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GENERAL EXCHANGE ECONOMY

TRAN QUOC CHIEN

In the paper a general version of exchange economy is described. Further, some known results, namely the equality of the core of free disposals and that of group rational disposals (see [2], [3]) and the equality of the core and the set of equilibria (see [1], [4]), are generalized and proved.

0. INTRODUCTION

The model studied here is a generalization of usual games and markets. The fundamental notion is coalitions which form a Boolean σ-algebra. The states of an economy, the objects under consideration, form a set of vector measures, the set of values of which is a topological vector space. The concept of preference relations is generalized with regard to the so-called direct democracy law (see [2]). Abilities of coalitions, subsets of states, form an admissibility system.

In Section 3 the core of an economy is defined and the equality of the core of free disposals and that of group rational disposals is formulated and proved.

In Section 4 equilibria of an economy are defined in terms of the cores and a theorem on the core and the set of equilibria is proved.

1. DEFINITIONS

(M 1) $\mathscr{B}$ denotes a Boolean σ-algebra of *coalitions*.

In the paper individual players are understood as coalitions composed of only one element. Let E be the unit and e be zero of $\mathscr{B}$. If E is finite, individual players are perceptible, but in the case when $\mathscr{B}$ is strongly atomless they are insignificant. From this reason we shall deal just with coalitions.

(M 2) $\mathscr{A}$ denotes the space of *states.*

Suppose there is given a topological vector space X ordered by a cone X_+. $\mathscr{A}$ is then a linear subspace of positive measures defined on $\mathscr{B}$ with the values in X. For $m, n \in \mathscr{A}$, $b \in \mathscr{B}$ we define the measure $m|b|\, n$ by the formula

$$\forall c \in \mathscr{B}(m|b|\, n)(c) = m(b \wedge c) + n(c - b).$$

Assume that $m|b|\, n \in \mathscr{A}$ for every $m, n \in \mathscr{A}$, $b \in \mathscr{B}$.

(M 3) $(\prec_b)_{b \in \mathscr{B}}$, in short $(\prec_b)$, denotes a *preference relations system*.
Every $b \in \mathscr{B}$ is associated with a binary relation $\prec_b$ in $\mathscr{A}$ fulfilling
(PRS 1) $\prec_E$ is nonempty.
If $\prec_E$ is empty there is nothing to consider.
(PRS 2) For every $m, n, k, l \in \mathscr{A}$, $b \in \mathscr{B}$ $m \prec_b n$ iff $m|b|\, k \prec_b n|b|\, l$.
This means the preference of b does not depend on this part of states which does not concern b.
(PRS 3) For every $m, n \in \mathscr{A}$, $b \in \mathscr{B}$ if $m \prec_b n$ then there exists a $c \subset b$, $c \neq e$ such that for every $c' \subset c$ $m \prec_c n$ and $m|b - c|\, n = n$.
This is somewhat like a vote, players in c prefer n to m and for $b - c$ m, n are the same.
(PRS 4) For every disjoint $b_1, b_2 \in \mathscr{B}$ and $m, n \in \mathscr{A}$ fulfilling $m \prec_c n$ $\forall c \subset b_1$ and $m \prec_c n$ $\forall c \subset b_2$ we have $m \prec_{b_1 \cup b_2} n$.
The sense of (PRS 4) is clear, if all subcoalitions of b_1 and b_2 prefer n to m then $b_1 \cup b_2$ prefer n to m too.

(M 4) $(\mathscr{M}(b))_{b \in \mathscr{B}}$, in short $\mathscr{M}$, is an *admissibility system*. For every $b \in \mathscr{B}$ $\mathscr{M}(b) \subset \subset \mathscr{A}$.
(AS 1) For every $b \in \mathscr{B}$ $\mathscr{M}(b)$ is nonempty.
(AS 2) For every $m, n \in \mathscr{A}$, $b \in \mathscr{B}$ $m \in \mathscr{M}(b)$ iff $m|b|\, n \in \mathscr{M}(b)$.
(AS 3) For every disjoint $b, c \in \mathscr{B}$ $\mathscr{M}(b) \cap \mathscr{M}(c) \subset \mathscr{M}(b \vee c)$.

$\mathscr{M}(b)$, $b \in \mathscr{B}$, consists of all these states which are admissible with respect to the b's internal acts (i.e. the states which can be reached by b itself). The admissibility of an $m \in \mathscr{A}$ does not depend on what happens in m outside of b (AS 2). Condition (AS 3) says that if for two disjoint coalitions b and c a state m is admissible then is also admissible for the join $b \vee c$.

Definition 1. A four-tuple $\mathscr{E} = (\mathscr{B}, \mathscr{A}, (\prec_b), \mathscr{M})$ composed of elements described in (M 1)–(M 4) is called an *exchange economy*, or simply an *economy*.

An admissibility system $\mathscr{M}$ is called *additive* iff for every disjoint $b, c \in \mathscr{B}$ $\mathscr{M}(b) \cap \cap \mathscr{M}(c) = \mathscr{M}(b \vee c)$.

A preference system $(\prec_b)$ is called *uniform* iff for every $m, n \in \mathscr{A}$ the set $\{b / m \prec_b n\}$ is an ideal.

Definition 2. Let $(\prec_b)$ be a preference system. We define the *modified preference system* $(\precsim_b)$ *assiciated with* $(\prec_b)$ by the formula

$$\forall m, n \in \mathscr{A}\ \forall b \in \mathscr{B}\ (m \precsim_b n \Leftrightarrow \forall c \subset b\, m \prec_c n)$$

Observe that

(1) $(\precsim_b)$ is uniform.

The economy $(\mathscr{B}, \mathscr{A}, (\precsim_b), \mathscr{M})$ is then indicated by $\bar{\mathscr{E}}$ and called the *modified economy associated* with $\mathscr{E}$.

2. COMPOSITION OF ADMISSIBILITY SYSTEMS WITH RELATION

We shall describe here a rule which allows to transform given admissibility systems to some new systems (such operation may, for instance, allow the free disposal or free reallocation of goods etc.).

A reflexive binary relation R in $\mathscr{A}$ is said to be *transferable* iff

$$\forall m, n, k, l \in \mathscr{A}\,, \quad \forall b \in \mathscr{B} : (m|b|\, k)\, R(n|b|\, k) \Leftrightarrow (m|b|\, l)\, R(n|b|\, l)\,.$$

For a reflexive and transferable relation R and a coalition $b \in \mathscr{B}$ we define the relation R^b in $\mathscr{A}$ by the formula

$$\forall m, n \in \mathscr{A}(mR^b n \Leftrightarrow \exists k \in \mathscr{A} : (m|b|\, k)\, R(n|b|\, k))\,.$$

Examples:

(2) Relation $=: m = {}^b n$ iff $m(c) = n(c) \quad \forall c \subset b$.

(3) Relation $\approx: m \approx n$ iff $m(E) = n(E)$ then $m \approx {}^b n$ iff $m(b) = n(b)$.

(4) Relation $\leqq: m \leqq {}^b n$ iff $m(c) \leqq n(c) \quad \forall c \subset b$.

(5) Relation $\lhd: m \lhd n$ iff $m(E) \leqq n(E)$, then $m \lhd {}^b n$ iff $m(b) \leqq n(b)$.

A binary relation R is said to be *additive* iff for any disjoint $b, c \in \mathscr{B}$ and $m, n \in \mathscr{A}$, $mR^b n$ and $mR^b n$ implies $mR^{b \vee c} n$.

Let $\mathscr{M}$ be an admissibility system over $\mathscr{A}$ and let R be a reflexive, transferable and additive binary relation in $\mathscr{A}$. By ${}^R\mathscr{M}$ we shall denote the mapping from $\mathscr{B}$ into the subsets of $\mathscr{A}$ such that

$$\forall b \in \mathscr{B} m \in {}^R\mathscr{M}(b) \quad \text{iff there exists} \quad n \in \mathscr{M}(b) \quad \text{with} \quad mR^b n\,.$$

It is easily checked that

(6) ${}^R\mathscr{M}$ is an admissibility system.

In general, changing an admissibility system for ${}^R\mathscr{M}$ means that we allow some new opportunities. In particular, all relations from examples (2)–(5) are additive. Changing $\mathscr{M}$ for ${}^{\leqq}\mathscr{M}$, could be interpreted as to allow the free disposal, if we change it for ${}^{\approx}\mathcal{M}$ we allow the internal free reallocation, if changing $\mathcal{M}$ for ${}^{\lhd}\mathcal{M}$ we allow both.

If R is a reflexive, transferable and additive relation in $\mathcal{A}$, we call an admissibility system $\mathcal{M}$ over $\mathcal{A}$ *R-semiadditive* iff $\mathcal{M} = {}^{R}\mathcal{N}$ for some additive system $\mathcal{N}$.

Roughly speaking, an R-semiadditive system is not much different from the additive one.

Observe that

(7) If a strictly positive numerical measure can be defined on $\mathcal{B}$, then for any $\mathcal{M}$

$$\text{we have } {}^{\leqq}[{}^{\approx}\mathcal{M}] = {}^{\lhd}\mathcal{M}\,.$$

$$(8) \qquad {}^{\lhd}\mathcal{M} = {}^{\lhd}[{}^{\approx}\mathcal{M}] \quad \text{for every} \quad \mathcal{M}\,.$$

3. CORE OF AN ECONOMY

Given an economy $\mathcal{E} = (\mathcal{B}, \mathcal{A}, (\prec_b), \mathcal{M})$, we define the *core* of $\mathcal{E}$(denoted by $C(\mathcal{M})$) by the formula

$$C(\mathcal{M}) = \{m \in \mathcal{M}(E) / \forall n \in \mathcal{A}\ \forall b\ 2\ \mathcal{B}\ m \prec^b n \Rightarrow n \notin \mathcal{M}(b)\}\,.$$

The core consists of just admissible states which cannot be improved in the sense that there is no coalition b and a state that is better for b (in the sense $\prec_b$) and simultaneously admissible for b.

Observe that

(9) If $\mathcal{M} \subset \mathcal{M}_0$ (in the sense that $\mathcal{M}(b) \subset \mathcal{M}_0(b)$ for every $b \in \mathcal{B}$)

$$\text{then } C(\mathcal{M}) \supset \mathcal{M}(E) \cap C(\mathcal{M}_0)\,.$$

Further, let $\bar{C}(\mathcal{M})$ denote the core of the modified economy $\bar{\mathcal{E}}$ associated with $\mathcal{E}$.

Obviously

$$(10) \qquad C(\mathcal{M}) \subset \bar{C}(\mathcal{M})\,.$$

Let $\prec$ be a preference relation and R be a reflexive and transferable relation in $\mathcal{A}$. We say $\prec$ is *R-transitive* iff $\prec \circ R = \prec$.

Theorem 1. Let R be a reflexive and transferable relation in $\mathcal{A}$ and let $\prec_b$ be R^b-transitive for every $b \in \mathcal{B}$. Then

$$C(\mathcal{M}) = \mathcal{M}(E) \cap C({}^{R}\mathcal{M})\,.$$

Proof. Let $m \in C(\mathcal{M})$ then $m \in \mathcal{M}(E) \subset {}^{R}\mathcal{M}(E)$. If $m \notin C({}^{R}\mathcal{M})$ then there exists a $b \in \mathcal{B}$, $n \in {}^{R}\mathcal{M}(b)$ so that $m \prec_b n$. But there exists a $n_0 \in \mathcal{M}(b)$ with $nR^b n_0$, hence $m \prec_b n_0$ for R^b-transitivity of $\prec_b$ and that contradicts to $m \in C(\mathcal{M})$. So $m \in C({}^{R}\mathcal{M})$. The inverse inclusion implies from (9) for $\mathcal{M} \subset {}^{R}\mathcal{M}$. □

Lemma 1. If R is a reflexive and transferable relation in $\mathscr{A}$ and if $(\prec_b)$ is a uniform preference system with $\prec_E \circ R = \prec_E$, then for every $b \in \mathscr{B}$ $\prec_b \circ R^b = \prec_b$.

Proof. Let $b \in \mathscr{B}$, $m, n \in \mathscr{A}$ with $m(\prec_b \circ R^b)\, n$. Then there is a $k \in \mathscr{A}$ with $m \prec_b k$ and $kR^b n$, so there are $l, l' \in \mathscr{A}$ with $(m|b|\, l) \prec_E (k|b|\, l')$ and $(k|b|\, l')\, R(n|b|\, l')$.
Then $(m|b|\, l) \prec_E (n|b|\, l')$, hence $m \prec_b n$.
Conversely if $m \prec_b n$ we have $m(\prec_b \circ R^b)\, n$ for $nR^b n$. □

Using Lemma 1 and Theorem 1 we have

(11) If $(\prec_b)$ is uniform and $\prec_E \circ E = \prec_E$, where R is a reflexive and transferable relation in $\mathscr{A}$, then

$$C(\mathscr{M}) = \mathscr{M}(E) \cap C(^R\mathscr{M}) .$$

Now let $\bar{\mathscr{A}}$ be the set of all measures on $\mathscr{B}$ with the values in X. $\bar{\mathscr{A}}$ is, with the usual additive and multiplicative operations, a linear space. Assume a topology τ is given in $\bar{\mathscr{A}}$ so that $(\bar{\mathscr{A}}, \tau)$ is a topological linear space.

Let $\mathscr{A}_+ \subset \mathscr{A}(\subset \bar{\mathscr{A}})$ be a closed, convex cone $0 \in \mathscr{A}_+$, consisting a strictly positive measure q (q is strictly positive iff $q(b) > 0$ for every $e \neq b \in \mathscr{B}$).

Let R_+ be the binary relation in $\bar{\mathscr{A}}$ generated by the cone $\mathscr{A}_+$.

Observe that

(12) $$R_+ \text{ is transitive}$$

(13) $$\text{If} \quad \lim_{k \to \infty} n_k = n \quad \text{and} \quad mR_+ n_k \; \forall k = 1, 2, \dots \quad \text{then} \quad mR_+ n .$$

$\bar{\mathscr{A}}$ is then ordered by R_+ (or by $\mathscr{A}_+$). We denote by $\mathscr{A}_+^*$ the set of all positive linear functionals on $\mathscr{A}_+$ ($f \in \mathscr{A}_+^* \Rightarrow f(n) \geqq 0$ for every $n \in \mathscr{A}_+$).

Let $M \subset \mathscr{A}$ be a closed, convex set, put

$$^{\bar{R}_+}M = \{n \in \bar{\mathscr{A}} | \exists m \in M \;\; nR_+ m\}$$

and

$$^{R_+}M = {^{\bar{R}_+}M} \cap \mathscr{A} .$$

Obviously

(14) $$^{R_+}M \quad \text{is also a convex and closed set.}$$

Lemma 2. Let $n_0 \notin {^{\bar{R}_+}M}$, then every linear functional f on $\bar{\mathscr{A}}$ separating n_0 and $^{\bar{R}_+}M$ is positive ($f \in \mathscr{A}_+^*$).

Proof. We have $f(n_0) \geqq f(m)$ for all $m \in {^{\bar{R}_+}M}$. Assume $f \notin \mathscr{A}_+^*$, so there exists $m_0 \in \mathscr{A}_+$ with $f(m_0) < 0$ and hence $m - \alpha m_0 \in {^{\bar{R}_+}M}$ for every $m \in M$ and $\alpha \geqq 0$. As for any fixed $m_1 \in M$, there exists $\alpha_0 > 0$ with $f(m_1 - \alpha_0 m_0) > f(n_0)$ we get a contradiction since $f(n_0) > f(m)$ for every $m \in {^{\bar{R}_+}M}$. □

It is easy to verify the following assertions

(15) Let $n_0 \in \mathscr{A} \setminus {}^{R+}M$. Then there exists an $f \in \mathscr{A}_+^*$ separating

$$n_0 \quad \text{and} \quad {}^{R+}M .$$

(16) Let $M \subset \mathscr{A}_+$ be a convex compact and let $f \in \mathscr{A}_+^*$. Then

$$\max_{m\in M} f(m) = \max_{m\in {}^{R+}M} f_q) = \max_{m\in {}^{R+}M} f(m) .$$

Now for an admissibility system $\mathscr{M}$ we define

$$\mathscr{M}_{R_+}(E) = \{n \in {}^{R+}\mathscr{M}(E)/\exists f \in \mathscr{A}_+^* \; f(n) = \max_{m\in\mathscr{M}(E)} f(m)\}$$

$$\mathscr{M}_{R_+}(b) = \mathscr{M}_{R_+}(E) \cap \mathscr{M}(b) \quad \forall b \in \mathscr{B} .$$

Obviously

(17) $\mathscr{M}_{R_+}$ is an admissibility system .

Theorem 2. Let the above defined relation R_+ be transferable and let $\mathscr{E} = (\mathscr{B}, \mathscr{A}, (\prec_b), \mathscr{M})$ be such an economy that $\prec_b$ is R_+^b-transitive for every $b \in \mathscr{B}$. Assume that $\mathscr{M}(E)$ is convex, compact and $\mathscr{M}(b)$ is closed for every $b \in \mathscr{B}$. Then

$$C(\mathscr{M}_{R_+}) = \mathscr{M}_{R_+}(E) \cap C({}^{R+}\mathscr{M})$$

Proof. If $m_0 \in C(\mathscr{M}_{R_+})$ then $m_0 \in \mathscr{M}_{R_+}(E)$. If $m_0 \notin C({}^{R+}\mathscr{M})$ there exists a $b \in \mathscr{B}$ and $n_0 \in {}^{R+}\mathscr{M}(E) \cap {}^{R+}\mathscr{M}(b)$ with $m_0 \prec_b n_0$. Put $M = \{n \in {}^{R+}\mathscr{M}(E) \cap {}^{R+}\mathscr{M}(b)/n_0 R_+^b n\}$. M is obviously nonempty, compact $(n_0 \in M)$ and ordered by R_+. Let n_1 be the maximal element of M with respect to R_+ (the existence of which will be proved later), then $n_1 \in \mathscr{M}_{R_+}(E)$ for n_1 is a boundary point of ${}^{R+}\mathscr{M}(E)$ $(n_1 = \lim_{\alpha\to 0+} (n_1 + \alpha q)$, $q \in \mathscr{A}_+$ strictly positive, and $n_1 + \alpha q \notin {}^{R+}\mathscr{M}(E)$ $\forall \alpha > 0$, otherwise there could exist $(n_1|b| (n_1 + \alpha q)) \in M$ with $n_1 R_+ (n_1|b| (n_1 + \alpha q))$, a contradiction with maximality of n_1). We obtain then $m_0 \prec_b n_1$ and it contradicts to $m_0 \in C(\mathscr{M}_{R_+})$. Thus $m \in C({}^{R+}\mathscr{M})$.

The inverse inclusion implies from (9) for $\mathscr{M}_{R_+} \subset {}^{R+}\mathscr{M}$.

Now we show that M has a maximal element (in the sense of R_+). It suffices, using Zorn's Lemma, to verify that every linearly ordered subset $P \subset M$ has a supremum. For this reason we denote $P_n = \{m \in P/nR_+ m\}$, $n \in P$, then $\bar{P}_n$(the closure of P_n) is a closed, nonempty subset of $\bar{P}$ ($\bar{P}$ is compact). Every finite system of $\{\bar{P}_n\}_{n\in P}$ has a nonempty intersection for R_+ is complete on P and it follows that $\bigcap_{n\in P} \bar{P}_n \neq \emptyset$. Every element of $\bigcap_{n\in P} \bar{P}_n$ is then a supremum of P. The proof is complete. □

A preference relation $\prec$ is said to be *monotonous* iff $m < n$ (in the sense $m(b) < n(b)$ for every $b \in \mathscr{B}$) implies $m \prec n$.

Theorem 2 implies the following theorem.

Theorem 3. Let all assumptions of Theorem 2 remain unchanged and let $\prec_E$ be monotonous. Then $C(\mathscr{M}_{R_+}) = C({}^{R_+}\mathscr{M})$ and $C({}^{R_+}\mathscr{M}) \subset \mathscr{M}_{R_+}(E)$.

We see that $\mathscr{M}_{R_+}(E)$ is somewhat like group rational set in the usual sense. The assertion of Theorem 3 says, roughly speaking, that the core of "free disposals" is equal to that of group rational disposals.

4. EQUILIBRIA OF ECONOMY

Given a topological vector space X ordered by a cone X_+ then by a *value operator* (or a *price system*) we understand any positive continuous linear functional $p \in X_+^*$.

An admissibility system $\mathscr{M}$ over a space $\mathscr{A}$ is called X_+*-bounded* iff for every $p \in X_+^*$, $b \in \mathscr{B}$ the set

$$\{pm(b)/m \in \mathscr{M}(b)\}$$

is bounded.

It is easily checked that

(18) If $\mathscr{M}$ is X_+-bounded, R is reflexive, transferable and nonexceeding (it means mRn implies $m(E) \leqq n(E)$) relation in $\mathscr{A}$, then ${}^R\mathscr{M}$ is also X_+-bounded.

For an X_+-bounded admissibility system $\mathscr{M}$ and a value operator $p \in X_+^*$ we define the system $\mathscr{M}_p$ by the formula

$$\forall b \in \mathscr{B}\ \mathscr{M}_p(b) = \{m \in \mathscr{A}/\exists n \in \mathscr{M}(b) : pm(b) \leqq pn(b)\}\ .$$

$\mathscr{M}_p(b)$ can be interpreted as the "*budget set*" of the coalition b associated with the value operator p.

Observe that

(19) $$\mathscr{M} \subset \mathscr{M}_p \quad \text{for every } p \in X_+^*\ .$$

(20) For any X_+-bounded admissibility system $\mathscr{M}$ and any reflexive, transferable and nonexceeding relation R $\mathscr{M}_p = [{}^R\mathscr{M}]_p$.

Now let an economy $\mathscr{E} = (\mathscr{B}, \mathscr{A}, (\prec_b), \mathscr{M})$ be given, then every element of the set

$$W(\mathscr{M}) = \mathscr{M}(E) \cap \bigcup_{p \in X_+^*} C(\mathscr{M}_p)$$

will be called an *equilibrium* of the economy $\mathscr{E}$.

Further the set of equilibria of the modified economy $\overline{\mathscr{E}}$ will be indicated by $\overline{W}(\mathscr{M})$.

Observe that

(21) $$W(\mathscr{M}) \subset C(\mathscr{M})\ ,$$

(22) $$W(\mathscr{M}) \subset \overline{W}(\mathscr{M})\ .$$

Supposing a preference system $(\prec_b)$ is given, an element $x \in X_+$ is said to be *desirable* iff for any $m \in \mathscr{A}$ there exists an $n \in \mathscr{A}$ with $n(E) = x$ and for every $b \in \mathscr{B}$ $m \prec_b m + n$.

Let X^0 denote the set of all desirable elements.

Theorem 4. Let $\mathscr{M}$ be $\lhd$-semiadditive and $0 \in X$ be a convergent point of X^0 (it means there exists a sequence of X^0 converging to 0). Then $\overline{W}(\mathscr{M}) = W(\mathscr{M})$.

Proof. It remains to prove $\overline{W}(\mathscr{M}) \subset W(\mathscr{M})$ (see (22)).

Let $m_0 \in \overline{W}(\mathscr{M})$ then there exists a $p \in X_+^*$ with $m_0 \in \mathscr{M}(E) \cap \overline{C}(\mathscr{M}_p)$. Choose some $e \neq b_0 \in \mathscr{B}$, $n_0 \in \mathscr{A}$ with $m_0 \prec_{b_0} n_0$. Let $v \in \mathscr{M}(b_0)$, then there exists some $u \in \mathscr{A}$ with $v(b) \leqq u(b)$ and $u \in \mathscr{M}(b)$ for any $b \in \mathscr{B}$ for the $\lhd$-semiadditivity of $\mathscr{M}$.

As it implies from (PRS 3), there exists some $e \neq c_0 \subset b_0$ with $m_0 \precsim_{c_0} n_0$ and $m_0 |b_0 - c_0|\, n_0 = n_0$. We have

$$pv(c_0) \leqq pu(c_0) < pn_0(c_0) \tag{*}$$

If $0 \leqq pn_0(b_0 - c_0) < pu(b_0 - c_0)$ then $b_0 - c_0 \neq e$. Choose some $x \in X^0$ with $px \leqq pu(b_0 - c_0) - pn_0(b_0 - c_0)$ and let $n_1 \in \mathscr{A}$ with $n_1(E) = x$ and for every $b \in \mathscr{B}$ $m_0 \prec_b m_0 + n_1$. Then $m_0 + n_1 \in \mathscr{M}_p(b_0 - c_0)$ and $m_0 \precsim_{b_0 - c_0} m_0 + n_1$, that contradicts to $m_0 \in \overline{W}(\mathscr{M})$.

Hence

$$pv(b_0 - c_0) \leqq pu(b_0 - c_0) \leqq pn_0(b_0 - c_0) \tag{**}$$

From (*) and (**) it follows $pv(b_0) < pn_0(b_0)$, i.e. $m_0 \in \mathscr{M}(E) \cap C(\mathscr{M}_p)$, and thus $m_0 \in W(\mathscr{M})$. □

Now for further development we recall some elements of measure theory.

An element $b \in \mathscr{B}$ is said to be an *atom* of a numerical measure m iff for every $c \leqq b$ either $m(c) = 0$ or $m(b - c) = 0$. A measure is atomless iff it has no atom.

A measure $m : \mathscr{B} \to X$ (X is a linear space) is *totolly atomless* iff $pm(pm(b) = p(m(b)))$ is atomless for every $p \in X^*$.

A Boolean σ-algebra $\mathscr{B}$ is said to be *strongly atomless* iff any measure $m : \mathscr{B} \to X$ is totally atomless.

Let a preference system $(\prec_b)$ be given and let $b \in \mathscr{B}$, $m, n \in \mathscr{A}$ with $m \prec_b n$. Then $\prec_b$ is called

right-continuous in (m, n) iff there exists an $\varepsilon > 0$ with $m \prec_b (1 - \delta)\, n$ for all $0 \leqq \delta \leqq \varepsilon$.

almost right-continuous in (m, n) iff there is an increasing sequence (b_i), $b_i \subset b$ with $\cup\, b_i = b$ and $\prec_{b_i}$ is right-continuous in (m, n) for all i.

partial right-continuous in (m, n) iff there is such a $e \neq c \subset b$ that $\prec_c$ is right-continuous in (m, n).

The system $(\prec_b)$ is (partial, resp. almost) right-continuous iff for every $b \in \mathscr{B}$, $m, n \in \mathscr{A}$ with $m \prec_b n$ $\prec_b$ is (partial, resp. almost) right-continuous in (m, n).

Theorem 5. If $(\prec_b)$ is partial right-continuous and if a numerical atomless measure μ can be determined on $\mathscr{B}$, then $(\precsim_b)$ is alomost right-continuous.

Proof. Let $e \neq b \in \mathscr{B}$, $m, n \in \mathscr{A}$ $m \precsim_b n$. Put

$$L = \{b' \in \mathscr{B} / b' \subset b, \precsim_{b'} \text{ is almost right-continuous in } (m, n)\} .$$

$L$is then ordered in the sense of operation "inclusion". Let $P \subset L$be a chain. Consider the set

$$A = \{\mu(b') / b' \in P\} .$$

We can choose a countable set $A_0 \subset A$, which is dense in A. As μ has no atom, we have $b_0 = \bigcup_{\mu(b') \in A_0} b'$, $b' \subset b_0$ for every $b' \in P$.

Obviously $b_0 \in L$, hence b_0 is a supremum of P. As it follows from Zorn's Lemma L has a maximum and it is easy to verify that the maximum is b. So $\precsim_b$ is almost right-continuous in (m, n). The proof is complete. □

An admissibility system $\mathscr{M}$ is called uniform iff for every sequence of disjoint elements $\{b_i\}$ and for every sequence $\{m_i\}$ of elements of $\mathscr{A}$ such that $m_i \in \mathscr{M}(b_i)$ for every i, the series $\sum m_i(b_i)$ converges.

Theorem 6. Let $\mathscr{E} = (\mathscr{B}, \mathscr{A}, (\prec_b), \prec)$ be such an economy that $\mathscr{B}$ is strongly atomless, X is a locally convex Hausdorff space, $(\prec_b)$ is partial right-continuous system and $\mathscr{M}$ is X_+-bounded, uniform and $\lhd$-semiadditive. If moreover $\forall m \in \mathscr{M}(E)\, \exists n \in \mathscr{M}(E)\; m(E) \leqq n(E)$, $n(E)$ lies in the interior of X_+ and there exists a set $X^0 \subset X$ composed of desirable elements for which 0 is a convergent point and

$$\forall e \neq b \in \mathscr{B}\; \forall m \in \mathscr{M}(b)\; \exists n \in \mathscr{M}(b)\; m(b) \leqq n(b) \in X^0 .$$

Then $C(\mathscr{M}) = W(\mathscr{M})$.

Proof. From Theorem 4, (21) and (10) it follows

$$\overline{W}(\mathscr{M}) = W(\mathscr{M}) \subset C(\mathscr{M}) \subset \overline{C}(\mathscr{M}) .$$

As $\mathscr{B}$ is strongly atomless, $\mathscr{B}$ has a numerical atomless measure, thus by Theorem 5 the system $(\precsim_b)$ is almost right-continuous. We have then $\overline{W}(\mathscr{M}) = \overline{C}(\mathscr{M})$ (see Theorem 5 of [1]) and it implies $W(\mathscr{M}) = C(\mathscr{M})$. □

Theorem 6 is a generalization of the well-known Theorem on the core and the set of equilibria of an economy with a continuous of players. All assumptions in this Theorem, except the assumption that 0 is a convergent point of X, are discussed in [1] (page 25). Above mentioned assumption is restrictive from the theoretical points of view, however, it is quite acceptable in practice, because every coalition surely prefers those allocations that ensure to it more foods than others, even though the arised gains can be very small $(m, n \in \mathscr{A},\; m(b) < 0\; \forall b \neq e \Rightarrow n \prec_b n + m\; \forall b \in \mathscr{B})$.

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RNDr. Tran Quoc Chien, matematicko-fyzikální fakulta UK (Faculty of Mathematics and Physics — Charles University), Malostranské nám. 25, 118 00 Praha 1. Czechoslovakia. Permanent address: *Department of Mathematics — Polytechnical School of Da-nang. Vietnam.*