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ON DIFFERENCE EQUATIONS AND DISCRETE SYSTEMS

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Discrete systems described by certain non-linear difference equations are studied in this paper. Such systems are represented by discrete Volterra-series. The paper recalls such notions as compositional, convolutional, time-invariant and stable systems and the paper defines the property of V-analyticity which depends on Volterra-series.

This paper deals with difference equations and discrete systems described by them. It is shown that under certain conditions it is possible to express such discrete systems by means of the so called discrete Volterra-series and denote them as V-analytical systems. This form of Volterra-series allows to represent the investigated system by a sequence of systems connected in parallel. Each one of the systems in the sequence has special properties which make it easy to analyse. The first system is a linear, the second one is of a quadratic nature, the third one is of a cubic nature etc. This representation of the non-linear system as a sequence of systems is suitable for the interpretation of the investigated system.

The considered equations have the form

$$L_n y + \varepsilon\varphi \circ y = x .$$

The first term on the left-hand side of the equation is a linear difference operator with constant coefficients and the second one is a non-linear operator determined by a power series

$$\sum_{j=2}^{\infty} a_j z^j .$$

This paper is connected with the general work of B. Pondělíček [1] and, where possible, the same notation is used.

1. GENERAL REMARKS; DISCRETE VOLTERRA-SERIES

Let $\mathscr{R}$ denote the set of all real numbers, $\mathscr{C}$ denote the set of all integers, $\mathscr{N}$ is the set of all natural numbers and let, for $i \in \mathscr{N}$, $\mathscr{C}^i$ is the cartesian power of $\mathscr{C}$. By $\mathscr{B}_i$ we denote the set of all bounded sequences of real numbers, defined on $\mathscr{C}^i$, for which the following condition holds:

$$h \in \mathscr{B}_i \Leftrightarrow \left(h(k_1, k_2, \ldots, k_i) \in \mathscr{R},\ k_l \in \{k_1, k_2, \ldots, k_i\},\right.$$
$$\left. k_l \leqq 0 \Rightarrow h(k_1, k_2, \ldots, k_i) = 0\right).$$

For all $x \in \mathscr{R}$ we define the following function $\boldsymbol{F} : \mathscr{R} \to \mathscr{N}$:

$$\boldsymbol{F}(x) = \begin{cases} 0 & \text{for} \quad x < 0 \\ n & \text{for} \quad x \in \langle n - 1, n) . \end{cases}$$

This function is non-decreasing, right-continuous and piecewise constant.

According to the definition of the function $\boldsymbol{F}$ we express summations using the integrals of Stieltjes. Let $f : \mathscr{R} \to \mathscr{R}$ be uniquely defined for $x = n \in \mathscr{N}$ and such that

$$\sum_{a \leqq n < b} |f(n)| < \infty ,$$

then

$$\int_a^b f(x)\, \mathrm{d}\boldsymbol{F}(x) = \sum_{a \leqq n < b} f(n) , \quad a, b \in \mathscr{R} . \tag{1}$$

The convolution $h * x$ of sequences $h \in \mathscr{B}_1$, $x \in \mathscr{B}_1$ can then be written in the form: for $k \in \mathscr{C}$

$$[h * x](k) = \sum_{\tau=0}^{k} h(k - \tau)\, x(\tau) = \sum_{\tau=0}^{k-1} h(k - \tau)\, x(\tau) =$$
$$= \int_0^k h(k - \tau)\, x(\tau)\, \mathrm{d}\boldsymbol{F}(\tau) = \int_0^\infty h(k - \tau)\, x(\tau)\, \mathrm{d}\boldsymbol{F}(\tau) ,$$

because the following implication holds:

$$\tau \geqq k \Rightarrow h(k - \tau) = 0 .$$

When $x \in \mathscr{B}_1$ we denote for $i \in \mathscr{N}$ $x^{*i}(k_1, k_2, \ldots, k_i) = \boldsymbol{x}(k_1)\, \boldsymbol{x}(k_2) \ldots x(k_i)$, then $x^{*i} \in \mathscr{B}_i$. If for each $i = 1, 2, \ldots$ the sequences $h_i \in \mathscr{B}_i$ and $x \in \mathscr{B}_1$, we define for $k \in \mathscr{C}$ the generalized convolution $h_i * x^{*i} \in \mathscr{B}_1$

$$[h_i * x^{*i}](k) = \tag{2}$$
$$= \sum_{\tau_1=0}^{k-1} \sum_{\tau_2=0}^{k-1} \ldots \sum_{\tau_i=0}^{k-1} h_i(k - \tau_1, k - \tau_2, \ldots, k - \tau_i)\, x(\tau_1)\, x(\tau_2) \ldots x(\tau_i) =$$
$$= \int_0^\infty \int_0^\infty \ldots \int_0^\infty h_i(k - \tau_1, k - \tau_2, \ldots, k - \tau_i) \prod_{k=1}^{i} x(\tau_k)\, \mathrm{d}\boldsymbol{F}(\tau_k) .$$

Every sequence $h_i \in \mathscr{B}_i$ determines a mapping $\boldsymbol{H}_i : \mathscr{B}_1 \to \mathscr{B}_1$, $\boldsymbol{H}_i x = h_i * x^{*i}$.

Definition 1. Let $h_i \in \mathscr{B}_i$, $k_j \in \mathscr{B}_j$, $x \in \mathscr{B}_1$. For mappings $\boldsymbol{H}_i$, $\boldsymbol{K}_j$ we define the following operations:

1) for $\alpha \in \mathscr{R}$ $\alpha \boldsymbol{H}_i x = \alpha h_i * x^{*i}$,
2) for $i = j$, $\boldsymbol{H}_i x + \boldsymbol{K}_i x = (\boldsymbol{H}_i + \boldsymbol{K}_i)\, x = (h_i + k_i) * x^{*i}$,
3) for $i, j \in \mathscr{N}$,

$$(\boldsymbol{H}_i x) \cdot (\boldsymbol{K}_j x) = (\boldsymbol{H}_i \cdot K_j)\, x = (h_i \cdot k_j) * x^{*i+j}.$$

Note. The operations 2 and 3 are evidently commutative.

Definition 2. Let for all members of the family $h_1, h_2, \ldots, h_i, \ldots$ hold $h_i \in \mathscr{B}_i$, let for $x \in \mathscr{B}_1$ every h_i determine the mapping $\boldsymbol{H}_i : \mathscr{B}_1 \to \mathscr{B}_1$ by the expression $\boldsymbol{H}_i x = h_i * x^{*i}$. The series $\sum_{i=0}^{\infty} \boldsymbol{H}_i x$, which is determined by the family $h_1, h_2, \ldots, h_i, \ldots$ is called *a discrete Volterra-series*.

2. LINEAR EQUATION

For $w \in \mathscr{N}$ we denote $\mathscr{V}_w \subset \mathscr{B}_1$, the linear subspace of all sequences $a \in \mathscr{B}_1$ for which $a(u) = 0$ if $u \leqq w$. Let $\boldsymbol{L}_n$ be a linear difference operator $\boldsymbol{L}_n : \mathscr{B}_1 \to \mathscr{B}_1$, for $y \in \mathscr{B}_1$ defined as follows:

$$\boldsymbol{L}_n y \equiv y(k + n) + b_{n-1}\, y(k + n - 1) + \ldots + b_0\, y(k)\,, \quad b_{n-1}, b_{n-4}, \ldots, b_0 \in \mathscr{R}\,. \tag{3}$$

If $x \in \mathscr{B}_1$ then the relation $\boldsymbol{L}_n y = x$ is a linear difference equation of order n.

Theorem 1. Let $h \in \mathscr{B}_1$, $h(1) = h(2) = \ldots = h(n - 1) = 0$, $h(n) = 1$ satisfy the equation $\boldsymbol{L}_n y = 0$. Then the sequence $u \in \mathscr{V}_n$ determined by the rule $u = h * x$ solves the initial-value problem $\boldsymbol{L}_n y = x$, $y \in \mathscr{V}_n$.

Proof. 1) $u = h * x \Rightarrow u \in \mathscr{V}_n$.

2) Substituting for u in the equation we get

$$u = h * x \Rightarrow \boldsymbol{L}_n u = x\,.$$

3) Since such an initial-value problem has exactly one solution, the theorem is proved. □

Corollary 1. The discrete system $\Phi : \mathscr{B}_1 \to \mathscr{V}_n$ described by the difference equation $\boldsymbol{L}_n y = x$, $y \in \mathscr{V}_n$, $x \in \mathscr{B}_1$ is linear, convolutional, causal and time-invariant. Furthermore if $\sum_{k=1}^{\infty} |h(k)| < \infty$, Φ is stable.

Proof. The first part of the assertion follows from Theorem 2.4 of [1].

Because $\sup_k \sum_\tau |h(k - \tau)| = \sum_{k=1}^{\infty} |h(k)| < \infty$, Φ is stable according to Theorem 3.1 of [1]. □

Corolarry 2. The discrete system from Corollary 1 is expressed in the form of the discrete Volterra-series determined by the family of sequences $h_1 = h$, $h_i = 0$ if $i \geqq 2$.

3. NON-LINEAR EQUATION

Let $a_j \in \mathscr{R}$ for $j = 2, 3, \ldots$ and let a real function of a real variable

$$\varphi(z) = \sum_{j=2}^{\infty} a_j z^j \tag{4}$$

be given as a power-series with the radius of convergence $\varrho > 0$. The function

$$\psi(z) = \frac{\mathrm{d}}{\mathrm{d}z}\,\varphi(z) = \sum_{j=2}^{\infty} j a_j z^j$$

as a power-series has the same radius of convergence.

For $a \in \mathscr{B}_1$ we shall use the norm $\|a\| = \sup_{k \in \mathcal{N}} |a(k)|$. The symbol $f \circ g$ means the composition of functions f and g.

In the next section we shall consider a non-linear difference equation

$$L_n y + \varepsilon \varphi \circ y = x\,, \tag{5}$$

$x \in \mathscr{B}_1$, $y \in \mathscr{V}_n$, ε is a real constant.
Let h be the sequence from Theorem 1. On applying the convolution to both sides of the equation (5) we obtain

$$h * (L_n y + \varepsilon \varphi \circ y) = h * x\,.$$

From the linearity of a convolution and from the properties of h, we have

$$y = h * x - \varepsilon h * \varphi \circ y\,. \tag{6}$$

It can be easily shown that a sequence $y \in \mathscr{V}_n$ satisfies (5) if and only if it satisfies (6).

4. CONTRACTIVITY OF THE MAPPING; CONDITIONS FOR THE UNIQUE SOLUTION

For a fixed $x \in \mathscr{B}_1$ we denote A the mapping from $\mathscr{V}_n$ to $\mathscr{V}_n$

$$Ay = h * x - \varepsilon h * \varphi \circ y\,. \tag{7}$$

The initial-value problem $y = Ay$, $y \in \mathscr{V}_n$ will be solved by using the Banach principle.

For a real R, $0 \leqq R < \varrho$ we define the function

$$\lambda(R) = \sup_{\|y\| \leqq R} \|\psi \circ y\|\,.$$

Lemma 1. The function $\lambda\colon \langle 0, \varrho) \to R$ is non-decreasing, continuous and $\lambda(0) = 0$.

Proof. It is easy to show that $\lambda(0) = 0$ and λ is non-decreasing. Furthermore let $\mathscr{A}$ and $\mathscr{B}$ be the sets defined as follows:

$$\mathscr{A} = \{x \in \mathscr{R} \mid x = \|\psi(y(k))\|,\ k \in \mathscr{N},\ \|y\| \leqq R\}\,,$$
$$\mathscr{B} = \{x \in \mathscr{R} \mid x = |\psi(z)|,\ |z| \leqq R\}\,.$$

Evidently $\mathscr{A} = \mathscr{B}$. Since $\mathscr{B}$ is a bounded and closed interval, $\sup \mathscr{B}$ exists and

$$\sup \mathscr{B} = \sup_{|z| \leqq R} |\psi(z)| = \sup_{\|y\| \leqq R} \|\psi \circ y\| = \lambda(R)\,.$$

If $|z| \leqq R < \varrho$, ψ is a continuous function on $\langle -R, R\rangle$. The continuity of λ is then a consequence of the uniform continuity of ψ. □

Lemma 2. Let y_1, y_2 be any pair of sequences from $\mathscr{B}_1$, such that $\|y_1\| \leqq R$, $\|y_2\| \leqq R$. Then

$$\|\varphi \circ y_1 - \varphi \circ y_2\| \leqq \lambda(R)\, \|y_1 - y_2\|\,. \tag{8}$$

Proof. It holds for the function φ : if $z \in (-\varrho, \varrho)$ then φ is continuous and differentiable. Let $z_1, z_2 \in \langle -R, R\rangle \subset (-\varrho, \varrho)$; it follows from the Mean Value Theorem that

$$|\varphi(z_1) - \varphi(z_2)| \leqq \sup_{|z| \leqq R} \left|\frac{\mathrm{d}}{\mathrm{d}z}\, \varphi(z)\right| |z_1 - z_2|\,.$$

Consider the pair y_1, y_2 from Lemma 2. For $k \in \mathscr{N}$, we denote $y_1(k) = z_{1,k}$, $y_2(k) = z_{2,k}$. Clearly $|z_{1,k}|, |z_{2,k}| \leqq R$. Therefore

$$|\varphi(z_{1,k}) - \varphi(z_{2,k})| \leqq \sup_{|z| \leqq R} |\psi(z)| \,.\, |z_{1,k} - z_{2,k}|$$

for every $k \in \mathscr{N}$.

This can be rewritten as

$$|\varphi(y_1(k)) - \varphi(y_2(k))| \leqq \sup_{|z| \leqq R} |\psi(z)| \,.\, |y_1(k) - y_2(k)| \leqq$$
$$\leqq \sup_{|z| \leqq R} |\psi(z)| \,.\, \sup_{|z| \leqq R} |y_1(k) - y_4(k)| = \lambda(R)\, \|y_1 - y_2\|$$

according to the definition of λ.

Hence

$$\|\varphi \circ y_1 - \varphi \circ y_2\| \leqq \lambda(R)\, \|y_1 - y_2\|\,. \tag{9}$$ □

We denote $\sum_{k=0}^{\infty} |h(k)| = H$; suppose $H < \infty$. Then for $\|h * x\|$, $h \in \mathscr{B}_1$, $x \in \mathscr{B}_1$ we have

$$\|h * x\| = \Big\| \sum_{\tau=0}^{k} h(k - \tau)\, x(\tau) \Big\| \leqq \Big\| \sum_{\tau=0}^{k} |h(k - \tau)|\, |x(\tau)| \Big\| \leqq$$
$$\leqq \Big\| \sum_{\tau=0}^{\infty} |h(k - \tau)| \sup_{\tau \in \mathscr{N}} |x(\tau)| \Big\| = \|x\| \sup_{k \in \mathscr{N}} \sum_{\tau=0}^{\infty} |h(k - \tau)| = \tag{10}$$
$$= \|x\| \sum_{k=0}^{\infty} |h(k)| = \|x\|\, H\,.$$

For any pair $y_1, y_2 \in \mathscr{B}_1$ such that $\|y_1\| \leqq R$, $\|y_2\| \leqq R$ we obtain

$$\begin{aligned} \|Ay_1 - Ay_2\| &= \|h * x - \varepsilon h * \varphi \circ y_1 - h * x + \varepsilon h * \varphi \circ y_2\| = \\ &= |\varepsilon| \, \|h * (\varphi \circ y_2 - \varphi \circ y_1)\| \leqq |\varepsilon| \, H \|\varphi \circ y_2 - \varphi \circ y_1\| \leqq \\ &\leqq |\varepsilon| \, H \, \lambda(R) \, \|y_2 - y_1\| \, . \end{aligned} \tag{11}$$

by using (8), (9) and (10).

Consequently the following theorem holds:

Theorem 2. Let $R_0 > 0$ be a real constant such that

$$\lambda(R_0) < \frac{1}{|\varepsilon| \, H} \, .$$

Then the mapping $A : \mathscr{V}_n \to \mathscr{V}_n$ (7) is a contraction on the sphere

$$\mathscr{U}_0 = \{y \in \mathscr{V}_n | \, \|y\| < R_0\}$$

with the constant of contractivity $K = |\varepsilon| \, \lambda(R_0) \, H$.

Proof. The proof follows from the formula (11). □

We shall use the Banach principle in this form:

Theorem 3. (Banach) Let $\mathscr{P}$ be a Banach space, let a mapping $A : \mathscr{P} \to \mathscr{P}$ be a contraction with a constant $K \in (0, 1)$ on a set $\mathscr{U} \subset \mathscr{P}$, with $\mathscr{U}$ non-empty. Let the closed sphere

$$\mathscr{S}_0 = \left\{ y \in \mathscr{P} | \, \|y - y_1\| \leqq \frac{K}{1 - K} \|y_1 - y_0\| \right\}, \tag{12}$$

determined by points $y_0 \in \mathscr{P}$ and $y_1 = Ay_0$ satisfy $\mathscr{S}_0 \subset \mathscr{U}$. Then the mapping A has in $\mathscr{S}_0$ exactly one fixed point $\bar{y}$ such that $\bar{y} = A\bar{y}$. The point $\bar{y}$ is the limit point of the sequence of iterations $y_{n+1} = Ay_n$, $n = 0, 1, 2, \ldots$.

Theorem 4. Let the mapping A defined by the relation (7) $Ay = h * x - \varepsilon h * \varphi \circ y$ be a contraction with a constant $K \in (0, 1)$ on a nonempty sphere

$$\mathscr{U}_0 = \{y \in \mathscr{V}_n \, | \, \|y\| \leqq R\} \, .$$

Then for every $x \in \mathscr{B}_1$ satisfying the condition

$$\|x\| \leqq \frac{1 - K}{H} R \tag{13}$$

the equation $y = Ay$ has exactly one solution $\bar{y} \in \mathscr{U}_0$.

Proof. If $y_0 = (0, 0, \ldots, 0, \ldots) \in \mathscr{B}_1$, then $y_1 = Ay_0 = h * x$. According to (10) we have $\|y_1\| < \|x\| H$. Substituting (13) into this relation we obtain

$$\|y_1\| < (1 - K) R , \tag{14}$$

hence $y_1 \in \mathscr{U}_0$. As in Theorem 3 let

$$\mathscr{S}_0 = \left\{ y \in \mathscr{V}_n \mid \|y - y_1\| \leqq \frac{K}{1 - K} \|y_1 - y_0\| \right\}.$$

Consider any $v \in \mathscr{S}_0$; then

$$\|v - y_1\| \leqq \frac{K}{1 - K} \|y_1 - y_0\| .$$

Therefore

$$\|v\| = \|v - y_1 + y_1\| \leqq \|v - y_1\| + \|y_1\| \leqq$$

$$\leqq \frac{K}{1 - K} \|y_1 - y_0\| + \|y_1\| .$$

As $y_0 = 0 \in \mathscr{B}_1$ we have

$$\|v\| \leqq \left(\frac{K}{1 - K} + 1\right) \|y_1\| \quad \text{and, using (14)},$$

$$\|v\| < \left(\frac{K}{1 - K} + 1\right)(1 - K) R = R ,$$

hence $v \in \mathscr{U}_0$.

Now it follows that $\mathscr{S}_0 \subset \mathscr{U}_0$, the mapping A is a contraction on $\mathscr{S}_0$ and, in accordance with Theorem 3, there exists exactly one $\bar{y} \in \mathscr{U}_0$ such that $\bar{y} = A\bar{y}$. $\square$

Note. If $x \in \mathscr{B}_1$ is bounded by the condition $\|x\| \leqq ((1 - K)/H) R$, then the equation $L_n y + \varepsilon\varphi \circ y = x$ has a solution $\bar{y}$ for which $\|\bar{y}\| \leqq R$ under all conditions contained in the text.

5. PROPERTIES OF THE ITERATIONS

In the same way as in Section 4, let $x \in \mathscr{B}_1$, $y \in \mathscr{V}_n$, let φ be a power-series (4), h be from Theorem 1 and let $A : \mathscr{V}_n \to \mathscr{V}_n$ be the mapping defined by the formula

$$Ay = h * x - \varepsilon h * \varphi \circ y .$$

We shall investigate the sequence of iterations $y_0 = 0 \in \mathscr{B}_1$, $y_{n+1} = Ay_n$, for $n = 0, 1, 2, \ldots$, which converges to the solution of the equation $y = Ay$ under the conditions of Section 4.

Lemma 3. If $y_0 = 0 \in \mathscr{B}_1$, then $y_1 = Ay_0$ and $y_2 = Ay_1$ are discrete Volterra-series.

Proof. When $y_0 = 0 \in \mathscr{B}_1$ then $y_1 = h * x$. We denote $h = h_1^{(1)} \in \mathscr{B}_1$, $h_k^{(1)} = 0 \in \mathscr{B}_k$ for $k = 2, 3, \ldots$. So we obtain the discrete Volterra-series for the first iteration $y_1 = \boldsymbol{H}_1^{(1)} x$.

For y_2 we have

$$y_2 = h * x - \varepsilon h * \varphi \circ y_1 = h * x - \varepsilon h * \sum_{j=2}^{\infty} a_j (\boldsymbol{H}_1^{(1)} x)^j .$$

By Definition 1 we have

$$y_2 = h * x - \varepsilon h * \sum_{j=2}^{\infty} a_j [(h_1^{(1)})^j * x^{*j}] .$$

For a fixed $j \geqq 2$ we get

$$- \varepsilon a_j \int_0^{\infty} h(k - \tau) \left[\int_0^{\infty} \int_0^{\infty} \ldots \int_0^{\infty} h(\tau - \tau_1) \, h(\tau - \tau_2) \ldots \right.$$
$$\left. \ldots h(\tau - \tau_j) \prod_{i=1}^{j} x(\tau_i) \, \mathrm{d}\boldsymbol{F}(\tau_i) \right] \mathrm{d}\boldsymbol{F}(\tau) .$$

The integrals express summation by Section 1. After a reordering of the finite sum in the last formula we obtain

$$\int_0^{\infty} \int_0^{\infty} \ldots \int_0^{\infty} \left[-\varepsilon a_j \int_0^{\infty} h(k - \tau) \, h(\tau - \tau_1) \ldots h(\tau - \tau_j) \, \mathrm{d}\boldsymbol{F}(\tau) \right] \prod_{i=1}^{j} x(\tau_i) \, \mathrm{d}\boldsymbol{F}(\tau_i) .$$

We note $h_1^{(2)} = h$, and for $j = 2, 3, \ldots$

$$h_j^{(2)}(k - \tau_1, k - \tau_2, \ldots, k - \tau_j) =$$
$$= -\varepsilon a_j \int_0^{\infty} h(k - \tau) \, h(\tau - \tau_1) \ldots h(\tau - \tau_j) \, \mathrm{d}\boldsymbol{F}(\tau) .$$

Then the family of sequences $h_1^{(2)}, h_2^{(2)}, h_3^{(2)}, \ldots$ determines the discrete Volterra-series of the second iteration. □

Lemma 4. If y_n is a discrete Volterra-series determined by the family of sequences $h_1^{(n)}, h_2^{(n)}, h_3^{(n)}, \ldots, h_i^{(n)} \in \mathscr{B}_i$, the next iteration $y_{n+1} = A y_n$ is a discrete Volterra-series too, with the determining family $h_1^{(n+1)}, h_2^{(n+1)}, h_3^{(n+1)}, \ldots$.

Proof. Suppose $y_n = \sum_{i=1}^{\infty} \boldsymbol{H}_i^{(n)} x$, $x \in \mathscr{B}_1$, then

$$y_{n+1} = h * x - \varepsilon h * \{ \sum_{j=2}^{\infty} a_j (\sum_{i=1}^{\infty} \boldsymbol{H}_i^{(n)} x)^j \} .$$

After reordering the sum enclosed in the braces according to degrees of the terms it may be shown that

$$y_{n+1} = h * x - \varepsilon h * \{ a_2 (\boldsymbol{H}_1^{(n)})^2 + \sum_{i=3}^{\infty} \{ a_i (\boldsymbol{H}_1^{(n)})^i + \tag{15}$$
$$+ \sum_{l=1}^{i-2} a_{i-l} [\sum_{p=1}^{\min(l, i-l)} (\boldsymbol{H}_1^{(n)})^{i-l-p} \sum_{\mathbf{P}} (\boldsymbol{H}_{r_1}^{(n)})^{v_1} (\boldsymbol{H}_{r_2}^{(n)})^{v_2} \ldots (\boldsymbol{H}_{r_\alpha}^{(n)})^{v_\alpha} \mathbf{C}_{v_1, v_2, \ldots, v_\alpha}^{i-l}]\}\} \, x$$

with the conditions for the summation

$$\mathbf{P} \equiv (1 < r_1 < r_2 < \ldots < r_\alpha), \quad (\nu_1 + \nu_2 + \ldots + \nu_\alpha = p), \quad (r_1\nu_1 + r_2\nu_2 + \ldots + r_\alpha\nu_\alpha = l + p)$$

and

$$\mathbf{C}^k_{\nu_1,\nu_2,\ldots,\nu_\alpha} = \binom{k}{\nu_1}\binom{k-\nu_1}{\nu_2}\ldots\binom{k-\nu_1-\nu_2-\ldots-\nu_{\alpha-1}}{\nu_\alpha}.$$

We put

(16) $$h_1^{(n+1)} = h_1^{(n)} = h\,.$$

For the second degree we rewritte for $k \in \mathcal{N}$

$$[-\varepsilon h * a_2(\boldsymbol{H}_1^{(n)})^2 x](k) =$$

$$= -\varepsilon a_2 \int_0^\infty h(k-\tau)\left[\int_0^\infty\int_0^\infty h(\tau-\tau_1)\,h(\tau-\tau_2)\,x(\tau_1)\,x(\tau_2)\,\mathrm{d}\boldsymbol{F}(\tau_1)\,\mathrm{d}\boldsymbol{F}(\tau_2)\right]\mathrm{d}\boldsymbol{F}(\tau),$$

and we reorder the integration

$$-\varepsilon a_2 \int_0^\infty\int_0^\infty\left[\int_0^\infty h(k-\tau)\,h(\tau-\tau_1)\,h(\tau-\tau_2)\,\mathrm{d}\boldsymbol{F}(\tau)\right]x(\tau_1)\,x(\tau_2)\,\mathrm{d}\boldsymbol{F}(\tau_1)\,\mathrm{d}\boldsymbol{F}(\tau_2)\,.$$

Put

$$h_2^{(n+1)}(k-\tau_1, k-\tau_2) = -\varepsilon a_2 \int_0^\infty h(k-\tau)\,h(\tau-\tau_1)\,h(\tau-\tau_2)\,\mathrm{d}\boldsymbol{F}(\tau)$$

Similarly for $i \geqq 3$ the ith term has the form

$$-\varepsilon h * (\{a_i(h)^i + \sum_{l=1}^{i-2} a_{i-l}[\sum_{p=1}^{\min(l,i-l)} (h)^{i-l-p}\,.$$
$$.\sum_{\mathbf{P}}(h_{r_1}^{(n)})^{\nu_1}\,(h_{r_2}^{(n)})^{\nu_2}\ldots(h_{r_\alpha}^{(n)})^{\nu_\alpha}\,\mathbf{C}^{i-l}_{\nu_1,\nu_2,\ldots,\nu_\alpha}]\} * x^{*i})\,.$$

If we reorder the summation in the last expression, we can denote

$$h_i^{(n+1)}(k-\tau_1, k-\tau_2, \ldots, k-\tau_i) =$$

(17) $$= -\varepsilon \int_0^\infty h(k-\tau)\,\{a_i(h)^i + \sum_{l=1}^{i-2} a_{i-l}[\sum_{p=1}^{\min(l,i-l)} (h)^{i-l-p}\,.$$
$$.\sum_{\mathbf{P}}(h_{r_1}^{(n)})^{\nu_1}\,(h_{r_2}^{(n)})^{\nu_2}\ldots(h_{r_\alpha}^{(n)})^{\nu_\alpha}\,\mathbf{C}^{i-l}_{\nu_1,\nu_2,\ldots,\nu_\alpha}]\}\,\mathrm{d}\boldsymbol{F}(\tau)$$

$$\mathbf{P} \equiv (1 < r_1 < r_2 < \ldots < r_\alpha), (\nu_1 + \nu_2 + \ldots + \nu_\alpha = p), \quad (r_1\nu_1 + r_2\nu_2 + \ldots + r_\alpha\nu_\alpha = l + p)\,.$$

The independent variables $(\tau - \tau_1, \tau - \tau_2, \ldots, \tau - \tau_i)$ are omitted in the braces.

We have found the family $h_1^{(n+1)}, h_2^{(n+1)}, h_3^{(n+1)}, \ldots$ which determines the discrete Volterra-series of the $(n + 1)$ th iteration. □

From Lemmas 3 and 4 it follows

Theorem 5. Let for a given $x \in \mathscr{B}_1$ there exist exactly one $\bar{y} \in \mathscr{V}_n$ satisfying the equation $y = Ay$. Let us set $y_0 = 0 \in \mathscr{B}_1$ and define $y_{k+1} = Ay_k$ for $k = 0, 1, 2, \ldots$. Then y_n is a discrete Volterra-series for all natural n.

A property of members of the family $h_1^{(n)}, h_2^{(n)}, h_3^{(n)}, \ldots$, determining the discrete Volterra-series is described by the following

Theorem 6. Let $h_1^{(n)}, h_2^{(n)}, h_3^{(n)}, \ldots, h_i^{(n)}, \ldots$ and $h_1^{(n+c)}, h_2^{(n+c)}, \ldots, h_i^{(n+c)}, \ldots, h_i^{(n)}$, $h_i^{(n+c)} \in \mathscr{B}_i$ for $i = 1, 2, 3, \ldots$, be the family determining the discrete Volterra-series of the nth and the $(n + c)$th iteration respectively. Then for all natural n and $c = 1, 2, 3, \ldots$ it holds

$$h_n^{(n)} = h_n^{(n+c)} .$$

Proof. According to (16) $h_1^{(n)} = h$ for all natural n. Let for $n \geqq 2$

$$h_{n-1}^{(n-1)} = h_{n-1}^{(n-1+c)} \quad \text{for} \quad c = 1, 2, 3, \ldots. \tag{18}$$

Using (17) we obtain

$$\begin{aligned} h_n^{(n+c)}(k - \tau_1, k - \tau_2, \ldots, k - \tau_n) = \\ = -\varepsilon \int_0^\infty h(k - \tau) \{a_n(h)^n + \sum_{l=1}^{n-2} a_{n-l}[\sum_{p=1}^{\min(l,n-l)} (h)^{n-l-p} . \\ . \sum_{\mathbf{P}} (h_{r_1}^{(n+c-1)})^{\nu_1} (h_{r_2}^{(n+c-1)})^{\nu_2}, \ldots, (h_{r_\alpha}^{(n+c-1)})^{\nu_\alpha} \mathbf{C}_{\nu_1,\nu_2,\ldots,\nu_\alpha}^{n-l}]\} \, \mathrm{d}F(\tau) \end{aligned} \tag{19}$$

with the conditions

$$\mathbf{P} \equiv (1 < r_1 < r_2 < \ldots < r_\alpha), \quad (\nu_1 + \nu_2 + \ldots + \nu_\alpha = p),$$
$$(r_1\nu_1 + r_2\nu_2 + \ldots + r_\alpha\nu_\alpha = l + p).$$

We shall find the greatest value the indices $r_1, r_2, \ldots, r_\alpha$ can reach.

Let $1 \leqq l_0 \leqq n - 2$. From the summation conditions $\mathbf{P}$ we obtain $\nu_1 + \nu_2 + \ldots + \nu_\alpha = p_0$, $r_1\nu_1 + r_2\nu_2 + + \ldots + r_\alpha\nu_\alpha = l_0 + p_0$. At least one number $\nu_1, \nu_2, \nu_3, \ldots, \nu_\alpha$ is non-zero; let $\nu_1 \neq 0$. Then

$$r_1 = \frac{1}{\nu_1} (l_0 + p_0 - r_2\nu_2 - r_3\nu_3 - \ldots - r_\alpha\nu_\alpha).$$

Because $r_i \geqq 2$ and $\nu_i \geqq 0$ for $i = 1, 2, \ldots, \alpha$, the number r_1 reaches its greatest value for $\nu_1 = 1$ ano $\nu_2 = \nu_3 = \ldots = \nu_\alpha = 0$. Then $p_0 = 1$ and $r_1 = l_0 + p_0 = l_0 + 1$.

Furthermore we put the greatest possible value of l_0 ($l_0 \leqq n - 2$), hence $r_1 =$

$= n - 2 + 1 = n - 1$. In compliance with the last result h_{r_1} can occur with $r_1 \leqq \leqq n - 1$ in the product

$$(h_{r_1}^{(n+c-1)})^{v_1} (h_{r_2}^{(n+c-1)})^{v_2}, \ldots, (h_{r_\alpha}^{(n+c-1)})^{v_\alpha}.$$

It follows that $r_i \leqq n - 1$ for every $i = 1, 2, \ldots, \alpha$. According to the assumption (18) we have

$$h_{n-1}^{(n+c-1)} = h_{n-1}^{(n-1)},$$

therefore

$$h_{r_i}^{(n+c-1)} = h_{r_i}^{(n-1)} \quad \text{for} \quad i = 1, 2, \ldots.$$

Putting this result into (19) we obtain

$$h_n^{(n+c)} = h_n^{(n)} \quad \text{for all natural } n \text{ and } c = 1, 2, \ldots.$$

Definition 3. A discrete system Φ is called *V-analytical* on a set $\mathcal{U} \subset \mathcal{B}_1$ if there exists a family of sequences $h_1, h_2, \ldots, h_i, \ldots, h_i \in \mathcal{B}_i$, determining a discrete Volterra-series such that for every $x \in \mathcal{U}$

$$\Phi(x) = \sum_{i=1}^{\infty} \boldsymbol{H}_i x$$

with $\boldsymbol{H}_i x = h_i * x^{*i}$.

6. CONCLUSION

Let a discrete system $\Phi: \mathcal{B}_1 \to \mathcal{B}_1$ be described by a non-linear difference equation

$$\boldsymbol{L}_n y + \varepsilon\varphi \circ y = x,$$

$\boldsymbol{L}_n$ is a linear difference operator defined by (3), φ is a power-series from (4), $x \in \mathcal{B}_1$, $y \in \mathcal{V}_n$, ε is a real constant.

Let $h \in \mathcal{V}_{n-1} \subset \mathcal{B}_1$ be the sequence from Theorem 1. Let $K \in (0, 1)$ be the constant of contractivity of the mapping $\boldsymbol{A}: \mathcal{V}_n \to \mathcal{V}_n$ from (7), R_0 be the constant from Theorem 2, $H = \sum_{k=0}^{\infty} |h(k)|$. Then the discrete system Φ is V-analytical on the set

$$\mathcal{U} = \left\{ x \in \mathcal{B}_1 \mid \|x\| < \frac{1 - K}{H} R_0 \right\}.$$

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