SOME NEW RESULTS IN STATE SPACE DECOUPLING OF MULTIVARIABLE SYSTEMS II

Extensions to Decoupling of Systems with $D \neq 0$ and Output Feedback Decoupling

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The results of Part I: "A Link Between Geometric Approach and Matrix Methods" are extended to apply to causal ($D \neq 0$) multivariable systems. Conditions for output feedback group decoupling, useful for engineering calculations, are derived and treated in conjunction with the structure of the poles of decoupled system.

1. DECOUPLING OF MULTIVARIABLE SYSTEMS WITH $D \neq 0$

The case $D \neq 0$ was treated in the sense of geometric approach by Morse [6]. This paper is purely abstract, without a note about the possibility of implementing the theoretic results in a convenient algorithmic form. Our geometric formulation of decoupling problem for systems with $D \neq 0$ seems to be simpler and throws light on an immediate matrix interpretation of derived results.

To zero system output, clearly state feedback has to zero the relevant output controllability subspace. In the case $D = 0$ the problem of zeroing the output is simpler, as output contrallable subspace $\mathscr{V}_0$ is a projection of the (state) controllable subspace $\mathscr{R}_0$. Really, if $\mathscr{R}_0 \subset \operatorname{Ker} C$ (resp. $\mathscr{R}_0 \subset \Upsilon^*$) then the controllable subspace $\mathscr{R}_0$ is mapped to a zero output subspace. Let a state feedback (F, G) zero output controllability subspace. Then the controllability subspace

$$\mathscr{R} = \sum_{j \in \bar{n}} (A + BF)^{j-1} \langle BG \rangle$$

has to be contained in $\operatorname{Ker}(C + DF)$ (resp. in Υ^*). Because of $D \neq 0$, in order for $\mathscr{V}$ to be zeroed, the controllability subspace $\mathscr{R}$ has to be restricted by the condition $\langle G \rangle \subset \operatorname{Ker} D$. Then the problem of zeroing the output may be formulated in the following way.

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Find a state feedback (F, G) that constructs controllability subspace $\mathscr{R}$,

$$\mathscr{R} = \sum_{j \in \bar{n}} (A + BF)^{j-1} \langle BG \rangle$$

contained in the maximal unobservable subspace Υ^* and satisfies the restricting condition

$$\langle G \rangle \subset \operatorname{Ker} D . \tag{32}$$

The decoupling problem is more complex than that of zeroing the output. Utilizing subsystems S_i^*, $i \in \bar{l}$, decoupling may be formulated as simultaneously zeroing the outputs of all subsystems S_i^*, $i \in \bar{l}$ in the sense of the foregoing formulation with a restriction on the ranks of output controllability matrices to be invariant under the state feedback.

Summarizing, if there exists a state feedback (F, G) which decouples the system S by (1), not necessarily completely output controllable, then controllability subspaces $\mathscr{R}_i$, generated by new inputs v_i, $i \in \bar{l}$:

$$\mathscr{R}_i = \sum_{j \in \bar{n}} (A + BF)^{j-1} (\langle B \rangle \cap \mathscr{R}_i) = \sum_{j \in \bar{n}} (A + BF)^{j-1} \langle BG_i \rangle , \quad i \in \bar{l} \tag{33}$$

have to satisfy the following requirements:

$$\mathscr{R}_i \subset \Upsilon_i^* , \tag{34}$$

$$\langle G_i \rangle \subset \operatorname{Ker} \bar{D}_i^* , \quad i \in \bar{l} , \quad G = [G_1 \vdots \ldots \vdots G_l] \tag{35}$$

and

$$(C_i + D_i F) \mathscr{R}_i + \langle DG_i \rangle = C_i (\sum_{j \in \bar{n}} A^{j-1} \langle B \rangle) + \langle D_i \rangle , \quad i \in \bar{l} . \tag{36}$$

Note that conditions (34), (35) ensure zeroing of the output of the i-th subsystem S_i^*, $i \in \bar{l}$, hence imply noninteraction. Condition (36) appears to be an extension of output controllability condition (6). Clearly for the controllability subspaces $\mathscr{R}_i$, $i \in \bar{l}$ to be simultaneously constructed by state feedback, their compatibility

$$\bigcap_{i \in \bar{l}} \mathbb{F}(\mathscr{R}_i) \neq \emptyset \tag{37}$$

is required.

As there are no necessary and sufficient conditions for compatibility of arbitrary controllability subspaces, the concept of maximal controllability subspaces will be used in order to find sufficient compatibility conditions.

Let $\mathscr{R}_i^*$ be the maximal controllability subspace contained in the maximal unobservable subspace Υ_i^*, $i \in \bar{l}$. Because of (A, B)-invariance and maximality of subspaces $\mathscr{R}_i^*$, $i \in \bar{l}$, they may be defined, using Theorem 1.1, to be:

$$\mathscr{R}_i^* = \sum_{j \in \bar{n}} (A + BF_{\alpha i})^{j-1} (\langle B \rangle \cap \Upsilon_i^*) = \sum_{j \in \bar{n}} (A + BF_{\alpha i})^{j-1} \langle BG_i \rangle , \quad F \in \mathbb{F}(\Upsilon_i^*) . \tag{38}$$

According to Construction 3 we have:

$$(39) \qquad \mathscr{R}_i^* = \sum_{j \in \bar{n}} (A + BF_{\alpha i})^{j-1} \langle B\bar{G}_i \rangle \,,$$

$$\langle G_i \rangle = \operatorname{Ker}(L_{\beta i}^* B) \,, \quad i \in \bar{l} \,,$$

where $F_{\alpha i}$ is a map constructing the maximal unobservable subspace Υ_i^*. By Theorem 3.1 subspaces $\mathscr{R}_i^*$, $i \in \bar{l}$ will be compatible if and only if subspaces Υ_i^*, $i \in \bar{l}$ are compatible, hence (20) is necessary and sufficient condition for the maximal controllability subspaces $\mathscr{R}_i^*$ contained in subspaces Υ_i^*, $i \in \bar{l}$ to be compatible. Clearly, by (38), controllability subspaces $\mathscr{R}_i^*$, $i \in \bar{l}$ satisfy condition (34). For (35) to hold validity of the condition

$$(40) \qquad \langle G_i \rangle \subset \operatorname{Ker} D_i^* \,, \quad i \in \bar{l}$$

is required. For controllability subspaces $\mathscr{R}_i^*$, $i \in \bar{l}$ restricted by (40) we get:

$$(41) \qquad \hat{\mathscr{R}}_i^* = \sum_{j \in \bar{n}} (A + BF_{\alpha 1})^{j-1} \langle BG_i \rangle \,,$$

where

$$\langle G_i \rangle \subset \operatorname{Ker}(L_{\beta i}^* B) \,, \quad i \in \bar{l} \,.$$

Finally, using (19) we have

$$(42) \qquad \langle G_i \rangle = \operatorname{Ker} \begin{bmatrix} D_i^* \\ \hline L_{\beta i} B \end{bmatrix} , \quad i \in \bar{l} \,.$$

Recall that matrices $F_{\alpha i}$ in (39)–(41) construct the subspaces Υ_i^*, $i \in \bar{l}$.

Summarizing, if (20) is true, a class of compatible controllability subspaces

$$(43) \qquad \hat{\mathscr{R}}_i^* = \sum_{j \in \bar{l}} (A + BF_\alpha)^{j-1} \langle BG_i \rangle \,, \quad \langle G_i \rangle = \operatorname{Ker} \bar{D}_{\alpha i}^* \,, \quad i \in \bar{l} \,,$$

where F_α is according to (21), satisfies noninteration conditions (34), (35). Matrix F_α constructs simultaneously all maximal unobservable subspace Υ_i^*, $i \in \bar{l}$. It is easy to prove that if $\mathscr{R}_i^*$ is the maximal controllability subspace contained in Υ_i^*, then subspace $\hat{\mathscr{R}}_i^*$ (41) is maximal among all controllability subspaces satisfying both conditions (34) and (35), and these subspaces are similar to $\mathscr{R}_i^*$, $i \in \bar{l}$ in the case $D = 0$. Then the condition

$$(44) \qquad \bigcap_{j \in \bar{l}} \Upsilon_j^* \neq \emptyset$$

appears to be a necessary condition for decoupling.

Similarly to the case $D = 0$, condition (20) appears to be necessary and sufficient only for subspaces $\hat{\mathscr{R}}_i^*$, $i \in \bar{l}$ to be compatible. If (20) fails, there may exist smaller controllability subspaces, compatible, large enough to preserve (after restriction (35)) the rank of output controllability matrix. Notice that from (41), (42) the class of compatible controllability subspaces satisfying noninteraction condition is constructed formally in the same way as the class of maximal controllability subspaces $\mathscr{R}_i^*$, $i \in \bar{l}$ when $D = 0$. As to the output controllability condition (36), it is evident that it may be rewritten into a matrix form:

$$\text{rank}\,[(C_i + D_i F_{\alpha i})(BG_i \vdots \ldots \vdots (A + BF_{\alpha i})^{n-1} BG_i) \vdots D_i G_i] = \tag{45}$$
$$= \text{rank}\,[C_i B \vdots \ldots \vdots C_i B A^{n-1} \vdots D_i]\,, \quad i \in \bar{l}\,.$$

Because of clear similarity to the case $D = 0$, the foregoing geometric formulation will be stated as an algorithm for state feedback decouping of systems with $D \neq 0$, not necessarily completely output controllable, with no further comments.

Algorithm 2.

1. Determine subsystems S_i^*, $i \in \bar{l}$ according to a desired configuration of output blocks.

2. Apply Construction 2 to every subsystem S_i^*, and derive matrices $\bar{C}_{\alpha i}^*$, $\bar{D}_{\alpha i}^*$, $i \in \bar{l}$.

3. Compute matrices $F_{\alpha i}$ of state feedback by (18), reducing maximally the observability of subsystems S_i^*, $i \in \bar{l}$.

4. Construct controllability subspaces $\hat{\mathscr{R}}_i^*$, $i \in \bar{l}$, according to (41), (42).

5. Test necessary condition (45). If it fails, decoupling is not possible and the algorithm terminates.

6. Test compatibility condition (20). If it fails, state feedback decoupling in the sense of subspaces $\hat{\mathscr{R}}_i^*$, $i \in \bar{l}$ is not possible. Otherwise compute matrices F_α, G of state feedback which decouple the system:

$$F_\alpha = -\bar{D}_\alpha^{*+} \bar{C}_\alpha^*\,, \tag{46}$$

$$G = [G \vdots \ldots \vdots G_l]\,, \quad G_i = \overline{\text{Ker}\,\bar{D}_{\alpha i}^*}\,, \quad i \in \bar{l} \tag{47}$$

Example 1.2. Given system S by (1) with matrices:

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \quad B = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix}$$

$$C = \begin{bmatrix} 1 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \end{bmatrix} \quad D = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

whose outputs are partitioned into 3 groups, such that $m_1 = 1$, $m_2 = 2$, $m_3 = 1$.

According to Construction 2 for subsystems S_1^*, S_2^*, S_3^*, we have:

$$\bar{D}_{\alpha 1}^* = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix} \qquad \bar{C}_{\alpha 1}^* = \begin{bmatrix} 1 & 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \end{bmatrix}$$

$$\bar{D}_{\alpha 2}^* = \begin{bmatrix} 1 & 1 & 0 & 3 \\ 0 & 0 & 1 & 0 \end{bmatrix} \qquad \bar{C}_{\alpha 2}^* = \begin{bmatrix} 0 & 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 \end{bmatrix}$$

$$\bar{D}_{\alpha 3}^* = \begin{bmatrix} 1 & 1 & 0 & 3 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \qquad \bar{C}_{\alpha 3}^* = \begin{bmatrix} 0 & 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 & 0 \end{bmatrix}$$

Condition (20) is satisfied, hence a state feedback with the matrix

$$F_\alpha = \begin{bmatrix} -4 & -1 & 4 & 1 & -1 \\ -1 & 0 & -1 & -1 & 0 \\ 0 & -1 & 0 & 0 & 0 \\ -1 & 0 & -1 & 0 & 0 \end{bmatrix}$$

constructs all maximal unobservable subspaces Υ_i^*, $i \in \bar{3}$. For matrices G_i, $i \in l$ and G, we get according to (42)

$$G_1' = [1 \ 0 \ 0 \ 0] \qquad G_2' = \begin{bmatrix} -3 & 0 & 0 & 1 \\ 1 & -1 & 0 & 0 \end{bmatrix} \qquad G_3' = [0 \ 0 \ 1 \ 0]$$

$$G = \begin{bmatrix} 1 & -3 & 1 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \end{bmatrix}$$

Since G is nonsingular, the necessary condition (45) has not to be tested. The transfer function of the decoupled system is:

$$S^{(\alpha)}(p) = \frac{1}{p^3 - 2p^4} \begin{bmatrix} p^4 - 2p^3 + p - 1 & 0 & 0 & 0 \\ 0 & 0 & 2p^4 - p^3 & 0 \\ 0 & p^3 - 2p^4 & 0 & 0 \\ 0 & 0 & 0 & p^3 - 2p^4 \end{bmatrix}$$

2. STABILITY OF THE DECOUPLED SYSTEM

The stability of decoupled system was completely solved by Wonham and Morse [8] and in the same way, also for systems with $D \neq 0$, by Silverman and Payne [7]. For that reason only final results with some new comments, connected with the further expressions, will be presented in this section.

According to Wonham and Morse [8], the poles of decoupled system are freely assignable by state feedback if the state space may be represented as a direct sum of maximal controllability subspaces $\mathscr{R}_i^*$, $i \in l$ (controllability of S is assumed), i.e.:

$$\sum_{i \in l} \oplus \mathscr{R}_i^* = \mathscr{X} . \tag{48}$$

Clearly, if (48) is true, compatibility is guaranteed too, because of $\mathscr{R}^{*\Sigma} = 0$, hence, (A, B)-invariance. This sufficient condition for decoupling (see also Remark 1.1) is concluded in the following theorem (Silverman and Payne [7]):

Theorem 1.2. If necessary condition (8) is fulfilled and the $(n \times \dim \mathscr{R}_i^*)$ matrices $\bar{R}_i^*$ formed from the bases of the maximal controllability subspaces $\mathscr{R}_i^*$, $i \in l$ satisfy the conditions:

$$\operatorname{rank} [\bar{R}_1^* \vdots \ldots \vdots \bar{R}_l^*] , \quad \sum_{i \in l} \operatorname{rank} \bar{R}_i^* = n , \tag{49}$$

then state feedback (2) with matrices:

$$F = \begin{bmatrix} - [\bar{D}_{\alpha 1}^{*+} \bar{C}_{\alpha 1}^* \bar{R}_1^* \vdots \ldots \ D_{\alpha l}^{*+} \bar{C}_{\alpha l}^* \bar{R}_l^*] + G \begin{bmatrix} F_1 & & 0 \\ & \ddots & \\ 0 & & F_l \end{bmatrix} \end{bmatrix} \bar{R}^{*-1} \tag{50}$$

$$G = [G_1 \vdots \ldots \vdots G_l] \qquad G = \overline{\operatorname{Ker} \bar{D}_{\alpha i}^*} , \quad i \in l \tag{51}$$

where $(\bar{r}_i \times \dim \mathscr{R}_i^*)$ matrices F_i, $i \in l$ are arbitrary, decouples the system and freely assigns all the poles of the decoupled system.

Applying state feedback (F, G) by (50), (51) and transforming the state by linear transformation

$$x = \bar{R}^* \bar{x} \tag{52}$$

where

$$\bar{R}^* = [\bar{R}_1^* \vdots \ldots \vdots \bar{R}_l^*]$$

we get the matrices A_c, B_c of the system closed by state feedback (F, G)

$$A_c = \begin{bmatrix} \bar{A}_1 & & 0 \\ & \ddots & \\ 0 & & \bar{A}_l \end{bmatrix} + \begin{bmatrix} \bar{B}_1 F_1 & & 0 \\ & \ddots & \\ 0 & & \bar{B}_l F_l \end{bmatrix} , \qquad B_c = \begin{bmatrix} \bar{B}_1 & & 0 \\ & \ddots & \\ 0 & & \bar{B}_l \end{bmatrix} . \tag{53}$$

The pairs $\bar{A}_i$, $\bar{B}_i$ are controllable, hence arbitrary pole assignment is provided for every subsystem.

If $\sum_{i \in l} \dim \mathscr{R}_i^* < n$, subspaces $\mathscr{R}_i^*$, $i \in l$ are not disjoint and all poles of decoupled system are not freely assignable by state feedback. In order to test for fixed poles, state transformation:

$$\bar{x} = \left[\bar{R}^* \vdots \overline{\operatorname{Ker} \bar{R}^{*\prime}}\right] x \tag{54}$$

where matrices $\bar{R}^*$ and $\overline{\operatorname{Ker} \bar{R}^*}$ are formed resp. by all linearly independent columns of $\bar{R}^*$ and by all basis vectors of complement $\bar{R}^{*\perp}$, has to be performed. The fixed poles will be canceled as a result of connected subspaces $\mathscr{R}_i^*$, $i \in l$ and observability of the decoupled system will be reduced. Clearly controllable systems with stable invariant zeros have all fixed poles stable, too. Despite of block diagonality of transfer function matrix, decoupled system will not consist of independent subsystems and internal (state) interaction will occur.

Remark 1.2. A sufficient condition for subspaces $\mathscr{R}_i^*$, $i = l$ to be disjoint is the independence of the relevant maximal unobservable subspaces $\mathscr{R}_i^*$, $i \in l$, hence perfect observability. Therefore, the maximal controllability subspaces Υ_i^*, $i \in l$ of system S with full row rank matrix C will be always disjoint. Conversely, when Υ_i^*, $i \in l$ are not disjoint, but $\mathscr{R}_i^*$, $i \in l$ are, the stabilization may be interpreted as "sending back" the controllable modes of the unobservable part to make them transmission zeros of decoupled system.

The foregoing remark will be illustrated by an example.

Example 2.2. Given system by (1) with matrices:

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 2 & 3 & 0 \\ 1 & 1 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad C' = \begin{bmatrix} 1 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix},$$

with partition of outputs such that $m_1 = m_2 = 1$. To solve the decoupling problem we apply Algorithm 1, finding the matrices of state feedback:

$$F_\alpha = \begin{bmatrix} 2 & 4 & 0 \\ 1 & 1 & 1 \end{bmatrix}, \quad G = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

The maximal controllability subspaces

$$\mathscr{R}_1^* = \left\langle \begin{bmatrix} 0 & 1 \\ 1 & -1 \\ 0 & 0 \end{bmatrix} \right\rangle, \quad \mathscr{R}_2^* = \left\langle \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\rangle$$

are disjoint. Transfer function matrix of the closed loop system S_c is:

$$S_c(p) = \begin{bmatrix} \frac{1}{p} & 0 \\ 0 & \frac{1}{p} \end{bmatrix}.$$

The cancelation of mode (-1) in transfer function matrix is due to the fact that, according to Algorithm 1, matrix F_α of state feedback constructs both maximal unobservable supspaces Υ_1^*, Υ_2^* as

$$\Upsilon_1^* = \left\langle \begin{bmatrix} 1 & 1 \\ -1 & 0 \\ 0 & 0 \end{bmatrix} \right\rangle, \quad \Upsilon_2^* = \left\langle \begin{bmatrix} 1 & 0 \\ -1 & 0 \\ 0 & 1 \end{bmatrix} \right\rangle.$$

These subspaces are not disjoint (they have common mode (-1)). The given system has an invariant zero in (-1) that is also an output decoupling zero of the decoupled system. Because of the invariance of invariant zeros under state feedback, assigning all poles of decoupled system to be, for example, in (-2), we have the invariant zero (-1) which occurs as a transmission zero of the decoupled system $S^{(a)}$:

$$S^{(a)}(p) = \begin{bmatrix} \dfrac{p+1}{(p+2)^2} & 0 \\ 0 & \dfrac{1}{p+2} \end{bmatrix}.$$

Note that this would be impossible when the maximal controllability subspaces were connected, because of the fixed position of common modes.

3. OUTPUT FEEDBACK DECOUPLING OF MULTIVARIABLE SYSTEMS

In comparison with the state feedback decoupling, the output feedback decoupling was given rather insignificant place in the literature. Some authors (Falb and Wolovich [2], Hazlerigg and Sinha [4]) examine necessary and sufficient condition for decoupling by output feedback as conditions for solution of the matrix equation:

$$F = KC\,, \tag{54}$$

because of the trivial substitution (for systems with $D = 0$):

$$u = Ky + Gv = KCx + Gv\,.$$

Such a solution of the problem is an immediate consequence of a more general decoupling problem and if the decoupled system is stable, then the output feedback will ensure stability, too. Condition (54) is quite strong and only restricted group of systems satisfies it. The only paper that solves the output feedback decoupling problem (Howze [5]) also in the case when for a given F (54) fails, is devoted to the classical Morgan's problem. In this section we derive conditions for the general output feedback decoupling problem defined in Part I. The method used will be rather different than that of Howze [5], as the results for Morgan's problem may not be mechanically extended to the problem of group decoupling. Only systems (1) with $D = 0$ will be considered.

Definition 1.2. Output feedback (K, G) by (3) trivially decouples a given system if there exists a state feedback (F, G) decoupling the system and the matrix K is a solution of matrix equation (54).

It is well known that the very problem of pole assignment by output feedback may not be generally solved. Hence, Definition 1.2 is quite unrealistic, as state feedback that decouples the system has to ensure not only noninteraction, but also pole assignment and output feedback has to ensure the same requirements, too. In this paper the output feedback decoupling problem will be solved by compromising with the ability to arbitrarily assign the poles of the decoupled system.

Clearly, from (2) and (3), if output feedback (K, G) decouples the system, then there exists state feedback (F, G) decoupling the system, too.

Definition 2.2. Out put feedback (K, G) decouples a given system if the state feedback (F, G), where $F = KC$, decouples the system.

Expression (54) has not to be true for every state feedback decoupling the system. In contrast to the trivial solution, defined for a fixed state feedback matrix F, Definition 2.2 determines a solution for the class of all matrices F decoupling the system. Evidently, the class of all matrices F defined by Definition 2.2 includes the class of all trivial solutions in the sense of Definition 1.2.

Remark 2.2. A simple consequence of Definition 2.2 is that solvability of state feedback decoupling problem is a necessary condition for output feedback decoupling. Another necessary condition follows from the invariance of observability under output feedback (3). Really, if state feedback, decoupling the system, reduces observability (the maximal controllability subspaces $\mathscr{R}_i^*$, $i \in \bar{l}$ are not independent), matrix K may not exist as the observability is invariant under output feedback.

The class of all matrices (F, G) which stable decouple the system in the sense of maximal controllability subspaces is given by Theorem 1.2:

$$(55) \qquad F = -\left[[\bar{D}_{\alpha 1}^{*+} \bar{C}_{\alpha 1}^* \bar{R}_1^* \vdots \ldots \vdots \bar{D}_{\alpha l}^{*+} \bar{C}_{\alpha l}^* \bar{R}_l^*] + G \begin{bmatrix} F_1 & & 0 \\ & \ddots & \\ 0 & & F_2 \end{bmatrix} \right] \bar{R}^{*-1}$$

$$G = [G_1 \vdots \ldots \vdots G_l] \qquad G_i = \overline{\operatorname{Ker} \bar{D}_{\alpha i}^*}, \quad i \in \bar{l},$$

where matrices F_i of dimensions $(\bar{r}_i \times \dim \mathscr{R}_i^*)$ are arbitrary, $\bar{R}_i^*$ are the bases of subspaces $\mathscr{R}_i^*$, $i \in \bar{l}$ and $\bar{R}^*$ is formed from $\bar{R}_i^*$, $i \in \bar{l}$ as $\bar{R}^* = [\bar{R}_1^* \vdots \ldots \vdots \bar{R}_l^*]$. Substituting (55) for F in (54) and multiplying (54) on the right by matrix $\bar{R}^*$, we get:

$$(56) \qquad KC\bar{R}^* = -[\bar{D}_{\alpha 1}^{*+} \bar{C}_{\alpha 1}^* \bar{R}_1^* \vdots \ldots \vdots \bar{D}_{\alpha l}^{*+} \bar{C}_{\alpha l}^* \bar{R}_l^*] + G \begin{bmatrix} F_1 & & 0 \\ & \ddots & \\ 0 & & F_l \end{bmatrix}.$$

Recall that subspaces $\mathscr{R}_i^*$, $i \in \bar{l}$ are determined at Step 3 of Algorithm 1 by the expression:

$$\mathscr{R}_i^* = \sum_{j \in \bar{n}} (A + BF_{\alpha i})^{j-1} \langle BG_i \rangle , \tag{57}$$

where

$$G_i = \overline{\text{Ker } \bar{D}_{\alpha i}^*} = \overline{\text{Ker } (L_{\beta i}^* B)} , \quad i \in \bar{l} \tag{58}$$

and are disjoint, so $\bar{R}^*$ is nonsingular.

According to the structure algorithm, applied to subsystem S_i^*, $i \in \bar{l}$ by (4),

$$\langle L_{\beta i}^{*\prime} \rangle \not\supset \langle C_i' \rangle$$

because

$$\langle C_i^{*\prime} \rangle = \sum_{\substack{j \in \bar{l} \\ j \neq i}} \langle C_j^{*\prime} \rangle , \quad i \in \bar{l} . \tag{59}$$

Note, that (59) fails when rows of C_i are linear combinations of rows belonging to other submatrices C_j, $i \neq j$, $i, j \in \bar{l}$, but it is easy to see that such a system pertains to the case of inherent interaction and may not be decoupled by any control law. Then from (57), (58), (59) we get:

$$KC\bar{R}^* = K \begin{bmatrix} \dot{C}_1 & & 0 \\ & \ddots & \\ 0 & & \dot{C}_l \end{bmatrix} \tag{60}$$

where $\dot{C}_i$ are nonzero matrices of dimensions $(m_i \times \dim \mathscr{R}_i^*)$, $i \in \bar{l}$. According to (31), we have for the ranks of matrices $\dot{C}_i$:

$$\text{rank } \dot{C}_i = g_i , \quad i \in \bar{l}$$

where g_i denotes the rank of output controllability matrix of subsystem S_i, $i \in \bar{l}$. Equation (56) will be transformed to the form:

$$K \begin{bmatrix} \dot{C}_1 & & 0 \\ & \ddots & \\ 0 & & \dot{C}_l \end{bmatrix} = -[\bar{D}_{\alpha 1}^{*+} \bar{C}_{\alpha 1}^* \bar{R}_1^* \vdots \ldots \vdots \bar{D}_{\alpha l}^{*+} \bar{C}_{\alpha l}^* \bar{R}_l^*] + G \begin{bmatrix} F_1 & & 0 \\ & \ddots & \\ 0 & & F_l \end{bmatrix} . \tag{61}$$

The conditions for solution of the above matrix equation for K will be discussed further.

Denote

$$K = [K_1 \vdots \ldots \vdots K_l]$$

where K_i are arbitrary submatrices of dimensions $(r \times m_i)$, $i \in \bar{l}$. From (61) we get:

$$K_i \dot{C}_i = -\bar{D}_{\alpha i}^{*+} \bar{C}_{\alpha i}^* \bar{R}_i^* + G_i F_i , \quad i \in \bar{l} . \tag{62}$$

Substitute:

$$U_i = -\bar{D}_{\alpha i}^{*+} \bar{C}_{\alpha i}^* \bar{R}_i^* + G_i F_i , \quad i \in \bar{l} .$$

Then the necessary and sufficient condition for equation (62) to have a solution K_i is:

$$\text{rank} \begin{bmatrix} \dot{C}_i \\ \hline U_i \end{bmatrix} = \text{rank } \dot{C}_i , \quad i \in \bar{l} . \tag{63}$$

Assuming that the columns of $\dot{C}_i$, $i \in \bar{l}$ are not redundant, the above condition is equivalent to the condition:

$$(64) \qquad \operatorname{rank} \dot{C}_i = \dim \mathscr{R}_i^* , \quad i \in \bar{l}$$

(by redundant columns of $\dot{C}_i$ we mean those columns which may be zeroed if a linear transformation is performed to zero a maximal number of column of matrix $[\dot{C}_i' \,\vdots\, U_i']'$). If some redundant columns are available and rank $\dot{C}_i = \hat{g}_i < g_i$, then (64) will be modified to be:

$$\operatorname{rank} \dot{C}_i = \hat{g}_i , \quad i \in \bar{l} .$$

For unity of description the redundance will not be considered.

Some important conclusions are derived from the foregoing results.

(i) Only very special types of stable decouplable systems are also output feedback decouplable for a fixed matrix F (in the sense of Definition 1.2). The maximal controllability subspaces $\mathscr{R}_i^*$, $i \in \bar{l}$ of these systems have to be of dimensions equal to the rank g_i, $i \in \bar{l}$ of relevant output controllability matrices. Availability of (64) with rank $\dot{C}_i = g_i$, $i = \bar{l}$ implies quite strong condition rank $C = n$.

(ii) It is seen from (63) that solvability of output feedback decoupling problem depends on matrix F and suitably selecting matrices F_i, $i \in \bar{l}$ (hence poles of the decoupled system), the solvability of (62) may be modified.

Further we deal with determination of restrictions on matrices F_i, $i \in \bar{l}$ in (55) in order to ensure (if possible) solvability of (62), hence of (61).

Let T_i be nonsingular $(\dim \mathscr{R}_i^* \times \dim \mathscr{R}_i^*)$ transformation matrix of the type:

$$T_i = [T_{i1} \,\vdots\, T_{i2}] , \quad T_{i1} = \dot{C}_i^+ , \quad T_{i2} = \overline{\operatorname{Ker} \dot{C}_i} , \quad i \in \bar{l}$$

Then the following partition of matrix $\dot{C}_i$ is obvious:

$$(65) \qquad \dot{C}_i T_i = [\underbrace{\dot{C}_i^{(1)}}_{g_i} \,\vdots\, 0] , \quad \operatorname{rank} \dot{C}_i^{(1)} = g_i , \quad i \in \bar{l}$$

Multiplying (62) on the right by matrix T_i we get:

$$(66) \qquad K_i[\dot{C}_i^{(1)} \,\vdots\, 0] = -\bar{D}_{\alpha i}^{*+} \bar{C}_{\alpha i}^* [\overbrace{\bar{R}_i^{*(1)}}^{g_i} \,\vdots\, \bar{R}_i^{*(2)}] + G_i [\overbrace{F_i^{(1)}}^{g_i} \,\vdots\, F_i^{(2)}] , \quad i \in \bar{l}$$

i.e.

$$(67) \qquad K_i \dot{C}_i^{(1)} = -\bar{D}_{\alpha i}^{*+} \bar{C}_{\alpha i}^* \bar{R}_i^{*(1)} + G_i F_i^{(1)}$$

$$(68) \qquad 0 = -\bar{D}_{\alpha i}^{*+} \bar{C}_{\alpha i}^* \bar{R}_i^{*(2)} + G_i F_i^{(2)} , \quad i \in \bar{l} .$$

As rank $\dot{C}_i^{(1)} = g_i$ (from (65)), the first equation has always the solution:

$$K_i = (-\bar{D}_{\alpha i}^{*+} \bar{C}_{\alpha i}^* \bar{R}_i^{*(1)} + G_i F_i^{(1)}) \dot{C}_i^{(1)+} , \quad i \in \bar{l} .$$

and, for any matrix F_i, $i \in l$, it is possible to determine a matrix K_i, $i \in l$ satisfying (67). The second equation is not generally satisfied. Rewrite it to the form:

$$G_i F_i^{(2)} = \bar{D}_{\alpha i}^{*+} \bar{C}_{\alpha i}^* \bar{R}_i^{*(2)}, \quad i \in l. \tag{69}$$

Let $\hat{T}_i$ be nonsingular transformation matrix of the form:

$$\hat{T}_i = \begin{bmatrix} \bar{D}_{\alpha i}^* \\ \hdashline G_i^+ \end{bmatrix}$$

where

$$G_i^+ = (G_i' G_i)^{-1} G_i'$$

and nonsingularity is guaranted by Step 3 of Algorithm 1:

$$G_i = \overline{\operatorname{Ker} \bar{D}_{\alpha i}^*}, \quad i \in l.$$

Multiplying (69) on the left by $\hat{T}_i$ we get:

$$\begin{bmatrix} 0 \\ \hdashline I_{\bar{r}_i} \end{bmatrix} F_i^{(2)} = \begin{bmatrix} I_{q_i} \\ \hdashline 0 \end{bmatrix} \bar{C}_{\alpha i}^* \bar{R}_i^{*(2)}, \quad i \in l. \tag{70}$$

Finally we have from (70):

$$0 = \bar{C}_{\alpha i}^* \bar{R}_i^{*(2)} \tag{71}$$

$$F_i^{(2)} = 0, \quad i \in l \tag{72}$$

and the necessary and sufficient conditions for validity of (68) follows.

Summarize the derived results:

$$\left.\begin{matrix} \left.\begin{matrix}(71)\\(72)\end{matrix}\right\} \Rightarrow (70) \Rightarrow (68) \\ (67) \text{ is always true} \end{matrix}\right\} \Rightarrow (66) \Rightarrow (62) \Rightarrow (62)$$

Hence, if (71) and (72) are true, there always exists an output feedback decoupling the system.

The foregoing results, together with Remark 2.2, are concluded in the following theorem.

Theorem 2.2. Given a system S by (1) with $D = 0$. Output feedback (K, G) by (3) decouples the system if:

(i) There exists state feedback (F, G) by (2):

$$F = \left[-[\bar{D}_{\alpha 1}^{*+} \bar{C}_{\alpha 1}^* \bar{R}_1^* \vdots \dots \vdots \bar{D}_{\alpha l}^{*+} \bar{C}_{\alpha l}^* \bar{R}_l^*] + G \begin{bmatrix} F_1 & & 0 \\ & \ddots & \\ 0 & & F_l \end{bmatrix} \right] \bar{R}^{*-1} \tag{73}$$

$$G = [G_1 \vdots \dots \vdots G_l], \quad G_i = \overline{\operatorname{Ker} \bar{D}_{\alpha i}^*}, \quad i \in l$$

stable decoupling the system S;

(ii) Holds:

(74) $$C_{\alpha i}\bar{R}_i^{*(2)} = 0\,, \quad i \in \bar{l}$$

where matrices $\bar{R}_i^{*(2)}$ are given by the transformation:

$$C_i\bar{R}_i^*T_i = C_i[\bar{R}_i^{*(1)} \vdots \bar{R}_i^{*(2)}] = [\dot{C}_i \vdots 0] \qquad T = [C^+ \vdots \operatorname{Ker} C_i]\,, \quad i \in \bar{l}\,.$$

Then the output feedback (K, G) with matrices:

(75) $$K = [K_1 \vdots \ldots \vdots K_l]\,, \quad K_i = [-\bar{D}_{\alpha i}^{*+}\bar{C}_{\alpha i}^*\bar{R}_i^{*(1)} + G_iF_i^{(1)}]\,\dot{C}_i^{(1)+}\,, \quad i \in \bar{l}$$

solves the decoupling problem. The poles of the decoupled system are identical to the poles of state feedback decoupled system (obtained by application of state feedback (73)), where matrices F_i are defined by the expression:

(76) $$F_i = [F_i^{(1)} \vdots F_i^{(2)}]\,T_i^{-1}$$

with elements of $F_i^{(1)}$ and $F_i^{(2)}$, $i \in \bar{l}$ resp. arbitrary and zero.

Proof. (i) is implied by Remark 2.2 and Theorem 1.2;

(ii) is implied by (71). Expression (75) is implied by (67). (76) is implied by (67) and (72). □

Clearly, selecting submatrix $F_i^{(1)}$ in (76) and since K_i in (75), all poles of the decoupled system may not be freely assigned. The fixed poles are determined by the condition $F_i^{(2)} = 0$ in (76).

Theorem 2.2 is based on Theorem 1.2 and for that reason gives only sufficient conditions for output feedback decoupling. The necessity of these conditions for the cases when the state feedback decoupling problem is solved in the sense of maximal controllability subspaces $\mathscr{R}_i^*$, $i \in \bar{l}$ (see also Remark 1.1) is obvious from the discussion foregoing Theorem 1.2.

In a rather different manner, using the geometric approach of Wonham and Morse [8], Denham [1] derived the necessary and sufficient conditions for output feedback decoupling to be (5), (6), (7) to which are added also:

(77) $$A(\mathscr{R}_i \cap \operatorname{Ker} C) \subset \mathscr{R}_i\,, \quad i \in \bar{l}\,,$$

(78) $$\mathscr{R}^\Sigma = 0\,,$$

where

$$\mathscr{R}^\Sigma = \bigcap_{i \in \bar{l}} \mathscr{R}_i^\Sigma\,, \quad \mathscr{R}_i^\Sigma = \sum_{\substack{j \in \bar{l} \\ j \neq i}} \mathscr{R}_{i}\,.$$

Obviously (78) expresses the necessary condition for observability and arbitrary pole assignment of the state feedback decoupled system. Condition (77) is only a geometric interpretation of (74). Recall that (5)–(7) represent geometrically the necessary and sufficient conditions for state feedback decoupling. For the reasons described in Part I, Section 3, these conditions are reduced to the case of solution in the sense of maximal controllability subspaces $\mathscr{R}_i^*$, $i \in l$ (Remark 1.1). Therefore, the geometric approach of Denham [1] gives a solution only for those systems for which the sufficient conditions of Theorem 2.2 are also necessary, hence just the same result. Note, in addition, that our approach outlines a concrete algorithm for solution of the output feedback decoupling problem and also completely treats the question of stability of the decoupled system in contrast to the approach of Denham [1] where these problems are not examined.

4. CONCLUDING REMARKS

The problem of state feedback group decoupling of linear multivariable systems is considered as a transformation of the geometric formulation of Wonham and Morse [8] into a matrix form, useful for engineering calculations and computer implementation.

The equivalence established between the conditions for compatibility of the maximal unobservable subspaces and the relevant maximal controllability subspaces appears to be the main result of Part I. On that ground:

(i) decoupling problem is formulated geometrically for system with $D \neq 0$;

(ii) general algorithm for state feedback decoupling, similar to that of Silverman and Payne [7] is derived;

(iii) the stabilisation of the decoupled system is proposed as a process of "sending back" the invariant zeros.

Decoupling of not completely output controllable systems is considered and a class of systems with inherent interaction is described.

Conditions for output feedback group decoupling are derived and treated in connection with the structure of the poles of the decoupled system.

It is clear that the described theory may not be applied when the maximal unobservable subspaces are not compatible and there exists "smaller" compatible controllability subspaces, preserving output controllability. The problem of outlining such controllability subspaces remains open and will be the subject of further investigations.

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