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ADDITIONAL SIGNALS IN LINEAR DISCRETE-TIME CONTROL SYSTEMS III

Additional Disturbance Feedforward

VÁCLAV SOUKUP

A linear discrete-time control system is treated in which a measurable external disturbance affecting the control process is added to the feedback loop. Using the polynomial approach, closed-loop stability is investigated and time optimal as well as least squares control problems are solved. The results are compared with the solutions of similar problems in a simple control system.

1. INTRODUCTION

Following both previous parts [8], [9] of the paper, this last part deals with the additional signal of disturbance feedforward (ASDF) in a linear, discrete-time (sampled-data), single variable control system.

Starting with the closed-loop stability investigation, the time optimal as well as least squares control problems are solved. The results are analyzed and compared with the simple system solutions. As the polynomial method is used again the reader is assumed to be acquainted with the fundamental symbols and operations of the algebraic theory ([1], [2]) which have been briefly presented in Sections 1 and 2 of [8].

Control systems with ASDF along with other types of interconnected systems using analogue control techniques have been described, e.g., in [3]–[6]. A brief analysis of their digital control applications based on Z-transform and state-space approach can be found in [7].

2. ADDITIONAL SIGNAL OF THE DISTURBANCE FEEDFORWARD IN A LINEAR DISCRETE-TIME CONTROL SYSTEM

The considered system structure is shown by the block diagram in Fig. 1. A closed-loop discrete-time system subjected to a reference input W is continuously affected at the same time by a disturbance (load) $\mathscr{V}$ through a part $\mathscr{S}_1$ of a controlled system

$\mathscr{S}$. Provided this disturbance input is measurable it can be sampled and fed through an additional controller R_2 to contribute to the control signal U. If the additional loop is omitted the considered structure is reduced into a simple feedback control system with only one digital controller R_1 operating on the error signal E. Samplers

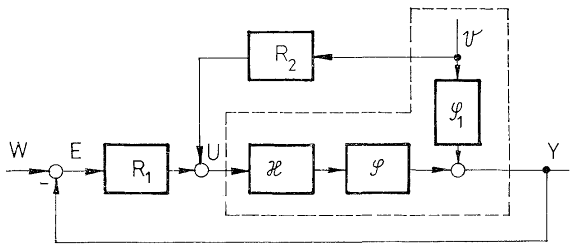

Fig. 1.

are not pictured in Fig. 1 for simplicity and the continuously operating part of the system is shown within the dash line.

Let us denote the sampled (and digitalized) disturbance input by V and the result-

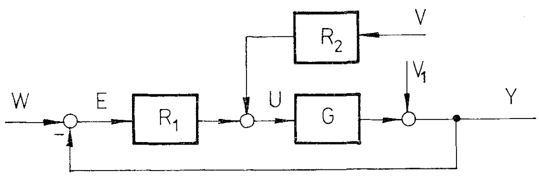

Fig. 2.

ing sampled-data effect of $\mathscr{V}$ after passing through $\mathscr{S}_1$ by V_1. Then the block diagram in Fig. 1 can be transformed for discrete values of time as shown in Fig. 2 where

(1) $$G = \frac{b}{a}, \quad (a, b) \sim 1, \quad z^{-1} \mid b,$$

represents the discrete-time transfer sequence of the system $\mathscr{S}$ being controlled (including a data reconstructor $\mathscr{H}$),

(2) $$R_1 = \frac{m_1}{n_1}, \quad (n_1, m_1) \sim 1, \quad \text{and} \quad R_2 = \frac{m_2}{n_2}, \quad (n_2, m_2) \sim 1,$$

are both controllers transfer sequences.

The following transfer relations hold:

$$Y = (1 + GR_1)^{-1} (GR_1 W + GR_2 V + V_1), \tag{3}$$

$$E = (1 + GR_1)^{-1} (W - GR_2 V - V_1), \tag{4}$$

$$U = (1 + GR_1)^{-1} (R_1 W + R_2 V - R_1 V_1). \tag{5}$$

Substituting (1) and (2) into equations (3)–(5) and writing them in the vector matrix form,

$$\begin{bmatrix} Y \\ E \\ U \end{bmatrix} = \begin{bmatrix} K_{W/Y} & K_{V/Y} & K_{V_1/Y} \\ K_{W/E} & K_{V/E} & K_{V_1/E} \\ K_{W/U} & K_{V/U} & K_{V_1/U} \end{bmatrix} \begin{bmatrix} W \\ V \\ V_1 \end{bmatrix} = \tag{6}$$

$$= (an_1 + bm_1)^{-1} n_{20}^{-1} \begin{bmatrix} bm_1 n_{20} & bm_2 n_{10} & an_0 \\ an_0 & -bm_2 n_{10} & -an_0 \\ am_1 n_{20} & am_2 n_{10} & -am_1 n_{20} \end{bmatrix} \begin{bmatrix} W \\ V \\ V_1 \end{bmatrix}$$

where

$$n_{10} = \frac{n_1}{(n_1, n_2)}, \quad n_{20} = \frac{n_2}{(n_1, n_2)} \quad \text{and} \quad n_0 = (n_1, n_2)\, n_{10} n_{20}. \tag{7}$$

Hence the pseudocharacteristic polynomial of the closed loop system ([1], [10]) is

$$l = (an_1 + bm_1)\, n_{20}. \tag{8}$$

It is important to note that relations (6)–(8) are true under the assumption that both control algorithms R_1 and R_2 are actually realized as a single two-input one-output system.

3. STABILITY AND CAUSALITY OF THE CLOSED-LOOP SYSTEM

Speaking about stability Liapunov's asymptotic stability (in the large) is assumed here.

Theorem 1. The closed-loop system with ASDF pictured in Fig. 2 and described by relations (1)–(2) is stable and causal if and only if

$$R_1 = M_1 N^{-1} \quad \text{and} \quad R_2 = M_2 N^{-1} \tag{9}$$

where M_1, M_2 and N are stable sequences,

$$aN + bM_1 = 1 \tag{10}$$

and N^{-1} is causal.

Proof.

1. At first it will be proved that the closed-loop system is stable if and only if the

closed-loop transfer sequences $K_{W/Y}$, $K_{W/E}$ and $K_{V/Y}$ in (6) can be written in the form

(11) $K_{W/Y} = bM_1, K_{W/E} = aN$ and $K_{V/Y} = bM_2$ where M_1, M_2 and N are stable sequences.

a) Only if: It follows from (6) and (8) that

(12) $$K_{W/Y} = bm_1 n_{20} l^{-1}, K_{W/Y} = an_0 l^{-1} \quad \text{and} \quad K_{V/Y} = bm_2 n_{10} l^{-1}.$$

Denoting

(13) $$M_1 = m_1 n_{20} l^{-1}, \quad M_2 = m_2 n_{10} l^{-1} \quad \text{and} \quad N = n_0 l^{-1}$$

then, really, $K_{W/E}$, $K_{W/E}$ and $K_{V/Y}$ are according to (11). The pseudocharacteristic polynomial l of a stable system is a stable polynomial and therefore M_1, M_2 and N must be stable sequences.

b) If: Suppose that M_1, M_2 and N are stable but $l = l^+ l^-$ is not stable with $l^- \nsim 1$. Then $l^- \mid m_1 n_{20}$, $l^- \mid m_2 n_{10}$ and $l^- \mid n_0$ at the same time according to (13). But $(m_1 n_{20}, m_2 n_{10}, n_0) \sim 1$ and consequently $l^- \sim 1$ is the only possibility.

Note that stability of the remaining closed-loop transfer sequences in (6)

(14) $$K_{V_1/Y} = -K_{V_1/E} = aN, \quad K_{V/E} = -bM_2,$$
$$K_{W/U} = -K_{V_1/U} = aM_1 \quad \text{and} \quad K_{V/U} = aM_2$$

follows from that of M_1, M_2 and N.

2. Using (13)

$$aN + bM_1 = an_0 l^{-1} + bm_1 n_{20} l^{-1} = (an_1 + bm_1) n_{20} l^{-1} = ll^{-1} = 1$$

and equation (10) is verified.

Relations (9) follow from (3)–(5), (11) and (14) since

$$R_1 = K_{W/U} K_{W/E}^{-1} = M_1 N^{-1} \quad \text{and} \quad R_2 = K_{V/U} K_{W/E}^{-1} = M_2 N^{-1}.$$

3. According to (9), N^{-1} must be causal to ensure causality of both controllers R_1 and R_2. Since G itself is causal the closed-loop system is then causal too. □

4. TIME OPTIMAL CONTROL

When solving the time optimal control (TOC) problem, linear control algorithms are to be found such that the closed-loop system is stable and the error sequence E is finite and as short as possible. At the same time the control sequence U is required to be either stable (stable TOC) or finite (finite TOC).

The following theorem gives the solution of both TOC problems under the assumption the system being controlled is initially at rest.

Theorem 2. Given the discrete-time system with ASDF pictured in Fig. 2, described by relations (1)–(2) and subjected to the inputs W. V and V_1 where $W - V_1 = f/h$, $(h, f) \sim 1$, and $V = p/q$, $(q, p) \sim 1$, then

a) the stable TOC is assured by the controllers (9) with

$$N = \frac{h_0 x}{a_0^+ f^+}, \quad M_1 = \frac{y}{b^+ f^+} \quad \text{and} \quad M_2 = \frac{q_0 f_{p1}^- v}{b_0^+ p^+} \tag{15}$$

where the triplet of polynomials x, y, v satisfies the equations

$$a_0^- x - b_{01}^- p_f^- v = (a_0, p_f)^- s \tag{16}$$

and

$$(b, q)^- (b_0, f_p)^- y + h p_f^- v = r \tag{17}$$

with x causal. The polynomials s, r represent the solution of the equation

$$h(a_0, p_f)^- s + b_{01}^- r = f^+ \tag{18}$$

with s of minimum degree.

The notations used in (15)–(18) are as follows:

$$a_0 = \frac{a}{(a, h)} \quad \text{and} \quad h_0 = \frac{h}{(a, h)}, \tag{19}$$

$$b_0 = \frac{b}{(b, q)} \quad \text{and} \quad q_0 = \frac{q}{(b, q)}, \tag{20}$$

$$f_p = \frac{f}{(f, p)} \quad \text{and} \quad p_f = \frac{p}{(f, p)}, \tag{21}$$

$$f_{p1} = \frac{f_p}{(b_0, f_p)} \quad \text{and} \quad b_{01} = \frac{b_0}{(b_0, f_p)}. \tag{22}$$

The error sequence (polynomial)

$$E = e = (a_0, p_f)^- f^- s \tag{23}$$

and the control sequence

$$U = \frac{a_0 (f, p)^- f_{p1}^- r}{h_0 b_0^+ (b, q)}. \tag{24}$$

The optimal solution exists if and only if $h_0 \sim h_0^+$ and $(b, q) \sim (b, q)^+$. The resulting optimal error (23) is unique while the optimal controllers R_1 and R_2 are not.

b) The finite TOC is assured by the controllers (9) with

$$N = \frac{h_0 x}{a_0^+ f^+}, \quad M_1 = \frac{y}{f^+} \quad \text{and} \quad M_2 = \frac{q_0 f_{p1}^- v}{p^+} \tag{25}$$

where the polynomials x, y, v satisfy the equations

$$a_0^- x - b_0^+ b_{01}^- p_f^- v = (a_0, p_f)^- s \tag{26}$$

and

$$(27) \qquad (b, q)(b_0, f_p)^- y + hp_f^- v = r$$

with x causal provided that s, r is the min deg s solution of the equation

$$(28) \qquad h(a_0, p_f)^- s + b_0^+ b_{01}^- r = f^+$$

and $a_0, h_0, b_0, q_0, f_p, p_f, f_{p1}$ and b_{01} are defined in (19)–(22), respectively.

The error polynomial has the form (23) and the control sequence

$$(29) \qquad U = \frac{a_0(f, p)^- f_{p1}^- r}{h_0(b, q)}.$$

The finite TOC problem is solvable if and only if $h_0 \sim 1$ and $(b, q) \sim 1$. The resulting optimal error is unique but the optimal controllers R_1 and R_2 are not.

Proof. According to (6), (11) and (14)

$$(30) \qquad E = aNW - bM_2V - aNV_1 = aN\frac{f}{h} - bM_2\frac{p}{q} =$$

$$= \frac{a_0 f}{h_0} N - \frac{b_0 p}{q_0} M_2 = (a_0 f, b_0 p) L$$

where

$$(31) \qquad L = \frac{s_0}{l_0} = \frac{a_0 f}{h_0(a_0 f, b_0 p)} N - \frac{b_0 p}{q_0(a_0 f, b_0 p)} M_2$$

is a sequence with polynomials s_0 and l_0 undetermined till now. Writing (31) in the form

$$(32) \qquad h_0 q_0 \frac{s_0}{l_0} = \frac{q_0 a_0 f}{(a_0 f, b_0 p)} N - \frac{h_0 b_0 p}{(a_0 f, b_0 p)} M_2$$

both sides of (32) must be stable sequences since N and M_2 are assumed to be stable. At the same time, the requirement that $E = e$ be a polynomial implies $l_0 \mid (a_0 f, b_0 p)$. But $(h_0 q_0, (a_0 f, b_0 p)) \sim 1$ and therefore $(h_0 q_0, l_0) \sim 1$. Hence the optimal $l_0 = l_0^+ = (a_0 f, b_0 p)^+$,

$$(33) \qquad L = \frac{s_0}{(a_0 f, b_0 p)^+}$$

and

$$(34) \qquad e = (a_0 f, b_0 p)^- s_0 = (f, p)^- (a_0, p_f)^- (b_0, f_p)^- s_0 .$$

a) In accordance with (30) the sequence L can be realized only through the sequences N and M_2, which must be chosen in such a way that the equation (cf. (31) and (33))

$$(35) \qquad \frac{a_0 f}{h_0} N - \frac{b_0 p}{q_0} M_2 = (a_0 f, b_0 p)^- s_0$$

be always solvable for any s_0. This condition is satisfied if

(36) $$M_2 = \frac{q_0 w}{b_0^+ p^+}$$

and N is given by (15); (35) then takes the form

(37) $$a_0^- f_{p1}^- x - b_{01}^- p_f^- w = (a_0, p_f)^- s_0$$

and is always solvable for any s_0. Using (10) the error can be also expressed as

(38) $$E = e = (1 - bM_1)\frac{f}{h} - bM_2\frac{p}{q}.$$

If (34) and (36) is substituted into (38) then

(39) $$bM_1 f = f - h(a_0 f, b_0 p)^- s_0 - hb_0^- p^- w.$$

Both sides of (39) must be polynomials and consequently the optimal stable M_1 is given by (15). Then (39) can be written in the form

$$h(a_0 f, b_0 p)^- s_0 + b^- f^- y + hb_0^- p^- w = f$$

and, after arrangement,

(40) $$h(a_0, p_f)^- s_0 + b_{01}^-(b, q)^- f_p^- y + hb_{01}^- p_f^- w = f^+ f_{p1}^-.$$

Equation (40) can be decomposed into two equations

(41) $$(b, q)^- f_p^- y + hp_f^- w = r_0$$

and

(42) $$h(a_0, p_f)^- s_0 + b_{01}^- r_0 = f^+ f_{p1}^-.$$

The control sequence follows from (6) and (14):

(43) $$U = \frac{a_0 f^- y}{h_0 b^+} + \frac{ap^- w}{b_0^+(b, q)} = \frac{a_0(f, p)^- r_0}{h_0\, b_0^+(b, q)}$$

and it is stable if and only if $h_0 \sim h_0^+$ and $(b, q) \sim (b, q)^+$. In this case equations (41) and (42) are always solvable since $((b, q)^- f_p^-, hp_f^-) \sim (f_p^-, hp_f^-) \sim 1$ and $(h(a_0, p_f)^-, b_{01}^-) \sim 1$, respectively.

Any optimal polynomial w must satisfy both equations (37) and (41). Let us write the corresponding w-part of the general solution x, w of (37) as

(44) $$w = w_0 + \frac{a_0^-}{(a_0, p_f)^-} f_{p1}^- t$$

where w_0 comes from a particular solution x_0, w_0 of (37) and t is an arbitary polynomial. When $w \neq 0$ given by (44) is substituted into (41) then

$$(45) \qquad (b, q)^- f_p^- y + hp_f^- \frac{a_0^-}{(a_0, p_f)^-} f_{p1}^- t = r_0 - hp_f^- w_0 .$$

Obviously (45) has a solution y, t if and only if $f_{p1}^- \mid (r_0 - hp_f^- w_0)$. But this is not generally fulfilled for any r_0 resulting from (42) and any w_0 in (44). To ensure this condition, $f_{p1}^- \mid r_0$ as well as $f_{p1}^- \mid w_0$ must be imposed. Thus the polynomials w, r_0 and s_0 are partially determined in advance:

$$(46) \qquad w = f_{p1}^- v , \quad r_0 = f_{p1}^- r \quad \text{and} \quad s_0 = f_{p1}^- s .$$

Applying relations (46) equations (37), (41) and (42) obtain their final forms (16), (17) and (18), respectively; M_2 is given by (15), e by (23) and U by (24). The optimal error (23) is unique since it is determined by the unique min deg s solution s, r of equation (18). Having found s, r and substituting them into (16) and (17) any solution x, y, v is acceptable. Therefore R_1 and R_2 are not unique. □

Note 1. The presented approach starts with the determination of polynomials s, r and it makes possible to determine the optimal error immediately. But another procedure is possible. Combining equations (16)–(18) we can simply prove that x and y satisfy the equation

$$(47) \qquad ha_0^- x + b^- y = f^+$$

which corresponds to the closed-loop stability equation (10) if M_1 and N are according to (15). It is well known that the min deg x solution x, y of (47) is looked for if the stable TOC problem is solved in a simple control system ($R_2 = 0$), with the resulting error being equal to

$$(48) \qquad e = a_0^- f^- x .$$

In the case of ASDF the optimal error is given by (23) or, using (16), by

$$(49) \qquad e = f^-(a_0^- x - b_{01}^- p_f^- v) .$$

Therefore, if the general or a particular solution x, y of (47) is determined and used in (49), a "corrective" polynomial v can be found (by trials) to minimize deg e. This procedure seems to be usually more simple when compared with the first one and especially suitable provided the optimal solution for $R_2 = 0$ has already been known and ought to be improved by ASDF. Then the controller (algorithm) R_2 can be added after the main feedback controller R_1 is tuned.

Nevertheless the first approach described above yields a more exact analysis of the problem and can be also simply applied for least squares control problem which will be solved in the next section. □

b) If the finite TOC problem is to be solved then M_1 and M_2 given by (15) do not ensure a finite control sequence U as seen in (24). Starting with equation (35) and choosing

$$M_2 = \frac{q_0 w}{p^+} \tag{50}$$

(L and N stay unchanged) the equation

$$a_0^- f_{p1}^- x - b_0^+ b_{01}^- p_f^- w = (a_0, p_f)^- s_0 \tag{51}$$

is obtained and always solvable for any s_0. Using (34) and (50) in (38) then

$$bM_1 f = f - h(a_0 f, b_0 p)^- s_0 - h b_0 p^- w . \tag{52}$$

If M_1 is given by (25) equation (52) can be written, after a small rearrangement, in the form

$$h(a_0, p_f)^- s_0 + b f_{p1}^- y + h b_0^+ b_{01}^- p_f^- w = f^+ f_{p1}^- . \tag{53}$$

The decomposition of (53) results in two equations

$$(b, q) f_p^- y + h p_f^- w = r_0 \tag{54}$$

and

$$h(a_0, p_f)^- s_0 + b_0^+ b_{01}^- r_0 = f^+ f_{p1}^- . \tag{55}$$

The control sequence is expressed as

$$U = \frac{a_0 f^- y}{h_0} + \frac{a p^- w}{(b, q)} = \frac{a_0 (f, p)^- r_0}{h_0 (b, q)} \tag{56}$$

and it is finite (polynomial) if and only $h_0 \sim 1$ and $(b, q) \sim 1$. Then equations (54) and (55) are always solvable. Since conditions (46) can be simply derived in a way similar to that used in the stable TOC case, the final form of M_2 stands in (25) and equations (51), (54) and (55) are transformed into (26), (27) and (28), respectively. The optimal error e has the final form (23) and U is given by (29). According to (23) the min deg s solution of (28) yields the unique optimal error. But any solution x, y, v satisfying equations (26) and (27) is allowed in order to solve the given problem and, therefore, R_1 and R_2 are not unique. □

Note 2. The identity

$$h a_0^- x + b y = f^+ \tag{57}$$

can be proved if (26)–(28) are combined. Hence the solution of the finite TOC problem can start with the general or a particular solution x, y of (57) the x-part of which is then considered in

$$e = f^- (a_0^- x - b_0^+ b_{01}^- p_f^- v) \tag{58}$$

and a suitable polynomial v is finally sought to minimize the degree of (58). □

5. LEAST SQUARES CONTROL

In the case of least squares control (LSC) the squared quadratic norm $\sigma_E = \|E\|^2$ of the error sequence is to be minimized. At the same time the closed-loop stability must not be destroyed and the control sequence U is required to be stable. The solution of the LSC problem is formulated and proved in the following theorem. The system is assumed to be initially at rest.

Theorem 3. Given the discrete-time system with ASDF pictured in Fig. 2, described by relations (1) and (2) and subjected to the inputs W, V and V_1 where $W - V_1 = f/h$, $(h, f) \sim 1$, and $V = p/q$, $(q, p) \sim 1$, then the LSC is ensured by the controllers (9) with

$$N = \frac{h_0 x}{a_0^+ f^* b_{01}^{-\sim}(a_0, p_f)^{-\sim}}, \quad M_1 = \frac{y}{b^+ f^* b_{01}^{-\sim}(a_0, p_f)^{-\sim}} \tag{59}$$

and

$$M_2 = \frac{q_0 f_{p1}^- v}{b_0^+ p^+ f^{-\sim} b_{01}^{-\sim}(a_0, p_f)^{-\sim}}$$

where a_0, h_0, b_0, q_0, f_p, p_f, f_{p1} and b_{01} are given by (19)–(22), respectively. The triplet of polynomials x, y, v satisfies the equations

$$a_0^- x - b_{01}^- p_f^- v = (a_0, p_f)^- s \tag{60}$$

and

$$(b, q)^- (b_0, f_p)^- y + h p_f^- v = r \tag{61}$$

with x causal where s, r represent the solution of the equation

$$h(a_0, p_f)^- s + b_{01}^- r = b_{01}^{-\sim} f^*(a_0, p_f)^{-\sim} \tag{62}$$

with $\deg s < \deg b_{01}^-$.

The error sequence

$$E = \frac{(a_0, p_f)^- f^- s}{(a_0, p_f)^{-\sim} f^{-\sim} b_{01}^{-\sim}}, \tag{63}$$

the control sequence

$$U = \frac{a_0 (f, p)^- f_{p1}^- r}{h_0 b_0^+ (b, q) f^{-\sim} b_{01}^{-\sim}(a_0, p_f)^{-\sim}} \tag{64}$$

and the optimal performance index

$$\sigma_{E\min} = \left\langle \frac{\bar{s}}{\overline{b_{01}^-}} \, \frac{s}{b_{01}^-} \right\rangle. \tag{65}$$

The LSC problem is solvable if and only if $h_0 \sim h_0^+$ and $(b, q) \sim (b, q)^+$. The resulting optimal error (63) is unique while the optimal controllers R_1 and R_2 are not.

Proof. Starting with equation (38)

$$(66) \qquad E = \frac{f}{h} - bM_1 \frac{f}{h} - bM_2 \frac{p}{q} =$$

$$= f\left(\frac{1}{h} - \frac{b}{h} M_1 - \frac{b_{01} p_f}{f_{p1} q_0} M_2\right) = f\left(\frac{1}{h} - b_{01} F\right)$$

where

$$(67) \qquad F = \frac{(b, q)(b_0, f_p)}{h} M_1 + \frac{p_f}{f_{p1} q_0} M_2 .$$

Let us define

$$(68) \qquad E^* = f^*\left(\frac{1}{h} - b_{01} F\right) = Q^* - f^* b_{01} F$$

where

$$(69) \qquad Q^* = \frac{f^*}{h} .$$

Then

$$(70) \qquad E^* = E \frac{f^{-\sim}}{f^{-}} \quad \text{and} \quad \bar{E} E = \overline{E^*} E^* .$$

Writing

$$\overline{E^*} E^* = (\overline{Q^*} - \overline{f^*} \overline{b_{01}} \bar{F})(Q^* - f^* b_{01} F) = (\bar{Z} - \overline{f^*} \overline{b_{01}^*} \bar{F})(Z - f^* b_{01}^* F)$$

where the sequence Z satisfies the identities

$$\bar{Z} Z = \overline{Q^*} Q^* , \quad b_{01}^* \bar{Z} = b_{01} \overline{Q^*} \quad \text{and} \quad \overline{b_{01}^*} Z = \overline{b_{01}} Q^* ,$$

then

$$(71) \qquad \overline{E^*} E^* = \left(\frac{b_{01}}{b_{01}^*} \overline{Q^*} - \overline{f^*} \overline{b_{01}^*} \bar{F}\right) \frac{\overline{c^\sim}}{\bar{c}} \left(\frac{\overline{b_{01}}}{\overline{b_{01}^*}} Q^* - f^* b_{01}^* F\right) \frac{c^\sim}{c} = \bar{E}_0 E_0$$

with the polynomial c undetermined till now. Note that (71) is verified since $c\bar{c} = c^* \overline{c^*} = c^\sim \overline{c^\sim}$ for any c. Obviously,

$$(72) \qquad E_0 = \frac{\overline{b_{01}} c^\sim}{\overline{b_{01}^*} c} Q^* - f^* b_{01}^* \frac{c^\sim}{c} F = \frac{b_{01}^{-\sim} c^\sim f^*}{b_{01}^{-} c h} - f^* b_{01}^* \frac{c^\sim}{c} F .$$

If the first term in (72) is decomposed as follows

$$(73) \qquad \frac{b_{01}^{-\sim} c^\sim f^*}{b_{01}^{-} c h} = \frac{s}{b_{01}^{-}} + \frac{r}{hc}$$

and the notation

$$(74) \qquad X = \frac{r}{hc} - f^* b_{01}^* \frac{c^\sim}{c} F$$

is used, then

$$E_0 = \frac{s}{b_{01}^-} + X \tag{75}$$

and

$$\sigma_E = \langle \bar{E}E \rangle = \langle \overline{E^*}E^* \rangle = \langle \bar{E}_0 E_0 \rangle = \left\langle \left(\frac{\bar{s}}{\overline{b_{01}^-}} + \bar{X} \right) \left(\frac{s}{b_{01}^-} + X \right) \right\rangle . \tag{76}$$

The decomposition (73) results in the equation

$$hcs + b_{01}^- r = b_{01}^{-\sim} f^* c^\sim . \tag{77}$$

If (77) is solvable, its general solution can be written in the form

$$s = s_2 - \frac{b_{01}^-}{(hc, b_{01}^-)} t \quad \text{and} \quad r = r_2 + \frac{hc}{(hc, b_{01}^-)} t \tag{78}$$

where s_2, r_2 is the particular solution having the property $\deg s_2 < \deg b_{01}^-$ and t is an arbitrary polynomial. Substituting s given by (78) into (76) one obtains

$$\sigma_E = \left\langle \frac{\bar{s}_2}{\overline{b_{01}^-}} \frac{s_2}{b_{01}^-} \right\rangle + \left\langle \left(\bar{X} - \frac{\bar{t}}{\overline{(hc, b_{01}^-)}} \right) \left(X - \frac{t}{(hc, b_{01}^-)} \right) \right\rangle \tag{79}$$

since

$$\frac{\bar{s}_2}{\overline{b_{01}^-}} = \frac{s_2^\sim}{b_{01}^{-\sim}} z^{-v} \quad \text{and} \quad \frac{s_2}{b_{01}^-} = \frac{\overline{s_2^\sim}}{\overline{b_{01}^{-\sim}}} z^v$$

where $v = \deg b_{01}^- - \deg s_2 > 0$. Expression (79) attains its minimum if $X = t/(hc, b_{01}^-)$. Then $E_0 = s_2/b_{01}^-$ and using (74)

$$F = \frac{r_2}{hf^* b_{01}^* c^\sim} . \tag{80}$$

Now if F is substituted into (68) and (77) is applied for $s_2 = s$ and $r_2 = r$, then

$$E^* = \frac{f^*}{h} - \frac{b_{01}^- r}{h b_{01}^{-\sim} c^\sim} = \frac{cs}{c^\sim b_{01}^{-\sim}} . \tag{81}$$

At the same time the error is

$$E = aN\frac{f}{h} - bM_2 \frac{p}{q} = f\left(\frac{a_0}{h_0} N - \frac{b_{01} p_f}{f_{p1} q_0} M_2 \right) = f(a_0, p_f)\, P \tag{82}$$

by (30), where

$$P = \frac{a_0}{(a_0, p_f)\, h_0} N - \frac{b_{01} p_f}{(a_0, p_f) f_{p1} q_0} M_2 . \tag{83}$$

Then

$$E^* = f^*(a_0, p_f)\, P \tag{84}$$

and, comparing (84) and (81),

$$P = \frac{cs}{c^{\sim} b_{01}^{-\sim} f^*(a_0, p_f)}. \tag{85}$$

Combining (83) and (85) yields

$$\frac{h_0 q_0 f_{p1} cs}{c^{\sim} b_{01}^{-\sim} f^*(a, p_f)} = \frac{a_0}{(a_0, p_f)} f_{p1} q_0 N - \frac{p_f}{(a_0, p_f)} b_{01} h_0 M_2 \tag{86}$$

and, in accordance with the assumption of stability for N and M_2, the left side of (86) must be a stable sequence. Since $((a_0, p_f), h_0 q_0 f_{p1}) \sim 1$, the choice $c = (a_0, p_f)^-$ is necessary. Then the optimal

$$P = \frac{s}{b_{01}^{-\sim} f^*(a_0, p_f)^*}, \tag{87}$$

$$F = \frac{r}{h b_{01}^* f^*(a_0, p_f)^{-\sim}}, \tag{88}$$

equation (77) obtains the form (62) and the error sequence (82) stands finally in (63). But the sequences P and F are realizable through N, M_1 and M_2 only. According to (83) and (87)

$$\frac{a_0}{h_0} N - \frac{b_{01} p_f}{f_{p1} q_0} M_2 = \frac{(a_0, p_f)^- s}{b_{01}^{-\sim} f^*(a_0, p_f)^{-\sim}} \tag{89}$$

and then N and M_2 must be chosen so as to make equation (89) solvable for any s. The choice (59) satisfies this requirement since it results in equation (60). Substituting M_2 and F into (67), the equation

$$(f, p)(b_0, f_p) b_{01}^* f^*(a_0, p_f)^{-\sim} M_1 + h p_f^- v = r \tag{90}$$

is obtained. If M_1 is chosen according to (59) equation (90) obtains form (61).

Then the control sequence following from (6) and (14) is given by (64). It is stable if and only if $h_0 \sim h_0^+$ and $(b, q) \sim (b, q)^+$. In this case equation (61) as well as (62) is always solvable and, moreover, the solution s, r of (62) with $\deg s < \deg b_{01}^-$ is unique and identical to the min deg s solution. Therefore the optimal error (63) is given uniquely. However, any triplet x, y, v satisfying both equations '60) and (61) is allowed and hence optimal controllers R_1 and R_2 are not unique. □

6. CONCLUSIONS

The results derived above are discussed by way of comparison with similar solutions in a simple system structure.

1. Regarding the solvability the investigated optimal control problems are not solvable using ASDF unless being solvable in a simple control system (with $R_2 = 0$).

The condition $h_0 \sim h_0^+$ for the stable TOC and LSC or $h_0 \sim 1$ for the finite TOC (which is sufficient in a simple system) is necessary for ASDF configuration too. Moreover the additional condition $(b, q) \sim (b, q)^+$ or $(b, q) \sim 1$ following from stability or fineteness of the second component of the control signal U must be satisfied. Usually (but not always) $q \mid h$ and then this second condition is redundant.

2. Let us consider the performance index λ_1 reached in a simple system structure and λ_2 attained by ASDF, $\lambda_2 \leqq \lambda_1$ [8]. The case $\lambda_2 = \lambda_1$ means that the optimal control process cannot be improved by any ASDF and consequently the optimal solution with $R_2 = 0$, i.e., $v = 0$, exists. Analyzing equations (16) = (60), (17) = (61), (26) and (27), the case $v = 0$ is possible if and only if $(a_0, p_f)^- \sim a_0^-$ and $(b_0, f_p)^- \sim 1$ for all the types of optimal control problems treated above. Hence ASDF is suitable to be applied in a linear discrete-time control system if

(91) either $(a_0, p_f)^- \sim a_0^-$

(92) or $(b_0, f_p)^- \sim 1$ or both.

If moreover $b_{01}^- \sim 1$ for the stable TOC or the LSC and $b_0^+ b_{01}^- \sim 1$ for the finite TOC, the optimal solution with $s = 0$ and consequently $e = 0$ can be attained.

By Notes 1 and 2 the feedback controller R_1 set up in the optimal way in a simple structure will always remain the optimal one if ASDF is applied for the stable or the finite TOC. But this is no longer true for the LSC problem. Combining equations (60)–(62), we obtain the equation

(93) $$h a_0^- x + b^- y = b_{01}^{-\sim} f^*(a_0, p_f)^{-\sim}$$

the special solution of which gives the optimal R_1 for the simple system ([1], [2], [8]) only if $b_{01}^{-\sim} \sim b^{-\sim}$ as well as $(a_0, p_f)^{-\sim} \sim a_0^{-\sim}$. Of course, this special case is possible even if (92) holds.

3. Only one additional sampler (with an analogue to digital converter) preceding R_2 is needed for ASDF application provided the controllers transfer sequences R_1 and R_2 are realized by computer programs.

EXAMPLE

Let us consider the discrete-time system shown in Fig. 1, where (in Laplace transforms)

$$\mathscr{S}(p) = \frac{2 \cdot 5\, e^{-p}}{p(p+1)}, \quad \mathscr{S}_1(p) = \frac{2 \cdot 5}{p+1}, \quad \mathscr{H}(p) = \frac{1 - e^{-p\tau}}{p} \quad \text{and} \quad \mathscr{V}(p) = \frac{1}{p+1}$$

and solve the optimal control problems if $\tau = 1$ sec and $W = z^{-1}/(1 - z^{-1})$.

At first the discrete-time transfer sequence

$$G = \frac{b}{a} = \frac{0 \cdot 9197 z^{-2} (1 + 0 \cdot 7181 z^{-1})}{(1 - z^{-1})(1 - 0 \cdot 3679 z^{-1})}$$

and the disturbance input sequences

$$V = \frac{p}{q} = \frac{1}{1 - 0{\cdot}3679z^{-1}} \quad \text{and} \quad V_1 = \frac{0{\cdot}9197z^{-1}}{(1 - 0{\cdot}3679z^{-1})^2}$$

are determined. Then

$$W - V_1 = \frac{f}{h} = \frac{0{\cdot}0803z^{-1}(1 + 2{\cdot}2907z^{-1} + 1{\cdot}6854z^{-2})}{(1 - z^{-1})(1 - 0{\cdot}3679z^{-1})^2}.$$

Since $h_0 = h_0^+ = 1 - 0{\cdot}3679z^{-1}$ and $(b, q) = 1$ the stable TOC and the LSC problems are solvable. One finds $a_0 = 1$, $b_0^- = b^- = z^{-2}$, $b_0^+ = b^+ = 0{\cdot}9197 \,.\, (1 + 0{\cdot}7181z^{-1})$, $f^+ = 0{\cdot}0803$, $f^- = z^{-1}(1 + 2{\cdot}2907z^{-1} + 1{\cdot}6854z^{-2})$, $p = 1$, $(f, p)^- = 1, f_p^- = f^-, p_f^- = 1, (a_0, p_f)^- = 1, q_0 = q_0^+ = q, b_{01}^- = z^{-1}, (b_0, f_p)^- = = z^{-1}$, $f_{p1}^- = 1 + 2{\cdot}2907z^{-1} + 1{\cdot}6854z^{-2}$, $f^{-\sim} = 1{\cdot}6854 + 2{\cdot}2907z^{-1} + z^{-2}$, $f^* = f^+ f^{-\sim}$, $b_{01}^{-\sim} = 1$, $(a_0, p_f)^{-\sim} = 1$.

As $(a_0, p_f)^- = a_0^- = 1$ but $(b_0, f_p)^- = z^{-1} \nsim 1$, an improvement in the control process by ASDF should be expected.

a) To solve the stable TOC problem, equation (18) is

$$(1 - z^{-1})(1 - 0{\cdot}3679z^{-1})^2 s + z^{-1}r = 0{\cdot}0803$$

and its min deg s solution is $s = 0{\cdot}0803$ and $r = 0{\cdot}1394 - 0{\cdot}0699z^{-1} + 0{\cdot}0109z^{-2}$. According to (23) the optimal error is $e = 0{\cdot}0803z^{-1} + 0{\cdot}1839z^{-2} + 0{\cdot}1353z^{-3}$. Equations (16) and (17) read

$$x - z^{-1}v = 0{\cdot}0803$$

and

$$z^{-1}y + (1 - z^{-1})(1 - 0{\cdot}3679z^{-1})^2 v = 0{\cdot}1394 - 0{\cdot}0699z^{-1} + 0{\cdot}0109z^{-2}$$

and have the general solution

$$x = 0{\cdot}0803 + 0{\cdot}1394z^{-1} + z^{-2}t\,, \quad v = 0{\cdot}1394 + z^{-1}t$$

and

$$y = 0{\cdot}1720 - 0{\cdot}1105z^{-1} + 0{\cdot}0189z^{-2} - (1 - z^{-1})(1 - 0{\cdot}3679z^{-1})^2\, t.$$

Putting $t = 0$, the optimal controllers

$$R_1 = \frac{0{\cdot}1720 - 0{\cdot}1105z^{-1} + 0{\cdot}0189z^{-2}}{0{\cdot}9197(1 - 0{\cdot}3679z^{-1})(1 + 0{\cdot}7181z^{-1})(0{\cdot}0803 + 0{\cdot}1394z^{-1})}$$

and

$$R_2 = \frac{0{\cdot}0122(1 + 2{\cdot}2907z^{-1} + 1{\cdot}6854z^{-2})}{(1 + 0{\cdot}7181z^{-1})(0{\cdot}0803 + 0{\cdot}1394z^{-1})}$$

and the optimal control sequence

$$U = \frac{(1 + 2{\cdot}2907z^{-1} + 1{\cdot}6854z^{-2})(0{\cdot}1394 - 0{\cdot}0699z^{-1} + 0{\cdot}0109z^{-2})}{0{\cdot}9197(1 + 0{\cdot}7181z^{-1})(1 - 0{\cdot}3679z^{-1})}.$$

Solving the given problem in a simple control system results in:

$$e = 0{\cdot}0803z^{-1} + 0{\cdot}3233z^{-2} + 0{\cdot}4546z^{-3} + 0{\cdot}2349z^{-4}\,,$$

$$R = R_1\,, \quad U = \frac{z^{-1}(1 + 2{\cdot}2907z^{-1} + 1{\cdot}6854z^{-2})(0{\cdot}1720 - 0{\cdot}1105z^{-1} + 0{\cdot}0189z^{-2})}{0{\cdot}9197(1 + 0{\cdot}7181z^{-1})(1 - 0{\cdot}3679z^{-1})}\,.$$

b) For the LSC problem, equation (62) reads

$$(1 - z^{-1})(1 - 0{\cdot}3679z^{-1})^2\, s + z^{-1}r = 0{\cdot}0803(1{\cdot}6854 + 2{\cdot}2907z^{-1} + z^{-2})$$

and has the solution with $\deg s = 0$:

$$s = 0{\cdot}1353\,, \quad r = 0{\cdot}4189 - 0{\cdot}0376z^{-1} + 0{\cdot}0183z^{-2}\,.$$

The error sequence (63)

$$E = \frac{0{\cdot}1353z^{-1}(1 + 2{\cdot}2907z^{-1} + 1{\cdot}6854z^{-2})}{1{\cdot}6854 + 2{\cdot}2907z^{-1} + z^{-2}} =$$

$$= 0{\cdot}0803z^{-1} + 0{\cdot}0748z^{-2} - 0{\cdot}0140z^{-3} - 0{\cdot}0254z^{-4} + \ldots\,; \quad \sigma_E = 0{\cdot}0183\,.$$

Solving (60) and (61) for the general solution gives

$$x = 0{\cdot}1353 + 0{\cdot}4189z^{-1} + z^{-1}t\,, \quad v = 0{\cdot}4189 + z^{-1}t$$

and

$$y = 0{\cdot}6894 - 0{\cdot}3466z^{-1} + 0{\cdot}0567z^{-2} - (1 - z^{-1})(1 - 0{\cdot}3679z^{-1})^2\, t\,.$$

Choosing $t = 0$ the simplest pair of optimal controllers is according to (9) and (59)

$$R_1 = \frac{0{\cdot}6894 - 0{\cdot}3466z^{-1} + 0{\cdot}0567z^{-2}}{0{\cdot}9197(1 + 0{\cdot}7181z^{-1})(1 - 0{\cdot}3679z^{-1})(0{\cdot}1353 + 0{\cdot}4189z^{-1})}$$

and

$$R_2 = \frac{0{\cdot}0366(1 + 2{\cdot}2907z^{-1} + 1{\cdot}6854z^{-2})}{(1 + 0{\cdot}7181z^{-1})(0{\cdot}1353 + 0{\cdot}4189z^{-1})}$$

and the optimal control sequence (64) is

$$U = \frac{(1 + 2{\cdot}2907z^{-1} + 1{\cdot}6854z^{-2})(0{\cdot}4189 - 0{\cdot}0376z^{-1} + 0{\cdot}0183z^{-2})}{0{\cdot}9197(1 + 0{\cdot}7181z^{-1})(1 - 0{\cdot}3679z^{-1})(1{\cdot}6854 + 2{\cdot}2907z^{-1} + z^{-2})}\,.$$

The results of the LSC problem solved in the simple control system are as follows:

$$E = \frac{z^{-1}(0{\cdot}1353 + 0{\cdot}4189z^{-1})(1 + 2{\cdot}2907z^{-1} + 1{\cdot}6854z^{-2})}{1{\cdot}6854 + 2{\cdot}2907z^{-1} + z^{-2}} =$$

$$= 0{\cdot}0803z^{-1} + 0{\cdot}3233z^{-2} + 0{\cdot}2176z^{-3} - 0{\cdot}0686z^{-4} - \ldots;\,; \quad \sigma_E = 0{\cdot}1938\,;$$

$$R = R_1 \text{ and } U = \frac{z^{-1}(1 + 2{\cdot}2907z^{-1} + 1{\cdot}6854z^{-2})(0{\cdot}6894 - 0{\cdot}3466z^{-1} + 0{\cdot}0567z^{-2})}{0{\cdot}9197(1 + 0{\cdot}7181z^{-1})(1 - 0{\cdot}3679z^{-1})(1{\cdot}6854 + 2{\cdot}2097z^{-1} + z^{-2})}\,.$$

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Ing. Václav Soukup, CSc., katedra řídicí techniky elektrotechnické fakulty ČVUT (Department of Automatic Control, Faculty of Electrical Engineering — Czech Technical University), Karlovo nám. 13, 121 35 Praha 2. Czechoslovakia.