KYBERNETIKA — VOLUME 19 (1983), NUMBER 1

PSEUDOMORPHISMS OF AUTOMATA

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A pseudomorphism between the automata (X, Q) and (X', Q') is defined as a class of pairs $\alpha: X \times Q \to X'$, $\beta: Q \to Q'$ which preserve the relation "q_2 may be obtained from q_1 by applying an elementary input". Pseudomorphisms provide simplicial maps between the complexes associated with this relation. They can be linked with the automata homomorphisms of the classical theory by means of cascade products.

Pseudo-isomorphisms are also discussed and an example given in which α is not surjective. Under some (pseudo-) homogeneity conditions an automaton is pseudo-isomorphic to a group quotient automaton.

1. INTRODUCTION

1.1. In general an automaton is defined as a quintuple $A = \{X, Y, Q, \delta, \lambda\}$ where Q is the state set, X the input set, Y the output set, $\delta : X \times Q \to Q$ the next state function and $\lambda : X \times Q \to Y$ the output function (Ginzburg [3]). In this paper we need only X, Q and δ. Thus we have, according to Ginzburg, a semi-automaton. For convenience this will henceforth be called an automaton.

1.2. Our aim is to study the relationship between automata, and in particular to identify two automata whose "patterns of behaviour" are the same even if there is no functional relation between the inputs which cause the behaviour.

1.3. An automaton $A = (X, Q)$ with input set X acting by right translation on the state space Q, $q \to q \,.\, x$, defines the *inertial relation* υ, $q_1 \,\upsilon\, q_2$ iff there exists $x_1 \in X$ such that $q_2 = q_1 \,.\, x_1\, q_1, q_2 \in Q$; i.e. q_1 is related to q_2 iff there exists an elementary input of X which sends q_1 to q_2 (Warner [2]).

1.4. Muir and Warner [4], Warner [5] have used Dowker's [2] homology of relations to describe a homology theory appropriate to this automaton A. In [4] we observe that a simplicial map from the simplicial complex $|Q|$ of $A = (X, Q)$ to the complex $|Q'|$ of $A' = (X', Q')$ is provided by a pseudomorphism (α, β), $\alpha : X \times Q \to X'$, $\beta : Q \to Q'$, satisfying for all $x \in X$, $q \in Q$

$$\beta(q) \cdot \alpha(x, q) = \beta(q \cdot x) \tag{1}$$

In terms of the relation υ of 1.3, iff (α, β) is such a pseudomorphism, then $q_1 \upsilon q_2 \Rightarrow \Rightarrow \beta(q_2) = \beta(q_1 \cdot x_1) = \beta(q_1) \cdot \alpha(x_1, q)$. Thus $\beta(q_1) \upsilon \beta(q_2)$ and β is a *morphism* of (Q, υ) to (Q', υ'). An alternative (cubical) chain complex is introduced in § 5 and the effect of pseudomorphisms on this structure is discussed.

1.5. In § 2 we generalise the above definition of pseudomorphism to a class $[(\alpha, \beta)]$ of pairs (α, β) satisfying relation (1) of 1.4 for a given β. A pseudomorphism will henceforth be taken to be such a class. Pseudo-isomorphisms are then defined and some properties established, for example pseudo-isomorphism is an equivalence relation on the set of automata.

1.6. We observe in § 3 that a pseudomorphism from A to A' can be regarded as a homomorphism in the sense of Ginzburg (3) from A to the cascade produce $A\alpha A'$. In fact when β is surjective the parallel product of A with itself 'covers' (Ginzburg [3]), or simulates $A\alpha A'$. Thus the state of A at any moment governs the input to A', there being an on-line connection between them. With this proviso A can be thought of a simulating A'.

1.7. The foregoing concepts are used by the author in § 4 and (Warner [6]) to establish pseudo-isomorphisms between a class of automata described as *pseudo-homogeneous* and group quotient automata.

1.8. In passing from the input actions to the relation some information is lost, but not too much. For an ordered pair of states (q_1, q_2) we know whether there exists an elementary input taking q_1 to q_2. This is an improvement on Arbib's [1] inertial tolerance automaton in which it is known only that there exists either an input taking q_1 to q_2 or an input taking q_2 to q_1. The imposition of symmetry loses information without reaping any advantage since it is still possible to mimic continuity with respect to time by defining $t_r \upsilon t_{r+1}$ on the quantized time set $T = \{0, 1, 2, \ldots\}$. Then with each input sequence $x^* = \{x_0, x_1, x_2, \ldots\}$ associate the motion $m_{x^*} : T \to \to Q$ by defining $m_{x^*}(0) = q_0$, an initial state, $m_{x^*}(t + 1) = m_{x^*}(t) \cdot x_t$. Clearly m_{x^*} is a morphism (preserves the relation υ).

1.9. The appendix provides an example of a pseudoisomorphism $[(\alpha, \beta)]$ in which α is not surjective. If $[(\alpha', \beta^{-1})]$ is the inverse of $[(\alpha, \beta)]$ then α' is also not surjective and $\alpha'(\alpha(x, q), \beta(q))$ does not coincide with x. Except when there is a possibility of confusion all relations will be written υ.

2. PSEUDOMORPHISMS

2.1. Definition. A pseudomorphism $[(\alpha, \beta)]$ from the automaton $A = (X, Q)$ to $A' = (X', Q')$ is an equivalence class of pairs of functions $\alpha : X \times Q \to X'$, $\beta : Q \to Q'$ such that

$$(1) \qquad \beta(q) \,.\, \alpha(x, q) = \beta(q \,.\, x) \quad \text{for all} \quad q \in Q, x \in X\,.$$

The equivalence relation $\sim$ is defined by $(\alpha, \beta) \sim (\alpha_1, \beta)$ if α, α_1 both satisfy (1) with respect to β. We refer to (α, β) as a representative of $[(\alpha, \beta)]$.

2.2. If A, A' are given inertial tolerance (Arbib [1]), then β is tolerance-preserving iff either $\beta(q) \,.\, \alpha(x, q) = \beta(q \,.\, x)$ or $\beta(q \,.\, x) \,.\, \alpha(x, q) = \beta(q)$. Thus pseudomorphisms preserve inertial tolerance.

2.3. If α does not depend on q (the state arrived at), then $\beta(q) \,.\, \alpha(x) = \beta(q \,.\, x)$ where $\alpha : X \to X'$. This is a *homomorphism* of automata (Ginzburg [3]). It is clearly not suitable in this case to pass to classes $[(\alpha, \beta)]$. If β is bijective and its inverse is a homomorphism we have an *isomorphism* between A and A'.

2.4. Definition. The pseudomorphism $[(\alpha, \beta)]$ is a *pseudo-isomorphism* represented by (α, β, α') if β is bijective and there exists $\alpha' : X' \times Q' \to X$ such that

$$\beta^{-1}(q') \,.\, \alpha'(x', q') = \beta^{-1}(q' \,.\, x')$$

where $x' \in X'$, $q' \in Q'$. Thus $[(\alpha', \beta^{-1})]$ is also a pseudo-isomorphism, and $q \,.\, \alpha'(\alpha \,.\, (x, q), \beta(q)) = q \,.\, x$. But α, α' are not necessarily surjective, and $\alpha'(\alpha(x,q), \beta(q))$ need not be x. Both these claims are illustrated in the example of the appendix. We denote the above pseudo-isomorphism by $[(\alpha, \beta, \alpha')]$.

2.5. Lemma. Pseudo-isomorphism is an equivalence relation $\simeq$ on the set of automata.

Proof. (i) $A \sim A$ is represented by $(u, 1, u)$ where $q \,.\, u(x, q) = q \,.\, x$, e.g. $\alpha(x, q) = x$. Then $[(u, 1, u)]$ is the identity pseudo-isomorphism.

(ii) If $A \sim A'$ is represented by (α, β, α'), then $(\alpha', \beta^{-1}, \alpha)$ represents $A' \simeq A$.

(iii) If (α, β, α') represents $A \simeq A'$ and $(\bar{\alpha}, \bar{\beta}, \bar{\alpha}')$ represents $A' \simeq A''$ then $A \simeq A''$ is represented by $(\hat{\alpha}, \bar{\beta}\beta, \hat{\alpha}')$ where

$$\hat{\alpha}(x, q) = \bar{\alpha}(\alpha(x, q), \beta(q))$$

$$\hat{\alpha}'(x'', q'') = \alpha'(\bar{\alpha}'(x'', q'')\,, \quad \bar{\beta}^{-1}(q'')) \quad \text{for} \quad x \in X\,, \quad x'' \in X''\,,$$

$q \in Q$, $q'' \in Q''$. Verification that $(\hat{\alpha}, \bar{\beta}\beta, \hat{\alpha}')$ does represent a pseudo-isomorphism follows without difficulty, e.g.,

$$\begin{aligned}\bar{\beta}\beta(q) \,.\, \hat{\alpha}(x, q) &= \bar{\beta}\beta(q) \,.\, \bar{\alpha}(\alpha(x, q), \beta(q)) \\ &= \bar{\beta}(\beta(q) \,.\, \alpha(x, q)) \\ &= \bar{\beta}\beta(q \,.\, x)\end{aligned}$$

Similarly, $(\hat{\alpha}', (\bar{\beta}\beta)^{-1})$ represents a pseudo-morphism.

2.6. Lemma. The set of pseudo-isomorphisms fiom A to itself forms a group.

Proof. Closure follows from transitivity (2.5 (iii)). The identity is defined in 2.5 (i). The inverse of $[(\alpha, \beta, \alpha')]$ is represented by $(\alpha', \beta^{-1}, \alpha)$ since $\beta^{-1}\beta = 1$, and in the notation of 2.5 (iii), $\hat{\alpha}(x, q) = \alpha'(\alpha(x, q), \beta(q))$, so $q \,.\, \hat{\alpha}(x, q) = q \,.\, x$. Thus $[(\hat{\alpha}, \beta^{-1}\beta, \hat{\alpha}')] = [(u, 1, u)] =$ identity.

It remains to verify associativity. Let $\gamma, \bar{\gamma}, \dot{\gamma} : A \simeq A$ be represented by (α, β, α'), $(\bar{\alpha}, \bar{\beta}, \bar{\alpha}')$, $(\dot{\alpha}, \dot{\beta}, \dot{\alpha}')$ respectively. Then, again using the notation of 2.5 (iii), $\gamma\bar{\gamma} = [(\hat{\alpha}, \bar{\beta}\beta, \hat{\alpha}')]$, and $(\gamma\bar{\gamma})\,\dot{\gamma} = [(\tilde{\alpha}, \dot{\beta}\bar{\beta}\beta, \tilde{\alpha}')]$ where

$$\begin{aligned}\tilde{\alpha}(x, q) &= \dot{\alpha}(\hat{\alpha}(x, q), \bar{\beta}\beta(q)) \\ &= \dot{\alpha}(\bar{\alpha}(\alpha(x, q), \beta(q)), \bar{\beta}\beta(q))\end{aligned}$$

And $\bar{\gamma}\dot{\gamma} = [(\alpha_\wedge, \dot{\beta}\bar{\beta}, \alpha'_\wedge)]$ with $\alpha_\wedge(x, q) = \dot{\alpha}(\bar{\alpha}(x, q), \bar{\beta}(q))$
Thus $\gamma(\bar{\gamma}\dot{\gamma}) = [(\alpha_\sim, \dot{\beta}\bar{\beta}\beta, \alpha'_\sim)]$ where

$$\begin{aligned}\alpha_\sim(x, q) &= \alpha_\wedge(\alpha(x, q), \beta(q)) \\ &= \dot{\alpha}(\bar{\alpha}(\alpha(x, q), \beta(q)), \bar{\beta}\beta(q))\end{aligned}$$

Similariy $\tilde{\alpha}'$ and $\alpha'_\sim$ have the same action.

3. HOMOMORPHISMS

3.1. Let $[(\alpha, \beta)]$ be a pseudomorphism from $A = (X, Q)$ to $A' = (X', Q')$. Let $\bar{Q} \subseteq Q \times Q$ be the diagonal $\{(q, q) : q \in Q\}$, and let $\omega(x, q) = x$.

We recall that a function $\alpha : X \times Q \to X'$ defines a *cascade product* $A\alpha A' = (X, Q \times Q')$ with input action $(q, q') \,.\, x = (q \,.\, x, q' \,.\, \alpha(x, q))$.

Define $f : \bar{Q} \to Q \times Q'$ by $f(q, q) = (q, \beta(q))$.

Lemma. (f, ζ) is a homomorphism from $(X, \bar{Q} \subseteq A\omega A)$ to $A\alpha A' = (X, Q \times Q')$, with ζ the identity on X.

Proof.

$$\begin{aligned} f(q, q) \cdot x &= (q, \beta(q)) \cdot x \\ &= (q \cdot x, \beta(q) \cdot \alpha(x, q)) \\ &= (q \cdot x, \beta(q \cdot x)) \\ &= f(q \cdot x, q \cdot x) \end{aligned}$$

Corollary. When β is surjective $A\omega A$ covers or simulates $A\alpha A'$ (Ginzburg [3]), i.e. f maps a subset of $Q \times Q$ onto $Q \times Q'$ and satisfies the relation established in the lemma.

Figure 1 illustrates the homomorphism $(f, 1)$

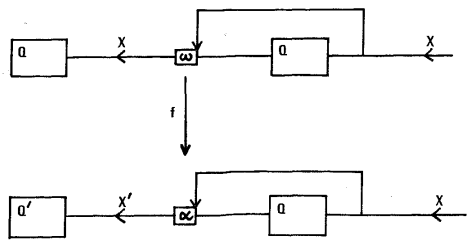

Fig. 1.

3.2. Lemma. A pseudo-isomorphism $[(\alpha, \beta, \alpha')]$ may be represented by an isomorphism f from the diagonal of $A\omega A\omega A = (X, Q \times Q \times Q)$ to $A\alpha A'\alpha' A = (X, Q \times Q' \times Q)$ given by $f(q, q, q) = (q, \beta(q), q)$.

Proof.

$$\begin{aligned} f(q, q, q) \cdot x &= (q, \beta(q), q) \cdot x \\ &= (q \cdot x, (\beta(q), q) \cdot \alpha(x, q)) \\ &= (q \cdot x, \beta(q) \cdot \alpha(x, q), \\ &\qquad q \cdot \alpha'(\alpha(x, q), \beta(q))) \\ &= (q \cdot x, \beta(q \cdot x), q \cdot x) \\ &= f(q \cdot x, q \cdot x, q \cdot x). \end{aligned}$$

4. PSEUDO-HOMOGENEITY

4.1. A *permutation automaton* $A = (X, Q)$ is defined in Warner [6] as having permutation inputs only. If G is the group generated by X then the above automaton is *homogeneous* if for all $q, q' \in Q$ there exists $g \in G$ such that $g(q) = q'$. Let H be the subgroup of G which fixes some $q_0 \in Q$. It is proved in [6] that the homogeneous automaton A is isomorphic to the *group quotient* automaton $(X, G/H)$ with action $[g] . x = [xg]$. Hence [ ] denotes equivalence class under the relation "$g \sim g'$ iff $g(q_0) = g'(q_0)$". The isomorphism $\tilde{\Phi} : G/H \to Q$ is well-defined by $\tilde{\Phi}[g] = g(q_0)$.

4.2. If in addition the permutation inputs of X in this homogeneous automaton and their inverses are *inertia-preserving* (1.3), i.e. $q_1 \upsilon q_2 \Rightarrow q_1 . x \upsilon q_2 . x, q_1 . x^{-1} \upsilon q_2 . . x^{-1}$, for all $x \in X$, then the group G above is a *υ-group* in which υ is preserved by left and right translations by group elements ([6], Lemma 4). The elements of G are isomorphisms of A with $\alpha = 1$.

4.3. A *pseudo-homogeneous* automaton is taken to be one which is indistinguishable in its υ-structure from a homogeneous automaton. Formally, $A = (X, Q)$ is *pseudo-permutational* if it is pseudo-isomorphic to a permutation automaton $\bar{A} = (\bar{X}, \bar{Q})$. Let $[(\alpha, \beta)]$ be such a pseudo-isomorphism. Without loss of generality we may take $\beta = 1$ and identify Q and $\bar{Q}$. Then if $q_2 = q_1 . x$ for some $x \in X, \alpha(x, q_1) = \bar{x} \in \bar{X}$ such that $q_2 = q_1 . \bar{x}$. Finally, such an automaton is pseudo-homogeneous if for all $q_1, q_2 \in Q$ there exists $g \in G$ (the group generated by $\bar{X}$) such that $g(q_1) = q_2$.

4.4. Now let the permutation inputs of $\bar{X}$ generate pseudomorphisms $[(\alpha, \beta)]$ of $\bar{A}$ to itself. Then for $\bar{x} \in \bar{X}, \beta(q) = q . \bar{x}$ and there exists $\alpha : \bar{X} \times \bar{Q} \to \bar{X}$ such that $(q . \bar{x}) . \alpha(\bar{x}_1, q) = q . \bar{x}_1\bar{x}$. Define $\beta^{-1}(q) = q . \bar{x}^{-1}$ even though $\bar{x}^{-1}$ may not be an element of $\bar{X}$. If there exists $\alpha' : \bar{X} \times \bar{Q} \to \bar{X}$ such that $(q . \bar{x}^{-1}) . \alpha'(x_1, q) = = q . \bar{x}_1\bar{x}$ then $[(\alpha, \beta)]$ is a pseudo-isomorphism. The group generated by such pseudo-isomorphisms is a subgroup of the group of all pseudo-isomorphisms of A described in Lemma 2.6.

4.5. Theorem. If each permutation input associated with a pseudo-homogeneous automaton $A = (X, Q)$ generates a pseudo-isomorphism then the automaton A is pseudo-isomorphic to a group quotient automaton whose group G is a υ-group.

Proof. A is pseudo-isomorphic to $\bar{A}$ which is homogeneous and therefore isomorphic to a group quotient automaton. The fact that the elements of $\bar{X}$ and their inverses are inertia-preserving follows immediately from 4.4. Let $\bar{x}, \bar{x}_1 \in \bar{X}$ and $q_2 = q_1 . \bar{x}_1$, so $q_1 \upsilon q_2$. Then $q_2 . \bar{x} = q_1 . \bar{x}_1\bar{x} = (q_1 . \bar{x}) . \alpha(\bar{x}_1, q)$, and $q_1 . \bar{x} \upsilon q_2 . \bar{x}$. Similarly for inverses $\bar{x}^{-1}$.

5. HOMOLOGY

5.1. It has already been pointed out (1.4) that a pseudo-morphism between automata provides a simplicial map between their associated Dowker simplicial complexes. In fact any homology theory based on the inertial relation υ is equally well served by pseudomorphisms.

5.2. Bearing in mind the definition of υ, namely $q_1 \upsilon q_2$, $q_1, q_2 \in Q$, iff there exists $x_1 \in X$ such that $q_2 = q_1 \cdot x_1$ we define a singular cubical homology theory on Q (cf. a similar treatment for tolerance spaces in Warner [4]).

5.3. *The standard n-cube* e_n is the subset of Hilbert space consisting of points $(u_i) = (u_1, u_2, \ldots, u_p, \ldots)$ such that $u_i = 0$, $i > n$, each $u_i = 0$ or 1, $i \leqq n$. The cube e_n has the relation $(u_i) \upsilon (u_i')$ iff the points (u_i), (u_i') differ in at most one co-ordinate, say the jth, and $u_j < u_j'$.

5.4. The *chain group* $C_n(Q)$ is the free abelian group generated by the set of morphisms $f: e_n \to Q$, called *singular n-cubes*. The boundary $\partial: C_n(Q) \to C_{n-1}(Q)$ is defined as usual by $\partial f = \sum_{i=1}^{n} (-1)^i (f^{i0} - f^{i1})$ where the *face* $f^{ij}: e_{n-1} \to Q$ is the composition of f with $v_{ij}: e_{n-1} \to e_n$, $v_{ij}(u_1, u_2, \ldots, u_{n-1}, 0, \ldots) = (u_1, u_2, \ldots, u_{i-1}, j, u_i, \ldots, u_{n-1}, 0, \ldots)$. Then $\partial\partial = 0$ and $(C(Q), \partial)$ is a chain complex. We normalise by factoring out degenerate cubes, viz. cubes which do not depend on all their co-ordinates. The normalised homology groups will be denoted by $H_n(Q)$. A brief résumé of the required terminology and concepts of classical homology theory is given in the appendix to [4].

5.5. Lemma. A pseudomorphism $[(\alpha, \beta)]$ between automata $A = (X, Q)$ and $A' = (X', Q')$ induces a chain homomorphism $\bar{\beta}: C(Q) \to C(Q')$.

Proof. β is a morphism from (Q, υ) to (Q', υ') by (1.4). Let $\bar{\beta}_n(f) = \beta f$ where $f \in C_n(Q)$. Then $\beta f \in C_n(Q')$. And

$$\partial \bar{\beta}_n(f) = \partial \beta f = \sum_{i=1}^{n} (-1)^i ((\beta f)^{i0} - (\beta f)^{i1}),$$

$$\bar{\beta}_{n-1}(\partial f) = \beta\, \partial f = \beta \sum_{i=1}^{n} (-1)^i (f^{i0} - f^{i1}),$$

while $\beta f^{ij}(e_{n-1}) = \beta f v_{ij}(e_{n-1}) = (\beta f)^{ij}(e_{n-1})$.
So $\bar{\beta}$ is a chain homomorphism.

5.6. In the function space Q'^{Q} of morphisms from Q to Q', let $\beta_1 \upsilon \beta_2$ iff $\beta_1(q) \upsilon \beta_2(q)$ for all $q \in Q$.

Lemma. If $\beta_1 \upsilon \beta_2$ then the induced chain homomorphisms are chain homotopic.

Proof. For all n, define $\Delta_n : C_n(Q) \to C_{n+1}(Q')$ as follows. Let $f : e_n \to Q$ be a singular n-cube of Q. Then $\Delta_n f = h$ where

$$h(u_1, u_2, \ldots, u_{n+1}) = \begin{cases} \beta_1 f(u, \ldots, u_{n+1}) & \text{when} \quad u_1 = 0 \\ \beta_2 f(u, \ldots, u_{n+1}) & \text{when} \quad u_1 = 1 \end{cases}$$

And $h \in C_{n+1}(Q')$ since h is a morphism.

This follows on the first co-ordinate from

$$\beta_1 f(u_2, \ldots, u_{n+1}) \, \upsilon \, \beta_2(f(u_2, \ldots, u_{n+1})$$

and on the other co-ordinates from the fact that β_1, β_2 are morphisms

$\{\Delta_n\}$ is the required chain homotopy, for if $f \in C_{n+1}$,

$$\begin{aligned} \Delta_n \partial f &= \Delta_n \sum_{i=1}^{n+1} (-1)^i (f^{i0} - f^{i1}) \\ &= \Delta_n f^{11} - \Delta_n f^{10} + \sum_{i=2}^{n+1} (-1)^i (\Delta_n f^{i0} - \Delta_n f^{i1}) \\ &= \beta_2 f - \beta_1 f - \partial \Delta_n f . \end{aligned}$$

5.7. By classical homology theory, then, β_1 and β_2 induce the same homomorphism of homology groups. But the inertial relation requires each input $x \in X$ to satisfy the relation $1 \upsilon x$ since $q \upsilon q \, . \, x$ for all q. We therefore deduce the following lemmas.

Lemma. Inertia-preserving inputs to an automaton $A = (X, Q)$ induce the identity homology homomorphism on $H_n(Q)$ for all n.

Lemma. A pseudomorphism of A to itself in which $1 \upsilon \beta$ induces the identity homology homomorphism on $H(Q)$.

5.8. α_n be the number of non-degenerate singular n-cubes of Q, and $\chi = \sum_i (-1)^i \alpha_i$. It follows immediately from the Hopf trace theorem (see [4], appendix) that if $\chi \neq 0$ every chain homomorphism chain homotopic to the identity maps a generator of some C_n to itself. Thus we have the following theorems.

Theorem. If $\chi \neq 0$, every inertia-preserving input to the automaton $A = (X, Q)$ maps some cube of Q to itself.

Theorem. Given an automaton $A = (X, Q)$ and the inertial normalised cubical singular chain complex $C(Q)$ of Q, then a pseudomorphism $[(\alpha, \beta)]$ of A to itself in which $q \upsilon \beta(q)$ for all $q \in Q$ maps some cube of $C(Q)$ to itself whenever $\chi \neq 0$.

APPENDIX

Example. Let $A = (X, Q)$ where $X = \{x_1, x_2, x_3\}$, $Q = \{q, q \cdot x_1 = q \cdot x_2, q, x_3\}$. The rest of the action of X on Q is given by

	q	$q \cdot x_1$	$q \cdot x_3$
x_1	$q \cdot x_1$	$q \cdot x_1$	$q \cdot x_3$
x_2	$q \cdot x_2$	q	$q \cdot x_3$
x_3	$q \cdot x_3$	q	$q \cdot x_3$

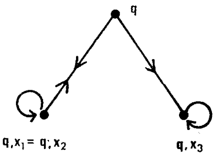

Fig. 2.

Let $A' = (X', Q')$ where $X' = \{x_1', x_2', x_3'\}$ and $Q' = \{\beta(q), \beta(q \cdot x_1), \beta(q \cdot x_3)\}$ with action

	$\beta(q)$	$\beta(q \cdot x_1)$	$\beta(q \cdot x_3)$
x_1'	$\beta(q \cdot x_1)$	$\beta(q \cdot x_1)$	$\beta(q \cdot x_3)$
x_2'	$\beta(q \cdot x_3)$	$\beta(q)$	$\beta(q \cdot x_3)$
x_3'	$\beta(q \cdot x_3)$	$\beta(q \cdot x_1)$	$\beta(q \cdot x_3)$

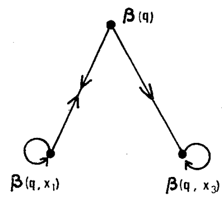

Fig. 3.

With $\beta : Q \to Q'$ the bijection given above, define $\alpha : X \times Q \to X'$ by

$$\alpha(x_1, q) = x_1' \quad \alpha(x_1, q \cdot x_1) = x_1' \quad \alpha(x_1, q \cdot x_3) = x_1'$$
$$\alpha(x_2, q) = x_1' \quad \alpha(x_2, q \cdot x_1) = x_2' \quad \alpha(x_2, q \cdot x_3) = x_1'$$
$$\alpha(x_3, q) = x_2' \quad \alpha(x_3, q \cdot x_1) = x_2' \quad \alpha(x_3, q \cdot x_3) = x_1'$$

Then $[(\alpha, \beta)]$ is a pseudo-isomorphism. We can check that $\beta(q \cdot x) = \beta(q) \cdot \alpha(\in, q)$ $\forall x \in X$, $q \in Q$.

And we can construct α' e.g. $\alpha'(x_1', \beta(q)) = x_1$, $\alpha'(x_2', \beta(q)) = x_3$, $\alpha'(x_3', \beta(q)) = x_3$ etc. Then α is not surjective (every action x_3' can be obtained from x_1' or x_2'). And $\alpha'(\alpha(x_2, q), \beta(q)) = \alpha'(x_1', \beta(q)) = x_1 \neq x_2$ although $q \cdot x_1 = q \cdot x_2$, so $q \cdot \alpha' \cdot (\alpha(x_2, q), \beta(q)) = q \cdot x_2$.

(Received October 15, 1981.)

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