} \). Put \( \mathcal{y}(z_i, l(x_i)) = x_i \). Notice that \( \mathcal{y} \) acts as a function, because if \( l(x_i) = l(x_j) \) we have \( z_i \neq z_j \).

The construction is possible and the motivation follows. Put \( l(x_i) = l(x_{j(1)}) = \ldots = l(x_{j(k)}) = l \). We have \( m_1 = m_\nu(x_1) \leq m_k = m_\nu(x_{j(k)}) \leq m_{k-1} = m_\nu(x_{j(k-1)}) \leq \ldots \leq m_1 = m_\nu(x_{j(1)}) \). For every natural \( r \in \{1, 2, \ldots, k\} \) let \( B_r = \{j(1), j(2), \ldots, j(s(l - m_r - 1))\} \). Notice that \( B_1 \subset B_2 \subset \ldots \subset B_k \subset B_i \) and \( B_i = B_i \{u < v \} \) iff \( m_u = m_v \). We shall try to describe in a detailed manner the action of \( \mathcal{y} \) and this will complete the motivation.

Clearly, \( z_{j(1)} = \mathcal{y}(s(l - m_1 - 1)) \). In order to obtain \( z_{j(2)} \), we distinguish two possible cases: a) \( m_1 > m_2 \) (and in this case \( z_{j(2)} = \mathcal{y}(s(l - m_2 - 1)) \); b) \( m_1 = m_2 \) (and in this case \( B_1 = B_2 \), so \( z_{j(2)} \) must be \( \mathcal{y}(s(l - m_2 - 1)) \)). It is to be seen that in case b) one has \( s(l - m_2 - 1) = 1 \geq 1 \) (in other words the construction is possible) because \( 2 \leq \text{card} \{x \in X^* \mid l(x) = l \text{ and } (x, m_2) \in V\} \leq (p^{r_m - 1} - 1) \). It is to be seen that strict inclusion occurs between the \( B_i \)'s being clearly favorable, we focus our attention to the "bad" situation \( m_k = m_{k+1} = m \ldots = m = m \geq 1 \leq r \leq i \). Here, in case \( h > 1 \), we consider \( m_{h-1} < m_1 \). We have \( B_h = B_{h+1} = \ldots = B_i \). The construction gives: \( z_{j(h)} = \mathcal{y}(s(l - m - 1)) \), \( z_{j(h+1)} = \mathcal{y}(s(l - m - 1) - 1) \), \( \ldots \), \( z_{j(h)} = \mathcal{y}(s(l - m - 1)) \). It remains to show that \( s(l - m - 1) - (r-h) \leq 1 \), i.e., \( r - h + 1 \leq (p^{r_m - 1} - 1)(p-1) \). This inequality follows from \( r - h + 1 \leq \text{card} \{x \in X^* : l(x) = l \text{ and } (x, m) \in V\} \leq (p^{r_m - 1} - 1)(p-1) \).

It is worth to add the fact that in case \( V \text{ is finite and the set } A \text{ (see the beginning of the proof) has } q \text{ elements, the procedure stops at step } q \).

The injectivity of \( \mathcal{y} \) is derived from the injectivity of \( g : \{x, m_\nu(x)\} \to \{x, m_\nu(x)\} \) iff \( x_1 \neq x_2 \) or \( m_\nu(x_1) \neq m_\nu(x_2) \). This implies that for different \( i \) and \( j \) one must obtain different values \( \mathcal{y}(x_i, l(x_i)) = x_i \) and \( \mathcal{y}(x_j, l(x_j)) = x_j \).

Our next task is to prove the inclusion \( V \subseteq V(\mathcal{y}) \). Indeed, in case \( (x, m) \in V \) let \( (x, m_\nu(x)) = (x, m_\nu(x)) \) in the enumeration given by \( g \). So \( m \leq m_\nu(x) \) and \( x = \mathcal{y}(z_i, l(x_i)) \) where the length of \( z_i \) is less than \( l(x_i) - m_\nu(x_i) - 1 \), which shows that \( K_0(x, l(x)) \leq l(x_i) - m_\nu(x_i) - 1 < l(x_i) - m_\nu(x_i) \), i.e., \( (x, m_\nu(x)) \in V(\mathcal{y}) \). Consequently, \( (x, m) \in \mathcal{M} - L \text{ test } V(\mathcal{y}) \).

Moreover, we can prove here also point (c), because it is seen that for every \( x \) in \( X^* \) such that \( (x, 1) \in V(\mathcal{y}) \) there exists a natural \( i \) such that \( x = x_i \) and \( (x_i, m_\nu(x_i)) \in V \), which implies \( (x, 1) \in V \).
All it remains to prove is point (b), i.e. the recursiveness of the graph of $\varphi$. This is seen taking arbitrarily $(z, t, x) \approx (z, t, x)$ in $X^* \times N \times X^*$ and checking if $(z, t, x)$ belongs to the graph of $\varphi$, according to the following decision algorithm:

1. If $m_\psi(x) = 0$, NO. Stop.
2. If $l(x) \neq t$, NO. Stop.
3. Choose $i$ such that $g(i) = (x, m_\psi(x))$ and $x = x_i$.
4. Run the first $i$ steps in the procedure defining $\varphi$ in order to find $z_i$.
5. If $z = z_i$, YES. Stop.
6. If $z \neq z_i$, NO. Stop.

Remark. It is obvious that for a given recursive $M - L$ test $V$ there are many p.r. functions $\varphi : X^* \times N \rightarrow X^*$ such that $V \in V(\varphi)$, e.g. our construction depends on the enumeration function $g$.

Theorem 3. Let $V$ be a $M - L$ test. Consider the following assertions:

(1) $V$ is representable.

(2) For every natural $m \geq 1$, one has

\[ \forall \ n \geq m + 1, \ \text{card} \{ x \in X^* \mid l(x) = n, m_\psi(x) = m \} \leq p^{n-m-1}. \]

Then (1) $\Rightarrow$ (2) and in case $V$ is recursive the implication (2) $\Rightarrow$ (1) holds too.

Proof. (1) $\Rightarrow$ (2). The hypothesis is that $V = V(\varphi)$ for some p.r. function $\varphi : X^* \times N \rightarrow X^*$.

Fix the natural numbers $n > m > 0$. For every $x$ in $X^*$ with $l(x) = n$ and such that $m_\psi(x) = m$ there exists $y$ in $X^*$ with $l(y) < l(x) - m$ and $\varphi(y, l(x)) = x$.

We have $l(y) \leq n - m - 1$.

We shall show that $l(y) = n - m - 1$. Supposing by contradiction $l(y) \leq n - m - 2$, let $l(y) = n - m - 1 - h$ with $h > 0$. This will lead us to the false relation $(x, m + h) \in V$. Indeed, $l(y) = n - m - h - 1 < n - m - h$ and $\varphi(y, l(x)) = x$ show that $(x, m + h) \in V(\varphi) = V$.

The just proved equality $l(y) = n - m - 1$ shows that

\[ \text{card} \{ x \in X^* \mid l(x) = n, m_\psi(x) = m \} \leq \text{card} \{ y \in X^* \mid l(y) = n - m - 1 \} = p^{n-m-1}, \]

and the assertion (2) is proved.

Assuming that $V$ is recursive we shall prove (2) $\Rightarrow$ (1). The hypothesis is that (1) holds for every $m \geq 1$. We shall show that $V = V(\varphi)$, where $\varphi$ is the p.r. function constructed in Theorem 2, namely we shall show that $V(\varphi) \subset V$.

Take $(x, m) \in V(\varphi)$. In any case $(x, 1) \in V$ (see Theorem 2). We shall prove that $(x, m) \in V$ by proving that $m_\psi(x) = m_{V(\varphi)}(x)$. Since $V \subset V(\varphi)$ (see Theorem 2) we have $m_{V(\varphi)}(x) \geq m_\psi(x)$ and all it remains to prove is that $m_\psi(x) \geq m_{V(\varphi)}(x)$.

Supposing the contrary, it follows that $(x, m_\psi(x) + 1) \in V(\varphi)$, hence there exists $z$ in $X^*$ with $l(z) = l(x) - m_\psi(x) - 1$ and $\varphi(z, l(x)) = x$. 

530
Let $g(i) = (x_i, m_F(x_i))$ where $x = x_i$ in the enumeration given by $g$ (see the construction of $\varphi$ in the proof of Theorem 2). We let the procedure giving $\varphi$ run $i$ steps and we obtain the string $z_i$ such that $\varphi(z_i, l(x_i)) = x_i$. We shall show that $l(z_i) = l(x_i) - m_F(x_i) - 1 = l(x) - m_F(x) - 1$, thus deriving a contradiction (in view of the injectivity of $\varphi$).

Now the reader must remember the action of $\varphi$ (see the proof of Theorem 2).

In case $l(x_i) = l(x_j)$ for all $j < i$, we have $l(z_i) = l(x_i) - m_F(x_i) - 1$, and the proof is finished in this case. In case $l(x_{j(1)}) = l(x_{j(2)}) = ... = l(x_{j(k)}) = l(x)$, we saw that $l(z_i) = l(x_i) - m_F(x_i) - 1$, and again the proof is finished. The most complicated case is when $m_F(x_{j(1)}) = m_F(x_{j(2)}) = ... = m_F(x_{j(k)})$, where $0 \leq r < k$. In this case we must put $z_i = y(s(l(x)) - m_F(x) - 1) - (r + 1)$.

In any case we have $r + 2$ elements $x$ such that $l(x) = n$ and $m_F(x) = m$ (we put $(x, 1) \notin V = V(\varphi)$). The next result establishes a precise connection between the Kolmogorov complexity $K$ and the critical level $m_F$ in case $V = V(\varphi)$.

**Theorem 4.** Let $V$ be a representable $M - L$ test and let $\varphi: X^* \times N \to X^*$ be a p.r. function such that $V = V(\varphi)$. The following assertions hold for all $x$ in $X^*$:

(a) $m_F(x) = 0$ iff $K(x \mid l(x)) \geq l(x) - 1$.

(b) If $m_F(x) > 0$, then $K(x \mid l(x)) = l(x) - m_F(x) - 1$.

In the particular case when $\varphi$ has the additional property that range $\varphi = \{x \in X^* \mid (x, 1) \in V\} = V_1$, point (a) can be stated more precisely, namely:

(a') $m_F(x) = 0$ iff $K(x \mid l(x)) = \infty$.

**Proof.** (a) Assume $m_F(x) = 0$, therefore $(x, 1) \notin V = V(\varphi)$. This shows that for every $y$ in $X^*$ with $l(y) < l(x) - 1$ we have $\varphi(y, l(y)) = x$. Then, either $\varphi(y, l(y)) = x$ for all $y$ in $X^*$ (which shows that $K_{\varphi}(x \mid l(x)) = \infty$), or there exists $y$ in $X^*$ with $\varphi(y, l(y)) = x$, but this $y$ must have $l(y) \geq l(x) - 1$. So, $K_{\varphi}(x \mid l(x)) \geq l(x) - 1$.

Assume now that $K_{\varphi}(x \mid l(x)) = \infty$. There are two cases:

i) if $K_{\varphi}(x \mid l(x)) = \infty$, then $\varphi(y, l(y)) = x$ for all $y$ in $X^*$ and then $(x, 1) \notin V(\varphi)$.

ii) if $K_{\varphi}(x \mid l(x)) < \infty$, then there exists at least one $y$ in $X^*$ with $\varphi(y, l(y)) = x$ and one must have $l(y) \geq l(x) - 1$. This shows that $(x, 1) \notin V(\varphi)$.

(b) According to the hypothesis, there exists $y$ in $X^*$ such that $\varphi(y, l(y)) = x$. 

531
We have: $m_y(x) = m_Y^r(x) = \max \{m \geq 1 \mid \text{there exists } y \text{ in } X^* \text{ with } l(y) < l(x) - m \text{ and } \phi(y, l(x)) = x\} = \max \{m \geq 1 \mid \text{there exists } y \text{ in } X^* \text{ with } m < l(x) - l(y) \text{ and } \phi(y, l(x)) = x\}$. The last maximum is attained for those $y$ in $X^*$ which have minimum length, i.e. for those $y$ in $X^*$ with $l(y) = l(x) - m$. In the particular case, all it remains to prove is the implication: $m_y(x) = 0 \Rightarrow m_y(x) = 0$.

**Corollary 5.** Let $V$ be a recursive representable $M-L$ test and let $\varphi : X^* \times N \to X^*$ be a p.r. function with the properties $V = V(\varphi)$ and range $\varphi = V$. Then the partial function $U_\varphi : X^* \to N$ given by $U_\varphi(x) = K_\varphi(x \mid l(x))$ is a p.r. function with recursive graph.

**Proof.** Relations (a') and (b) in Theorem 4 applied to the present function $\varphi$ show that $U_\varphi$ is a p.r. function. The graph of $U_\varphi$ is recursive because the pair $(x, m) \in$ graph $(U_\varphi)$ if and only if $m_y(x) > 0$ and $m = l(x) - m_y(x) - 1$. Here we made use of the recursive­ness of the function $m_y$ (see Theorem 1).

**Remark.** The p.r. function $\varphi$ given by the proof of Theorem 3 is a function satisfying the property that range $\varphi = V$.

**Theorem 6.** Let $\varphi : X^* \times N \to X^*$ be a p.r. function such that range $\varphi = V$. Then the following assertions are equivalent:

(i) For all natural $n > m - 1$, we have:

$$\text{card } \{x \in X^* \mid l(x) = n, \ (x, m) \in V\} = \frac{(V_\varphi - 1)}{(p - 1)}.$$
(ii) For all natural \( n > m \geq 1 \), we have:

\[
\text{card } \{ x \in X^* \mid l(x) = n, m_v(x) = m \} = p^{n-m-1}.
\]

2. If one of the above conditions (i) or (ii) is fulfilled for a set \( V \) having properties (a) and (b), then \( V \) is a recursive representable \( M-L \) test. Such \( M-L \) tests will be called full.

**Proof.** 1. (i) \( \Rightarrow \) (ii). The conditions (a), (b) and (i) insure that \( V \) is a \( M-L \) test, hence \( m_v \) takes only finite values.

On the other hand, for every natural \( j \geq 0 \) and \( n \geq j + 1 \) one can see, using condition (b), that

\[
\{ x \in X^* \mid l(x) = n, m_v(x) = n - (j + 1) \} = \{ x \in X^* \mid l(x) = n, (x, n-j-1) \in V \}.
\]

Consequently, card \( \{ x \in X^* \mid l(x) = n, m_v(x) = n - (j + 1) \} = \frac{(p^{n-(j+1)} - 1)(p - 1)}{(p - 1)} = \frac{p^n - 1}{p - 1} \)

(\( (i) \) \( \Rightarrow \) (ii)). For every natural \( n > m \geq 1 \) we have the equality

\[
\text{card } \{ x \in X^* \mid l(x) = n, (x, m) \in V \} = \text{card } \{ x \in X^* \mid l(x) = n, m_v(x) = m \} \cup \\
\cup \{ x \in X^* \mid l(x) = n, m_v(x) = m + 1 \} \cup \ldots \cup \{ x \in X^* \mid l(x) = n, m_v(x) = n - 1 \}.
\]

In fact, \( A_u = \{ x \in X^* \mid l(x) = n, m_v(x) = n \} = \emptyset \), because \( A_u \subset A_{u-1} = \{ x \in X^* \mid l(x) = n, m_v(x) = n - 1 \} \), according to condition (b) and card \( A_{u-1} = 1 \).

If \( A_u \) were non-empty, then card \( A_u = 1 \), so \( A_u = A_{u-1} \) and this is impossible. Again condition (b) guarantees also that \( A_u = \emptyset \), where \( u > n \). Thus the proof of (***) is complete.

Consequently, (***) yields

\[
\text{card } \{ x \in X^* \mid l(x) = n, (x, m) \in V \} = \\
= \sum_{j=m}^{n-1} \text{card } \{ x \in X^* \mid l(x) = n, m_v(x) = j \} = \sum_{j=m}^{n-1} p^{n-j-1} = \frac{(p^n - 1)(p - 1)}{(p - 1)}.
\]

2. All it remains to prove is that (i) implies the recursiveness of \( V \) (because in this case \( V \) will be a recursive \( M-L \) test satisfying condition (2) in Theorem 3).

The case when \( V \) is finite is obvious.

Assume therefore that \( V \) is infinite and let \( g : (N - \{0\}) \to X^* \times N \) be an injective recursive function such that \( g(N - \{0\}) = V \). Put \( g(i) = (x_i, m_i) \) for all natural \( i \geq 1 \).

We take arbitrarily \((x, m)\) in \( X^* \times N \) and we describe an algorithm for testing if \((x, m)\) is in \( V \). Put \( l(x) = n \). There exists a natural \( q \geq 1 \) such that the set \( G = \{ g(1), g(2), \ldots, g(q) \} \) contains all the elements \((y, m)\) in \( V \) with \( l(y) = n \). Moreover, \( q \) can be effectively found. For instance, \( q \) can be taken to be the least natural number \( h \) such that the set \( \{ g(1), g(2), \ldots, g(h) \} \) contains exactly \( (p^{n-h} - 1)(p - 1) \) pairs \((y, m)\) with \( l(y) = n \). If \((x, m) \in G\), then \((x, m) \in V\) and if \((x, m) \notin G\), then \((x, m) \notin V\).
Example 8. We shall exhibit an example of $M - L$ test $V$ which is full and we shall also construct the associate p.r. function $\varphi$ such that $V = V(\varphi)$ given by Theorem 3.

a) In order to give the $M - L$ test $V$ we shall denote, for every $n > m \geq 1$, by $A(n, m)$ the set $\{(x, m) \in V \mid l(x) = n\}$. It is clear that the $M - L$ test $V$ will be completely determined if we shall give all the sets $A(n, m)$.

Put $A(n, m) = \{(y(s(n - 1) + i), m) \mid i = 1, 2, \ldots, s(n - m - 1)\}$ (see Section 1).

It is seen that for every $m > 1$ one has

$$V_m = \bigcup_{i=m+1}^{n} \{y(s(n - 1) + i) \mid i = 1, 2, \ldots, s(n - m - 1)\}.$$

The reader can see now that this $V$ is a full $M - L$ test. Moreover, an elementary computation gives the form of the function $m_V$. We have for all $n \geq 2$:

$$m_V(y(s(n - 1) + 1)) = n - 1,$$

and

$$m_V(y(s(n - 1) + i)) = n - k - 1,$$

for every $1 \leq k \leq n - 2$, where $i \in \{s(k - 1) + 1, s(k - 1) + 2, \ldots, s(k)\}$; also

$$m_V(x) = 0,$$

for the other $x$ in $X^*$.

An inspection of $A(n, 1)$ shows that for $n \geq 2$ one has:

$$\text{card} \{x \in X^* \mid l(x) = n, \text{there exists } m > 1 \text{ such that } (x, m) \in V\} = s(n - 2).$$

b) In order to do the construction indicated in the proof of Theorem 2, we shall choose an enumeration function $g$ for the set $A = \{(x, m_V(x)) \mid x \in V\}$. This $g$ will satisfy the conditions u), v) required in the proof of Theorem 2 and it possesses the supplementary property (which completely determines $g$):

w) if for $i < j$ one has $l(x_i) = l(x_j)$ and $m_V(x_i) = m_V(x_j)$, then $x_i > x_j$ in lexicographical order. This means that for every $n > m \geq 1$, the set $\{x \in X^* \mid l(x) = n, m_V(x) = m\}$ is ordered by the inverse of the lexicographical order.

The p.r. function $\varphi : X^* \times N \rightarrow X^*$ produced by the proof of Theorem 2 is given by

$$\varphi(y(i), n) = y(s(n - 1) + i),$$

for every $n \geq 2$ and $i = 1, 2, \ldots, s(n - 2)$.

An alternative of Theorem 7 (which was based upon the equalities (i) and (ii) guaranteeing the recursiveness of $V$) will be the following theorem. Here we shall actually replace the equalities (i) and (ii) in Theorem 7 by inequalities and we shall assume the recursiveness of $V$.

Theorem 9. Let $V \subseteq X^* \times N$ be a set having the following properties:

(a) The set $V$ is recursive.

(b) For every natural $m \geq 1$, we have the inclusion $V_{m+1} \subseteq V_m$.
(c) For all natural $n > m \geq 1$, we have

$$\text{card } \{x \in X^* \mid l(x) = n, m_\nu(x) = m\} \leq p^{-n-1}. $$

Under these assumptions the set $V$ is a representable $M - L$ test.

**Proof.** In view of Theorem 3, all it remains to prove is the fact that $V$ is a $M - L$ test. This can be done using the equality (***) in the proof of Theorem 6, which yields

$$\text{card } \{x \in X^* \mid l(x) = n, (x, m) \in V\} = \sum_{j=m}^{n-1} \text{card } \{x \in X^* \mid l(x) = n, m_\nu(x) = j\} \leq \sum_{j=m}^{n-1} p^{-j-1} = (p^{n-m} - 1)/(p - 1). \quad \square$$

### 3. REMARKS AND OPEN PROBLEMS

Our representability theory (see also [2]) is an attempt to compare Kolmogorov’s complexity theory of strings which uses algorithms [3] with P. Martin-Löf’s complexity theory of strings which uses $M - L$ tests [5]. We have already seen that there exist non-representable $M - L$ tests [2], i.e. these theories are not equivalent. For instance, take $p = 2$, $X = \{0, 1\}$ and $V = \{(000, 1), (010, 1), (111, 1)\}$.

In this direction we could obtain the following result concerning recursive sets:

If we call $K$-test a set $V \subset X^* \times N$ having properties (a), (b) and (c) in Theorem 9, then Kolmogorov’s complexity theory and P. Martin-Löf’s complexity theory done only with $K$-tests are equivalent. This means that for every p.r. function $\psi : X^* \times \rightarrow X^* \times N \rightarrow X^*$ we can obtain the $K$-test $V(\psi)$ and for every $K$-test $V \subset X^* \times N$ we can obtain a p.r. function $\varphi : X^* \times X^* \rightarrow X^*$ such that $V = V(\varphi)$ (see Example 10 in [1], Theorems 3, 4 and 9).

We set the following natural open problems:

A) Does the equivalence $(1) \Leftrightarrow (2)$ in Theorem 3 hold also for non-recursive $M - L$ tests $V$? Equivalently, does the result in Theorem 9 hold also for non-recursive $M - L$ tests $V$?

B) Does the result in Theorem 2 hold also for non-recursive $M - L$ tests $V$?

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