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2-D POLYNOMIAL EQUATIONS

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Linear equations in two-dimensional (2-D) real polynomials are investigated. Existence of a minimum degree solution is studied. Various necessary and sufficient conditions of solvability are given which are suited for practical testing. Three computational algorithms are described.

1. INTRODUCTION

During recent years, a growing interest has been developed into problems involving systems and/or signals which depend on more than one variable (for reviews of this trend see, e.g., [2], [14] and references therein).

These *multidimensional* systems and/or signals have been investigated in relation to several modern engineering fields such as multidimensional system synthesis, the control of delay-differential systems, multidimensional digital filtering, multivariable network realizability, digital picture processing, seismic data processing, X-ray enhancement, gravity and magnetic field mapping, image deblurring, the enhancement and analysis of aerial photographs for detection of forest fires or crop damage, the analysis of satellite weather photographs, etc.

In this paper we focus our attention on *two-dimensional* (2-D) systems. The 2-D systems can be properly described by polynomials in two indeterminates (2-D polynomials). (See, for example, [3], [7], [8], [10], [12], [13], [14], [15] and references therein.) These indeterminates may represent operations of differentiation and/or delaying. Furthermore, they can also represent the dependence of the system on parameters or the aging of the system when it depends on time.

A growing presence of algebraic methods can be observed in the control theory in the last decade. Specifically, polynomial equations have become a useful tool for synthesis of 1-D control systems (see, e.g., [16], [18], [9] and [6]). From an analogy, 2-D polynomial equations are believed to play the same role in the area of 2-D systems.

For one example of a 2-D problem solved via 2-D polynomial equations the reader is referred to [15]. As another example, 2-D polynomial equations with zero coprime left-hand side polynomials appear when applying polynomial equations over rings used recently in [5] for "split" systems over rings.

The aim of this paper is to look at the 2-D polynomial equations from the control theoretical point of view. First, the existence of a "minimum degree solution" is studied. Then the classical fundamental results on solvability are mentioned briefly. Since these are not well suited for practical computation, two different new necessary and sufficient conditions of solvability are derived which are practically applicable. Finally, several computational algorithms are described which are, in fact, generalizations of the well-known 1-D algorithms.

An alternative way to the solution of a slightly less general type of the 2-D polynomial equations seems to be in the application of the method by Emre for the 2-D case (see [4], Remark 3.10).

2. PRELIMINARIES

This paper is starred by real polynomials in two indeterminates: v and w. Denoting the field of real (or complex) numbers by $\mathscr{R}$ (or $\mathscr{C}$) then $\mathscr{R}[v]$ and $\mathscr{R}(v)$ mean the ring of real polynomials in v and the field of real rational functions in v, respectively. The ring of real polynomials in two indeterminates is naturally denoted by $\mathscr{R}[v, w]$. This ring can also be viewed as $\mathscr{R}[v]\,[w]$, that is, the ring of polynomials in w with coefficients in $\mathscr{R}[v]$. Finally, $\mathscr{R}(v)\,[w]$ denotes the ring of polynomials in w with coefficients from $\mathscr{R}(v)$. The ring $\mathscr{R}(v)\,[w]$ is useful since, unlike $\mathscr{R}[v, w]$, it possesses the Euclidean division algorithm, which is an important feature of $\mathscr{R}[w]$. Rings $\mathscr{R}[w]$, $\mathscr{R}(w)$, $\mathscr{R}[w, v]$ and $\mathscr{R}(v)\,[w]$ can be defined in an analogous manner.

Properties of the above rings are described in any textbook of algebra, e.g., in [17], while 1-D polynomial matrices are treated in detail, e.g., in [9]. Here we just introduce some notation briefly.

The *w-degree* of a polynomial $P \in \mathscr{R}[v, w]\,(\mathscr{R}(v)\,[w])$ denoted by $\deg_w P$, is the greatest power of w occurring in P. A *greatest common divisor* of polynomials P and Q is denoted by $\gcd(P, Q)$. $(s, z) \in \mathscr{C} \times \mathscr{C}$ is the *zero* of a polynomial $P(v, w) \in \mathscr{R}[v, w]$ iff $P(s, z) = 0$. Polynomials $P, Q \in \mathscr{R}[v, w]$ are *factor coprime* iff $\gcd(P, Q) = 1$ and *zero coprime* iff they have no zero in common. Finally, $P \mid Q$ means P divides Q.

In this paper $\mathscr{R}[v]\,[w]$ and $\mathscr{R}(v)\,[w]$ are preferred for brevity. However, analogous procedures employing $\mathscr{R}[w]\,[v]$ and $\mathscr{R}(w)\,[v]$ can be obtained simply by interchanging the roles of v and w.

3. BASIC PROPERTIES

Consider the equation

$$AX + BY = C \tag{1}$$

where A, B and C are given polynomials of $\mathscr{R}[V, w]$ such that A and B are factor coprime. This will be called a (linear) 2-D polynomial equation and by its solution we mean any pair of polynomials X, $Y \in \mathscr{R}[v, w]$ satisfying (1).

The influence of a possible common factor of A, B is well understood from the 1-D case (simply gcd (A, B) must divide C) and therefore we suppose for simplicity that any common factor has already been cancelled from (1).

Equation (1) is linear and hence its general solution can be obtained from a particular one as follows.

Theorem 1. Let X', Y' be a solution of (1). Then the general solution of (1) is

$$\begin{aligned} X &= X' + BT \\ Y &= Y' - AT \end{aligned} \tag{2}$$

for an arbitrary polynomial $T \in \mathscr{R}[v, w]$.

Proof. The proof follows along the same line as the proof of the corresponding 1-D theorem given in [9]. □

In the case of 1-D, the application of the Euclidean division algorithm to relation (2) shows the existence of the (unique) "minimum degree solution" of (1). Since $\mathscr{R}[v, w]$ is not the Euclidean ring, however, one would not expect any similar result in 2-D. Surprisingly, this holds true also in 2-D under very general conditions.

Consider A, B and C as polynomials of $\mathscr{R}[v][w]$

$$\begin{aligned} A &= a_0 + a_1 w + \dots + a_m w^m \\ B &= b_0 + b_1 w + \dots + b_n w^n \\ C &= c_0 + c_1 w + \dots + c_p w^p \end{aligned} \tag{3}$$

where $m = \deg_w A$, $n = \deg_w B$, $p = \deg_w C$ and all a_i, b_i and $c_i \in \mathscr{R}[v]$. Further write

$$d = \gcd(a_m, b_n) \in \mathscr{R}[v]$$

so that $a_m = \bar{a}_m d$, $b_n = \bar{b}_n d$ for $\bar{a}_m$, $\bar{b}_n \in \mathscr{R}[v]$. Now we are in a position to state the main result of this section.

Theorem 2. Let the equation (1) be solvable. If

$$p = \deg_w C \leqq m + n - 1 \tag{4}$$

and if either

(5a) $$d = 1$$

or, at least,

(5b) $$\gcd(d, a_{m-1}\bar{b}_n - \bar{a}_m b_{n-1}) = 1 \quad \text{and} \quad d \mid c_p$$

then there is a unique solution X, Y of (1) such that

$$\deg_w X \leqq n - 1$$

and, consequently,

$$\deg_w Y \leqq m - 1$$

Proof. Write

$$X' = x'_0 + x'_1 w + \ldots + x'_k w^k$$

$$Y' = y'_0 + y'_1 w + \ldots + y'_l w^l$$

where all x'_i, $y'_j \in \mathscr{R}[v]$, for a solution X', Y' of (1) and suppose that $k > n - 1$. In view of (2) any X of the form

(6) $$X = x'_0 + x'_1 w + \ldots + x'_k w^k + (b_0 + b_1 w + \ldots + b_n w^n)\, t$$

with an arbitrary $t \in \mathscr{R}[v, w]$ is also a solution of (1). An X such that $\deg_w X < k$ can evidently be found iff

(7) $$b_n \mid x'_k$$

We now wish to show that this is always the case when both (4) and (5) are satisfied. To do this we equate the coefficients of the highest $(k + m)$ power of w in (1). This yields

(8) $$a_m x'_k + b_n y'_l = 0$$

and hence

(9) $$b_n \mid a_m x'_k$$

Now when (5a) is satisfied then (9) implies (7) immediately. If, on the other hand, only (5b) is satisfied then (9) yields just $\bar{b}_n \mid x'_k$ so that

(10) $$x'_k = \bar{b}_n u$$

for a polynomial $u \in \mathscr{R}[v]$ whereby

(11) $$y'_l = -\bar{a}_m u$$

as well. Nevertheless, equating further the coefficients of w^{n+k-1} in (1) gives

$$a_{m-1}x'_k + a_m x'_{k-1} + b_{n-1}y'_l + b_n y'_{l-1} = \begin{cases} 0 & \text{for} \quad k > n \\ c_p & \text{for} \quad k = n \end{cases}$$

and, inserting (10) and (11),

$$(a_{m-1}\bar{b}_n - b_{n-1}\bar{a}_m)\, u + d(\bar{a}_m x_{l-1} + \bar{b}_n y_{k-1}) = \begin{cases} 0 \\ c_p \end{cases}$$

from which, due to (5b),

$$d \mid u$$

so that, in view of (10), (9) results again.

We summarize what was done as follows. For any solution X', Y' for which $\deg_w X' > n - 1$ we can employ (6) to compute another solution X, Y such that $\deg_w X < \deg_w X'$. After repeating this procedure until $\deg_w X \leqq n - 1$ the minimum w-degree solution results. It is easy to see that under (4) always $l = m + k - n$ so that for the desired X, Y we have $\deg_w Y \leqq m - 1$. □

In fact, the assumptions (4) and (5) make it possible to estimate the minimum w-degree of a solution and thereby to simplify both solvability conditions and algorithms. The conditions (4) and (5) are satisfied in most practical problems. Even, if for given polynomials A, B and C, they do not hold in $\mathscr{R}[v][w]$, they may still be satisfied in $\mathscr{R}[w][v]$. That is why, from now on, *we assume that A, B and C in* (1) *satisfy* (4) and (5). The more general cases are postponed until Section 7.

4. SOLVABILITY CONDITIONS

We begin this section by recalling that the 1-D version of equation (1) (with coprime A, B) has always a solution. Is this true for 2-D polynomial equations as well? Unfortunately, the answer is no. As an example, notice that the equation

$$v^2x + wy = v$$

has no solution regardless of the fact that $\gcd(v, w) = 1$. The reason is, roughly speaking, that the common zero A and B does not possess a proper "multiplicity" in C.

It is our aim to investigate the situation more deeply. The fundamental results go back to Hilbert, Noether [11] and Bertini [1]. Our brief exposition is due to van der Waerden [17].

Hilbert's Nullstellensatz. If a polynomial $F \in \mathscr{R}[v, w]$ vanishes at all zeros common to polynomials A and B then there is an integer p such that the equation

$$AX + BY = F^p$$

has a solution (and conversely).

Proof. See, e.g., [17]. □

Hilbert's Nullstellensatz has an easy but for the control theory important corollary.

Theorem 3. An equation

$$AX + BY = 1$$

has a solution if and only if the polynomials A and B are zero coprime.

The relation $AX + BY = 1$ is often referred to as the *Bezout identity*.

For the purposes of the next theorem we need the following additional notions: The ideal $\mathfrak{I}(G, H)$ generated by polynomials $G, H \in \mathscr{R}[v, w]$ (defined as the set of all expressions of the form $EG + FH$ where $E, F \in \mathscr{R}[v, w]$), the greatest common divisor gcd $(\mathfrak{H}, \mathfrak{L})$ of ideals $\mathfrak{H}$ and $\mathfrak{L}$ (which consists of all sums $K + L$, where $K \in \mathfrak{H}$, $L \in \mathfrak{L}$) and the power $\mathfrak{L}^p$ of an ideal $\mathfrak{L}$ (consisting of all sums $\sum L_1 L_2 \dots L_p$ where $L_1, \dots, L_p \in \mathfrak{H}$). For the precise definitions, the reader is referred to any textbook of algebra (for example, [17]).

Fundamental Theorem of Noether. Let $\mathfrak{M} = \mathfrak{I}(A, B)$ and for every zero (s, z) common to both A and B, let p denote the smallest integer such that

$$\mathfrak{L}^p \subset \gcd(\mathfrak{M}, \mathfrak{L}^{p+1})$$

where $\mathfrak{L} = \mathfrak{I}(v - s, w - z)$. If C satisfies

$$C \in \operatorname{gdc}(\mathfrak{M}, \mathfrak{L}^p)$$

for all such $\mathfrak{L}$, then $C \in \mathfrak{M}$ or, equivalently, the equation (1) has a solution.

Proof. See [17]. □

The above theorems provide a good theoretical insight into the problem. On the other hand, they are not suited for practical testing of solvability, for the very reason of computing the common zeros of 2-D polynomials.

Now we wish to propose two methods which provides necessary and sufficient conditions suited for practical computation. In addition, as will be shown in the next section, they directly yield a minimum w-degree solution. The underlaying philosophy of our approach is that, instead of testing the existence of any solution, we test just the existence of the minimum w-degree solution. If this minimum w-degree solution fails to exist, Theorem 2 implies that there is no solution at all.

The first way consists in converting the given 2-D polynomial equation to a 1-D polynomial matrix equation, properties of which are now well understood (see, e.g., [9]). To do it write again

$$A = a_0 + a_1 w + \dots + a_m w^m$$
$$B = b_0 + b_1 w + \dots + b_n w^n$$
$$C = c_0 + c_1 w + \dots + c_p w^p$$

and

$$X = x_0 + x_1 w + \dots + x_{n-1} w^{n-1}$$
$$Y = y_0 + y_1 w + \dots + y_{m-1} w^{m-1}$$

where all coefficients are from $\mathscr{R}[v]$, and form the 1-D polynomial matrices (with entries from $\mathscr{R}[v]$)

$$\mathbb{A} = \left.\begin{bmatrix} a_0, a_1, \dots, a_m & & \\ & a_0, a_1, \dots, a_m & \\ & \ddots & \\ & & a_0, a_1, \dots, a_m \end{bmatrix}\right\} n$$

$$\mathbb{B} = \left.\begin{bmatrix} b_0, b_1, \dots, b_n & & \\ & b_0, b_1, \dots, b_n & \\ & \ddots & \\ & & b_0, b_1, \dots, b_n \end{bmatrix}\right\} m$$

$$\mathbb{C} = [c_0, c_1, \dots, c_p]$$
$$\mathbb{X} = [x_0, x_1, \dots, x_{m-1}]$$
$$\mathbb{Y} = [y_0, y_1, \dots, y_{n-1}]$$

The composite square polynomial matrix

$$\begin{bmatrix} \mathbb{A} \\ \mathbb{B} \end{bmatrix}$$

is called the polynomial *Sylvester* matrix and its determinant is called the *resultant*.

Recall that, since Theorem 2 is to be employed, conditions (4)–(5) are assumed. Now we are ready to state the necessary and sufficient condition of solvability of (1).

Theorem 4. An equation (1) has a solution if and only if $\begin{bmatrix} \mathbb{A} \\ \mathbb{B} \end{bmatrix}$ is a right divisor of $\mathbb{C}$.

Proof. In view of Theorem 2, (1) has a solution iff it has the minimum w-degree solution. Substituting this solution and using the notation above, (1) reads

$$[\mathbb{X}\mathbb{Y}]\begin{bmatrix} \mathbb{A} \\ \mathbb{B} \end{bmatrix} = \mathbb{C}$$

and the theorem follows at once from the theory of 1-D polynomial matrix equation (see [9]). □

The solvability condition can be stated in still another way. Considering (1) as a polynomial equation in $\mathscr{R}(v)[w]$, then it has always a unique minimum degree solution $\tilde{X}$, $\tilde{Y} \in \mathscr{R}(v)[w]$ (for which $\deg_w \tilde{X} < \deg_w B$ and $\deg_w \tilde{Y} < \deg_w A$). This is true by standard results ([9]) for $\mathscr{R}(v)[w]$ is a Euclidean domain. On the other hand, in view of Theorem 2, (1) has a unique minimum w-degree solution X, $Y \in \mathscr{R}[v, w]$ (if it has a solution in $\mathscr{R}[v, w]$ at all). Now $\mathscr{R}[v, w] \subset \mathscr{R}(v)[w]$ so that every solution in $\mathscr{R}[v, w]$ is clearly also the solution in $\mathscr{R}(v)[w]$ and, consequently, the two minimum degree solutions above, $\tilde{X}$, $\tilde{Y}$ and X, Y, must coincide. This is summarized in the following.

Theorem 5. Let $\tilde{X}$, $\tilde{Y} \in \mathscr{R}(v)[w]$ be the minimum degree solution of the equation (1) in $\mathscr{R}(v)[w]$. Then (1) has a solution in $\mathscr{R}[v, w]$ if and only if $\tilde{X}$, $\tilde{Y} \in \mathscr{R}[v, w]$.

5. COMPUTATIONAL ALGORITHMS

The two preceding theorems yield directly two ways of solving the 2-D polynomial equation. The following algorithms both result in the minimum w-degree solution. If any other solution is desired, it can be obtained from (2).

According to the proof of Theorem 4, the equation (1) can be rewritten as

$$[\mathbb{X}\mathbb{Y}]\begin{bmatrix}\mathbb{A}\\ \mathbb{B}\end{bmatrix} = \mathbb{C} \tag{12}$$

This 1-D polynomial matrix equation of a special type can be solved, for example, in the following steps.

Algorithm 1.

1) Using elementary row operations carry out the reduction

$$\left[\begin{array}{c|cc}\mathbb{A} & \mathrm{I} & 0\\ \mathbb{B} & 0 & \mathrm{I}\end{array}\right] \to [\mathbb{D} \mid \mathbb{E}]$$

where $\mathbb{D}$ is a square upper triangular polynomial matrix.

2) Find a polynomial matrix $\overline{\mathbb{C}}$ such that

$$\mathbb{C} = \overline{\mathbb{C}}\mathbb{D}$$

Denoting

$$\overline{\mathbb{C}} = [\bar{c}_0, \bar{c}_1, \ldots, \bar{c}_{m+n-1}]$$

$$\mathbb{D} = \begin{bmatrix} d_{11}, d_{12}, \ldots, d_{1m+n} & & \\ & d_{22} & \\ & & \ddots \\ 0 & & d_{m+n,m+n}\end{bmatrix}$$

the entries of $\overline{\mathbb{C}}$ can be calculated from

$$\bar{c}_i = \frac{1}{d_{i+1,i+1}}\left(c_i - \sum_{k=0}^{i-1} \bar{c}_k d_{k+1,i+1}\right) \tag{13}$$

When some $\bar{c}_i$, $i = 0, \ldots, m + n - 1$, is not a polynomial of $\mathscr{R}[v]$ but merely a polynomial fraction, then the equation has no solution.

3) Otherwise, the desired solution is given by

$$[\mathbb{X}\mathbb{Y}] = \overline{\mathbb{C}}\mathbb{E} \tag{14}$$

Alternatively, Theorem 5 offers to solve the equation

$$A\tilde{X} + B\tilde{Y} = C \tag{15}$$

for $\tilde{X}$, $\tilde{Y} \in \mathscr{R}(v)[w]$ instead of (1). This can be done as follows.

Algorithm 2.

1) Using elementary row operations (now in $\mathscr{R}(v)[w]$) perform the reduction

$$(16)\qquad \begin{bmatrix} A & 1 & 0 \\ B & 0 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & \tilde{E}_1 & \tilde{E}_2 \\ 0 & \tilde{E}_3 & \tilde{E}_4 \end{bmatrix}$$

where $\tilde{E}_1$, $\tilde{E}_2$, $\tilde{E}_3$ and $\tilde{E}_4 \in \mathscr{R}(v)[w]$.

2) Carry out the division

$$(17)\qquad \frac{\tilde{E}_1 C}{B} = \tilde{Q} + \frac{\tilde{R}}{B}$$

to obtain a quotient $\tilde{Q} \in \mathscr{R}(v)[w]$ and a reminder $\tilde{R} \in \mathscr{R}(v)[w]$ such that

$$\deg_w \tilde{R} < \deg_w B$$

and put

$$(18)\qquad \begin{aligned} X &= \tilde{R} \\ Y &= E_2 + AQ \end{aligned}$$

3) Now either X, Y are polynomials of $\mathscr{R}[v, w]$ (and not merely of $\mathscr{R}(v)[w]$) and then they form the desired solution of (1) or they are not of $\mathscr{R}[v, w]$ (they are polynomial fractions in v) which means that (1) has no solution.

For an alternative procedure solving (15) the reader is referred to [4].

6. EXAMPLE

As an example let us take the equation (1) with

$$\begin{aligned} A &= 1 + vw + w^2 + vw^2 \\ B &= w + vw \\ C &= 2 + v - w \end{aligned}$$

Since conditions (4)–(5b) are satisfied, let us start with Algorithm 1.
The matrices $\mathbb{A}$, $\mathbb{B}$ and $\mathbb{C}$ read

$$\mathbb{A} = \begin{bmatrix} 1 & v & 1 + v \end{bmatrix}$$

$$\mathbb{B} = \begin{bmatrix} 0 & 1 + v & 0 \\ 0 & 0 & 1 + v \end{bmatrix}$$

$$\mathbb{C} = \begin{bmatrix} 2 + v & -1 & 0 \end{bmatrix}$$

1) Since $\begin{bmatrix} \mathbb{A} \\ \mathbb{B} \end{bmatrix}$ itself is an upper triangular matrix, take directly

$$\mathbb{D} = \begin{bmatrix} \mathbb{A} \\ \mathbb{B} \end{bmatrix} = \begin{bmatrix} 1 & v & 1 + v \\ 0 & 1 + v & 0 \\ 0 & 0 & 1 + v \end{bmatrix}$$

$$\mathbb{E} = I$$

2) Using (13) recurrently, $\mathbb{C}$ is computed to be

$$\mathbb{C} = [2 + v \quad -1 - v \quad -2 - v]$$

3) For $\mathbb{E} = I$, take $[\mathbb{X}\mathbb{Y}] = \mathbb{C}$ so that the desired solution of (1) is

$$\begin{aligned} X &= \quad 2 + v \\ Y &= -1 - v - 2w - vw \end{aligned} \tag{19}$$

Alternatively, Algorithm 2 can be employed:

1) Carrying out the reduction (16)

$$\begin{bmatrix} 1 + vw + (1 + v) w^2 & 1 & 0 \\ (1 + v) w & 0 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 1 & -\dfrac{v + (1 + v) w}{1 + v} \\ 0 & -(1 + v) w & 1 + vw + (1 + v) w^2 \end{bmatrix}$$

We get $\tilde{E}_1 = 1$ and $\tilde{E}_2 = -\dfrac{v}{1 + v} - w$.

2) Performing the division (17)

$$\frac{2 + v - w}{(1 + v) w} = \tilde{Q} + \frac{\tilde{R}}{(1 + v) w}$$

and we have

$$\tilde{Q} = -\frac{1}{1 + v} \quad \text{and} \quad \tilde{R} = 2 + v\,.$$

3) Inserting this in (18) gives (19) again.

Finally, if the general solution is desired, it is of the form

$$\begin{aligned} X &= \quad 2 + v & &+ (w + vw)\, T \\ Y &= -1 - v - 2\, w - vw & &- (1 + v\, w + w^2 + vw^2)\, T \end{aligned}$$

with an arbitrary $T \in \mathscr{R}[v, w]$ as a parameter.

7. GENERAL CASE

Up to now just 2-D polynomial equations with A, B and C satisfying (4)–(5) have been focussed on. In spite of the fact that such equations are believed to cover lots of control problems, the other cases are worth studying as well. They are treated in this part.

We begin by removing the restriction (4) on the w-degree of the right-hand side polynomial C. Let now

$$\deg_w C = p > m + n - 1 \tag{20}$$

Denoting $q = p - (m + n - 1)$, form 1-D polynomial matrices

$$\mathbb{A}_q = \left.\begin{bmatrix} a_0, a_1, \dots, a_m & & \\ & a_0, a_1, \dots, a_m & \\ & \ddots & \\ & & a_0, a_1, \dots, a_m \end{bmatrix}\right\} n + q$$

$$\mathbb{B}_q = \left.\begin{bmatrix} b_0, b_1, \dots, b_n & & \\ & b_0, b_1, \dots, b_n & \\ & \ddots & \\ & & b_0, \dots, b_n \end{bmatrix}\right\} m + q$$

The same type of reasoning as in Theorem 2 and Theorem 4 gives the following results.

Theorem 6. The equation (1) with (5) and (20) has a solution if and only if a greatest common right divisor of matrices $\mathbb{A}_q$ and $\mathbb{B}_q$ is a right divisor of $\mathbb{C}$.

Identifying $\mathbb{A}_0 = \mathbb{A}$ and $\mathbb{B}_0 = \mathbb{B}$, Theorem 6 is clearly consistent with Theorem 4. By denoting

$$\mathbb{X}_q = [x_0, x_1, \dots, x_{n-1+q}]$$
$$\mathbb{Y}_q = [y_0, y_1, \dots, y_{m-1+q}]$$

the equation (1) reads

(21) $$\mathbb{X}_q \mathbb{A}_q + \mathbb{Y}_q \mathbb{B}_q = \mathbb{C}$$

This 1-D polynomial matrix equation is a generalization of (12). It can be solved as follows.

Algorithm 3.

1) Employing elementary row operations carry out the reduction

$$\begin{bmatrix} \mathbb{A}_q & I & 0 \\ \mathbb{B}_q & 0 & I \end{bmatrix} \to \begin{bmatrix} \mathbb{D} & \mathbb{E}_1 & \mathbb{E}_2 \\ 0 & \mathbb{E}_3 & \mathbb{E}_4 \end{bmatrix}$$

where $\mathbb{D}$ is an upper triangular 1-D polynomial matrix.

2) As in Algorithm 1, find a polynomial matrix $\overline{\mathbb{C}}$ such that

$$\mathbb{C} = \overline{\mathbb{C}}\mathbb{D}$$

3) Then every $\mathbb{X}$, $\mathbb{Y}$ of the form

$$\mathbb{X} = \overline{\mathbb{C}}\mathbb{E}_1 + \mathbb{T}\mathbb{E}_3$$
$$\mathbb{Y} = \overline{\mathbb{C}}\mathbb{E}_2 + \mathbb{T}\mathbb{E}_4$$

for an arbitrary 1-D polynomial matrix $\mathbb{T}$ of appropriate dimensions produce X, Y which is a solution of (1).

Hence the violation of (4) causes no problems since we are still able to estimate the minimum w-degree of a solution. No such estimation is known, however, when (5) is violated by

$$\gcd(d, a_{m-1}\bar{b}_n - \bar{a}_m b_{n-1}) \neq 1$$

as it is, for example, in the case of the equation

$$(22) \qquad (1 + w + (v - 1) w^2) X + (vw + (v - 1) w^2) Y = (1 + v) w^3$$

The w-degree of its solution

$$X = w^3$$
$$Y = w^2 - w^3$$

cannot be further reduced.

In such a case we can only take a "sufficiently high" l and search for a solution of w-degree less or equal to l by means of the 1-D equation

$$\mathbb{X}_{l-(n-1)}\mathbb{A}_{l-(n-1)} + \mathbb{Y}_{l-(m-1)}\mathbb{B}_{l-(n-1)} = \mathbb{C}$$

If this equation is not solvable, however, there can still be a solution of (1) (with $\deg_w X > l$).

Notice that, fortunately, (22) satisfies both (4) and (5) in v (i.e., when all A, B, C are considered of $\mathscr{R}[w][v]$) so that, of course, the use of Algorithm 1 (Algorithm 2) in $\mathscr{R}[w]$ ($\mathscr{R}(w)[v]$) would be preferred.

8. CONCLUSION

Linear equations in 2-D polynomials have been investigated. First, conditions ensuring the existence of a unique minimum w-degree solution were given (Theorem 2). These conditions made it possible to derive various new necessary and sufficient conditions of solvability which are practically testable (Theorems 4, 5 and 6). Finally, computational algorithms were described using either a 1-D polynomial matrix equation (Algorithm 1, 3) or a 1-D polynomial equation over the Euclidean domain $\mathscr{R}(v)[w]$ (Algorithm 2).

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REFERENCES

[1] E. Bertini: Zum Fundamentalsatz aus der Theorie der algebraischen Funktionen. Math. Ann. *34* (1889), 447—449.

[2] N. K. Bose: Multidimensional Systems: Theory and Applications. IEEE Press, New York 1979.

[3] R. Eising: 2-D Systems — An Algebraic Approach. Ph. D. Dissertation, Eindhoven University of Technology, Eindhoven 1979.

[4] E. Emre: The polynomial equation $QQ_c + RP_c = \emptyset$ with application to dynamic feedback. SIAM J. Control Optim. *18* (1980), 6, 611—620.

[5] E. Emre and P. P. Khargonekar: Regulation of split linear systems over rings: coefficient-assignment and observers. IEEE Trans. Automat. Control *AC-27* (1982), 1, 104—113.

[6] T. Kailath: Linear Systems. Prentice-Hall, Englewood Cliffs 1980.

[7] E. W. Kamen: A note on the representation of lumped-distributed networks, delay-differential systems and 2-D systems. IEEE Trans. Circuits and Systems *CAS-27* (1980), 5, 430—432.

[8] E. W. Kamen: Linear systems with comensurate time delays: stability and stabilization independent of delay. IEEE Trans. Automat. Control *AC-27* (1982), 2, 367—375.

[9] V. Kučera: Discrete Linear Control: The Polynomial Equation Approach. Wiley, Chichester 1979.

[10] A. S. Morse: Ring models for delay-differential systems. Automatica *12* (1976), 5, 529—531.

[11] M. Noether: Über einen Satz aus der Theorie der algebraischen Funktionen, Math. Ann. *6* (1873), 351—359.

[12] A. W. Olbrot and S. H. Żak: Controllability and observability problems for linear functional-differential systems. Foundations of Control Engineering *5* (1980), 2, 79—89.

[13] P. N. Paraskevopoulos: Feedback design techniques for linear multivariable 2-D systems. In: Analysis and Optimization of Systems (A. Bensoussan and J. L. Lions, eds.), Springer-Verlag, Berlin—Heidelberg—New York 1980.

[14] Special issue on multidimensional systems. Proc. IEEE *65* (1977), 6.

[15] M. Šebek: 2-D Exact model matching. IEEE Trans. Automat. Control *AC-28* (1983), 2, 215—217.

[16] L. N. Volgin: The Fundamentals of the Theory of Controlling Machines (in Russian). Soviet Radio, Moscow 1962.

[17] B. L. van der Waerden: Modern Algebra, 4th ed. (2 volumes). Frederic Ungar Publishing Co., New York 1964.

[18] W. A. Wolovich: Linear Multivariable Systems. Springer-Verlag, New York—Heidelberg—Berlin 1974.

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