KYBERNETIKA — VOLUME 18 (1982), NUMBER 4

OPTIMAL AND ANALYTICAL RESULTS OF $M/E_k/1:\infty$(FC FS) QUEUEING MODEL

RAKESH KUMAR VERMA

An $M/E_k/1 : \infty$ (FC FS) queueing model has been considered for its analysis. The optimal value of the arrival rate of the customers has been found. Some interesting theorems have been proved.

1. INTRODUCTION

The object is to analyse and optimize the well known $M/E_k/1 : \infty$(FC FS) queueing model. It is assumed that the user of the facility receives the reward from the service of his customers. On the other hand, it is also assumed that the user of the facility will incur a waiting loss when the arrival rate is non-zero. For the user this loss equals to $C > 0$ per customer per unit time. The corresponding long-run average loss per unit time is $C\lambda W(\lambda, \mu, k)$, where $W(\lambda, \mu, k)$ is the expected waiting time (in the system) of a customer with λ, μ, and k as the arrival rate, the service rate, and the number of phases of service of a customer respectively. The objective of the facility owner is to maximize per unit time net profit defined as the reward minus the waiting loss.

Hadidi [7] derived the results showing the system's operational improvements by making the provisions for changes in the arrival rate and the service rate upon a change in the rate of demand. Tilt [4] introduced the concept of duopsony and duopoly in the optimization of an $M/M/1 : \infty$(FC FS) queueing system. Gupta, Srinivasan and Yu [1], Lippman [2], Man [3], Rolfe [5], and Young [6] and among others adjusted the mean arrival rate for simple queueing models. My aim is to present the analytical and optimal results for the system under consideration. Following section deals with the optimal and analytical results.

2. OPTIMAL AND ANALYTICAL RESULTS

The service facility owner (the user) receives a reward from the service of a customer. Let $R(\lambda)$ be the reward to the facility owner obtained per unit arrival such that $R(\lambda) > R(0)$ for the $\lambda > 0$. It is assumed that $R(\lambda)$ is continuous and twice differentiable on $[0, \infty)$ with continuous derivatives, and that there exists an upper bound on the first derivatives. Thus for $\lambda \in [0, \infty)$, $M' \equiv \sup R'(\lambda)$, where primes denote the differentiation with respect to λ.

The facility owner will incur a waiting loss only when $\lambda > 0$. Let C be the loss per customer per unit time. Standard queueing theory reveals that the expected number of customers in an $M/E_k/1 : \infty(\text{FC FS})$ system is given by the following expression:

$$\text{(1)} \qquad \mathsf{E}(L_s) = \frac{k+1}{2k} \frac{\lambda^2}{\mu(\mu - \lambda)} + \frac{\lambda}{\mu}.$$

Hence, the objective of the facility owner is to maximize the function $[R(\lambda) - C\,\mathsf{E}(L_s)]$. Thus the objective function is given below:

$$\text{(2)} \qquad F(\lambda, \mu, k, C) = \begin{cases} R(0) & (\text{for } \lambda = 0) \\ R(\lambda) - C\,\mathsf{E}(L_s) & (\text{for } \lambda < \mu) \\ -\infty & (\text{for } \lambda \geqq \mu) \end{cases}$$

It is seen that the facility owner must only consider the values of λ on the interval $[0, \mu)$. Since we have assumed that $R(\lambda)$ is continuous and twice differentiable on $[0, \infty)$, therefore, $R(\lambda)$ is also continuous and twice differentiable on $[0, \mu)$. Since $R'(\lambda) < M'$, therefore, $\lim_{\lambda \to \mu} F(\lambda, \mu, k, C) = -\infty$. Thus there exists a $\bar{\lambda}$, $0 < \bar{\lambda} < \mu < k\mu$ such that $F(\lambda, \mu, k, C) > F(0, \mu, k, C)$. The optimal value of the arrival rate is defined by $\bar{\lambda}$ and the corresponding value of the objective function will be $\bar{F}(\lambda, \mu, k, C)$.

2(I). Necessary and sufficient conditions

Differentiating $F(\lambda, \mu, k, C)$ with respect to λ, one gets

$$\text{(3)} \qquad F'(\lambda, \mu, k, C) = R'(\lambda) - C\left[\frac{k+1}{2k}\left\{\frac{2\lambda\mu - \lambda^2}{\mu(\mu - \lambda)^2}\right\} + \frac{1}{\mu}\right].$$

$$\text{(4)} \qquad F''(\lambda, \mu, k, C) = R''(\lambda) - C\left[\frac{k+1}{k}\left\{\frac{\mu}{(\mu - \lambda)^3}\right\}\right].$$

Obviously, $F'(\lambda, \mu, k, C) = 0$ is the necessary condition, and $F'(\lambda, \mu, k, C) = 0$ and $F''(\lambda, \mu, k, C) = 0$ are the sufficient conditions for $F(\lambda, \mu, k, C)$ to possess a local maximum at $\lambda \in [0, \mu)$. Later on I shall impose some conditions on $R''(\lambda)$ for which $F''(\lambda, \mu, k, C) = 0$ is sufficient condition for a unique global maximum at $\bar{\lambda}$.

2(II). Representation of upper bound on $\bar{\lambda}$

From $F'(\lambda, \mu, k, C) = 0$ and $R'(\lambda) < M'$, one gets

$$F'(\lambda, \mu, k, C) = M' - C\left[\frac{k+1}{2k}\left\{\frac{2\lambda\mu - \lambda^2}{\mu(\mu-\lambda)^2}\right\} + \frac{1}{\mu}\right]. \tag{5}$$

Now solving $F'(\lambda, \mu, k, C) = 0$ for λ results in λ_1 and λ_2, where

$$\lambda_1 + \lambda_2 = 2\mu \tag{6}$$

$$\lambda_1\lambda_2 = \frac{2k\mu^2(\mu M' - C)}{2k\mu^2(\mu M' - C) + C(k+1)\mu} \tag{7}$$

Since $F'(\lambda, \mu, k, C) < 0$ precludes the optimality, except $\lambda = 0$, we conclude that $\bar{\lambda} \leqq \max(0, \lambda_i)$, $i = 1, 2$.

At this stage, I have placed few constraints on R and not much of value can be said on determination $\bar{\lambda}$ and $\bar{F}(\lambda, \mu, k, C)$. Now I am particularly interested in what way $\bar{F}(\lambda, \mu, k, C)$ and $\bar{\lambda}$ respond to change in the values of μ and C.

Theorem 1. For given $R(\lambda)$, C and k, there exists μ^0, $0 < \mu^0 < \infty$ such that $\bar{\lambda} = 0$ if and only if $\mu < \mu^0$. The function $\Phi(\lambda, \mu, k, C) = F(\lambda, \mu, k, C) - \bar{\lambda}(\mu)$ is a function of bounded variation on $[\mu^0, \mu^*] \subseteq [\mu^0, \infty)$, where $\mu^0 < \mu^* < \infty$.

Proof. Let $S_\infty = \{\lambda : 0 < \lambda < \mu < \infty,\ R(\lambda) > R(0)\}$. Obviously the set S_∞ is non-empty set. By virtue of the first mean value theorem there exists a unique μ, say $\mu(\lambda) > \lambda$, such that $F(\lambda, \mu(\lambda), k, C) = R(0)$. Let $\mu^0 = \inf_{\lambda \in S_\infty} \mu(\lambda)$. In what follows, it is shown that $\mu^0 > 0$.

Since $R'(\lambda) \leqq M'$ (for all $\lambda \in S_\infty$ and $0 < M' < \infty$), therefore, for $0 < \lambda < \mu$ one can get

$$\begin{aligned} F(\lambda, \mu, k, C) &= R(\lambda) - C\,\mathsf{E}(L_s) \\ &\leqq R(0) + \lambda\left[M' - \frac{C}{\lambda}\mathsf{E}(L_s)\right] \\ &< R(0) + \lambda\left[M' - \frac{C(k+1)}{2k\mu^2} - \frac{C}{\mu}\right] \end{aligned} \tag{8}$$

So $F(\lambda, \mu, k, C) < R(0)$ for $0 < \lambda < \mu < \mu^i$ $(i = 1, 2)$, where

$$\mu^1 = \{2kC + \sqrt{[4k^2C^2 + 8kM'\,C(k+1)]}\}/4kM'$$

$$\mu^2 = \{2kC - \sqrt{[4k^2C^2 + 8kM'\,C(k+1)]}\}/4kM'$$

In particular $F(\lambda, \mu, k, C) < R(0)$ for all $\lambda \in S_\infty$ and all $\mu < \mu^1$ and $\mu < \mu^2$. Therefore, for all $\lambda \in S_\infty$, $\mu(\lambda) > \mu^i$, where μ^i, $i = 1, 2$, are positive. Clearly by definition of μ^0,

$$\bar{\lambda}(\mu) = \begin{cases} 0 \quad (\text{for} \quad \mu \leqq \mu^0) \\ \text{a positive quantity greater than } \mu^0 \quad (\text{for } \mu > \mu^0) \end{cases} \tag{9}$$

Now the question is: what is $\bar{\lambda}(\mu^0)$? If there does not exist $\lambda^0 \in S_\infty$ such that $\mu(\lambda^0) = \mu^0$, it becomes clear that $F(\lambda, \mu^0, k, C) < F(0, \mu^0, k, C) = R(0)$. Now if there exists $\lambda^0 \in S_\infty$ such that $\mu(\lambda^0) = \mu^0$, it becomes clear that $F(\lambda^0, \mu^0, k, C) = F(0, \mu^0, k, C) = R(0)$. But by the assumption, I have to choose the smallest λ satisfying $F(\lambda, \mu, k, C) = \bar{F}(\lambda, \mu, k, C)$, so again $\bar{F}(\lambda, \mu^0, k, C) = R(0)$ and $\lambda(\mu^0) = 0$. Thus I have shown that $\bar{\lambda}(\mu) = 0$ if and only if $\mu \leqq \mu^0$ for given R, k and C.

The above analysis of $F(\lambda, \mu, k, C)$ and $\lambda(\mu)$ shows that $\bar{F}(\lambda, \mu, k, C) > \bar{F}(\lambda, \mu^0, k, C) = R(0)$ and $\lambda(\mu) > \lambda(\mu^0) = 0$ for all $\mu > \mu^0$.

Now to prove that $\bar{\Phi}(\lambda, \mu, k, C)$ is a function of bounded variation on $[\mu^0, \mu^*]$, I have to prove that $\bar{F}(\lambda, \mu, k, C)$ and $\bar{\lambda}(\mu)$ are strictly increasing on $[\mu^0, \mu^*]$.

Keeping in view the above analysis, I have to prove only that $\bar{F}(\lambda, \mu, k, C)$ and $\bar{\lambda}(\mu)$ are increasing functions on (μ^0, ∞), i.e., $\bar{F}(\lambda, \mu_1, k, C) < \bar{F}(\lambda, \mu_2, k, C)$ and $\bar{\lambda}(\mu_1) < \bar{\lambda}(\mu_2)$ for $\mu^\circ < \mu_1 < \mu_2$.

The steady-state condition on an $M/E_k/1$ queue shows that

$$(10) \qquad 0 < \lambda(\mu_1) < \mu_1 < \mu_1 k$$

$$(11) \qquad 0 < \lambda(\mu_2) < \mu_2 < \mu_2 k$$

By the definition of $F(\lambda, \mu, k, C)$, and (10) and (11) it becomes clear that

$$(12) \qquad \bar{F}(\lambda, \mu_1, k, C) \equiv F(\bar{\lambda}(\mu_1), \mu_1, k, C) < F(\bar{\lambda}(\mu_1), \mu_2, k, C) < F(\bar{\lambda}(\mu_2), \mu_2, k, C) \equiv \bar{F}(\lambda, \mu_2, k, C).$$

Now it remains to prove that $\bar{\lambda}(\mu_1) < \bar{\lambda}(\mu_2)$, for $\mu^0 < \mu_1 < \mu_2$.
Clearly

$$(13) \qquad F(\bar{\lambda}(\mu_1), \mu_1, k, C) > F(\lambda, \mu_1, k, C) \quad (\text{for } \lambda < \bar{\lambda}(\mu_1)).$$

Since the rate of change of waiting loss function with respect to the service rate μ is a negative quantity, therefore, this function is decreasing function of μ on (λ, ∞).

Again since $\mathrm{d}(\mathrm{d}(C\, \mathsf{E}(L_s)/\mathrm{d}\mu)/\mathrm{d}\lambda$ increases as the value of the arrival rate increases. Hence, the rate of increase in $F(\lambda, \mu, k, C)$ depends upon λ.

Consequently, for $\lambda < \bar{\lambda}(\mu_1)$, the following strict inequality is obtained:

$$(14) \qquad [F(\bar{\lambda}(\mu_1), \mu_2, k, C) - F(\bar{\lambda}(\mu_1), \mu_1, k, C)] > [F(\lambda, \mu_2, k, C) - F(\lambda, \mu_1, k, C)]$$

On adding (13) and (14), the following strict inequality is obtained:

$$(15) \qquad F(\bar{\lambda}(\mu_1), \mu_2, k, C) > F(\lambda, \mu_2, k, C).$$

But

$$(16) \qquad F(\bar{\lambda}(\mu_2), \mu_2, k, C) \geqq F(\bar{\lambda}(\mu_1), \mu_2, k, C).$$

From (15) and (16) the following strict inequality is obtained:

$$F(\bar{\lambda}(\mu_2), \mu_2, k, C) > F(\lambda, \mu_2, k, C)\,. \tag{17}$$

From the above strict inequality it is deduced that $\bar{\lambda}(\mu_1) \leqq \bar{\lambda}(\mu_2)$. The possibility $\bar{\lambda}(\mu_1) = \bar{\lambda}(\mu_2)$ can be omitted as follows:

$$F'(\lambda, \mu_2, k, C)\big|_{\lambda=\lambda(\mu_1)} = R'(\bar{\lambda}(\mu_1)) - C[\mathrm{E}'(L_s)] \quad \begin{cases}\lambda = \lambda(\mu_1)\\ \mu = \mu_2\end{cases}$$

$$> R'(\bar{\lambda}(\mu_1)) - [C\mathrm{E}'(L_s)] \quad \begin{cases}\lambda = \lambda(\mu_1)\\ \mu = \mu_1\end{cases}$$

$$(\text{since } \bar{\lambda}(\mu_1) < \mu_1 < \mu_2)$$

$$= 0 \;(\text{since } \bar{\lambda}(\mu_1) = 0 \Rightarrow F'(\lambda(\mu_1), \mu_1, k, C) = 0)\,.$$

Thus it is concluded that $\bar{\lambda}$ is strictly increasing on (μ^0, ∞), and hence on $[\mu^0, \mu^*] \subseteq [\mu^0, \infty)$.

Let $P = \{\mu^0 = \mu_0 < \mu_1 < \ldots < \mu_n = \mu^*\}$ be the partition of $[\mu^0, \mu^*]$. Let $V(\Phi, \mu^0, \mu^*)$ be the total variation of $\Phi(\lambda, \mu, k, C)$ on $[\mu^\circ, \mu^*]$. Then

$$V(\Phi, \mu^0, \mu^*) = \sum_{i=0}^{n-1} \left|\bar{\Phi}(\lambda, \mu_{i+1}, k, C) - \bar{\Phi}(\lambda, \mu_i, k, C)\right|.$$

Also

$$\begin{aligned}
&\left|\bar{\Phi}(\lambda, \mu_{i+1}, k, C) - \bar{\Phi}(\lambda, \mu_i, k, C)\right| = \\
&= \left|[\bar{F}(\lambda, \mu_{i+1}, k, C) - \bar{\lambda}(\mu_{i+1})] - [\bar{F}(\lambda, \mu_i, k, C) - \bar{\lambda}(\mu_i)]\right| \\
&= \left|[\bar{F}(\lambda, \mu_{i+1}, k, C) - \bar{F}(\lambda, \mu_i, k, C)] - [\bar{\lambda}(\mu_{i+1}) - \bar{\lambda}(\mu_i)]\right| \\
&= \left|[\bar{F}(\lambda, \mu_{i+1}, k, C) - \bar{F}(\lambda, \mu_i, k, C)]\right| + \left|[\bar{\lambda}(\mu_{i+1}) - \bar{\lambda}(\mu_i)]\right| \\
&= [\bar{F}(\lambda, \mu_{i+1}, k, C) - \bar{F}(\lambda, \mu_i, k, C)] + [\bar{\lambda}(\mu_{i+1}) - \bar{\lambda}(\mu_i)]
\end{aligned}$$

(for $\bar{F}$ and $\bar{\lambda}$ are increasing).

Thus

$$\sum_{i=0}^{n-1} \left|\Phi(\lambda, \mu_{i+1}, k, C) - \Phi(\lambda, \mu_i, k, C)\right| <$$

$$< \sum_{i=0}^{n-1} [\bar{F}(\lambda, \mu_{i+1}, k, C) - \bar{F}(\lambda, \mu_i, k, C)] + \sum_{i=0}^{n-1} [\bar{\lambda}(\mu_{i+1}) - \bar{\lambda}(\mu_i)] =$$

$$= \bar{F}(\lambda, \mu^*, k, C) - \bar{F}(\lambda, \mu^0, k, C) + \bar{\lambda}(\mu^*) - \bar{\lambda}(\mu^0) = \text{a finite number.}$$

Therefore $V(\bar{\Phi}, \mu^0, \mu^*)$ is less than or equal to a finite quantity. Hence $\bar{\Phi}(\lambda, \mu, k, C)$ is a function of bounded variation on $[\mu^0, \mu^*] \subseteq [\mu^0, \infty)$. □

Theorem 2. For given $R(\lambda)$, μ and k, there exists C^*, $0 \leqq C^* < \infty$, such that $\bar{\lambda} = 0$ if and only if $C \geqq C^*$. The function $\bar{\Psi}(\lambda, \mu, k, C) = \bar{F}(\lambda, \mu, k, C) - \bar{\lambda}(\mu)$ is a function of bounded variation on $[\varepsilon, C^*] \subseteq (0, C^*]$, where ε is a sufficiently small positive number.

Proof. The proof is somewhat identical to that of Theorem 1. Let $S_\mu = \{\lambda : 0 < \lambda < \mu < \mu k, R(\lambda) > R(0)\}$. It is clear that $\bar{\lambda}(\mu)|_C = 0$ for all $C \geqq C^* = 0$ when S_μ is a null set.

Now consider the case when S_μ is a non-empty set. By the definition of $F(\lambda, \mu, k, C)$, it is clear that $F(\lambda, \mu, k, C)$ is a decreasing function of C.

It can easily be seen that there exists an upper bound to $C(\lambda)$. Let $C^* \equiv \sup C(\lambda)$ for $\lambda \in S_\mu$ denote the least upper bound (l.u.b.). It is also clear that $C^* > 0$ as long as $C(\lambda) > 0$ for all $\lambda \in S_\mu$. The definition of C^* implies that $\bar{\lambda}(\mu)|_C = 0$ if $C > C^*$ and $\bar{\lambda}(\mu)|_C = 0$ if $C < C^*$. From these arguments and those employed in the proof of $\bar{\lambda}(\mu^0) = 0$ in Theorem 1, it can be shown that $\bar{\lambda}(\mu)|_C > \bar{\lambda}(\mu)|_{C^*} = 0$ and $\bar{F}(\lambda, \mu, k, C)|_C > \bar{F}(\lambda, \mu, k, C)|_{C^*} = R(0)$ for all $C < C^*$.

Thus I need to show that $\bar{F}(\lambda, \mu, k, C)$ and $\bar{\lambda}(\mu)$ are strictly decreasing in C on $(0, C^*)$.

Following the procedure of Theorem 1 and using the definition of $\bar{\lambda}$, given and used above, one can get

$$\begin{aligned} \bar{F}(\lambda, \mu, k, C)|_{C_1} &\equiv F(\lambda_1, \mu, k, C)|_{C_1} \geqq \\ &\geqq F(\lambda_2, \mu, k, C)|_{C_1} > \\ &> F(\bar{\lambda}_2, \mu, k, C)|_{C_2} \equiv \bar{F}(\lambda, \mu, k, C)|_{C_2} . \end{aligned}$$

Thus $\bar{F}(\lambda_1, \mu, k, C_1) > \bar{F}(\lambda, \mu, k, C_2)$ proves that $\bar{F}(\lambda, \mu, k, C)$ is strictly decreasing in C on $(0, C^*)$.

Now proceeding along the lines of the proof of Theorem 1, one sees that $\bar{\lambda}(C)$ is strictly decreasing on $(0, C^*)$, and hence on $(0, C^*]$.

Now I am in position to show that $\Psi(\lambda, \mu, k, C)$ is a function of bounded variation on $[\varepsilon, C^*]$. For convenience, I shall use $\bar{\Psi}(C)$ for $\bar{\Psi}(\lambda, \mu, k, C)$, and $\bar{F}(C)$ for $\bar{F}(\lambda, \mu, k, C)$.

Let $P = \{\varepsilon = C_0 < C_1 < \ldots < C_n = C^*\}$ be the partition of $[\varepsilon, C^*]$ and let $V(\bar{\Psi}, \varepsilon, C^*)$ be the total variation of $\bar{\Psi}(C)$ on $(\varepsilon, C^*]$. Now

$$\begin{aligned} |\Psi(C_{i+1}) - \Psi(C_i)| &= |[\bar{F}(C_{i+1}) - \bar{\lambda}(\mu, C_{i+1})] - \\ - [\bar{F}(C_i) - \bar{\lambda}(\mu, C_i)]| &< [\bar{F}(C_i) - \bar{F}(C_{i+1})] + [\bar{\lambda}(C_i) - \bar{\lambda}(C_{i+1})] \\ &\text{(for } \bar{F}(C) \text{ and } \bar{\lambda}(\mu, C) \text{ are decreasing).} \end{aligned}$$

Thus $\sum_{i=0}^{n-1} |\bar{\Psi}(C_{i+1}) - \bar{\Psi}(C_i)|$ is a finite number. □

2(III). Some conditions on R(λ)

Here I shall impose certain conditions on $R(\lambda)$, which will ensure that $F(\lambda, \mu, k, C)$ has a unique global maximum for ready reference.

Condition I: $R''(\lambda) < \left[\frac{k+1}{k}\left\{\frac{C\mu}{(\mu-\lambda)^3}\right\}\right]$ (for all $\lambda < \mu < \mu k$)

Condition II: $R''(\lambda) < \left[\frac{k+1}{k}\left\{\frac{C\mu}{(\mu-\lambda)^2}\right\}\right]$ (for all $\lambda < \mu < \mu k$)

Condition III: $R''(\lambda) \leqq 0$ (for all $\lambda \geqq 0$).

Condition I implies that $F(\lambda, \mu, k, C)$ has a unique global maximum for some $\bar{\lambda} \in [0, \mu)$. If $F'(0, \mu, k, C) \leqq 0$, $\bar{\lambda} = 0$. And if $F'(0, \mu, k, C) > 0$, $\bar{\lambda} = 0$ and it satisfies $F'(\bar{\lambda}, \mu, k, C) = 0$.

2(IV). The optimal arrival rate

It is assumed that Condition I holds. Hence, as previously stated, $F(\lambda, \mu, k, C)$ will possess a unique global maximum at $\bar{\lambda} \in [0, \mu)$. Now there arise two cases:

Case I: $F'(0, \mu, k, C) \leqq 0$

Case II: $F'(0, \mu, k, C) > 0$.

In Case I, $R'(0) \leqq C/\mu$ and it is clear that $F(0, \mu, k, C) > F(\lambda, \mu, k, C)$, for all $\lambda > 0$, since $F(\lambda, \mu, k, C)$ is strictly concave for all $\lambda > 0$. We have $\bar{\lambda} = 0$ for $\mu \leqq \mu^0$, where μ^0 is defined by $\mu^0 \equiv C/R'(0)$.

In Case II, $F(\lambda, \mu, k, C)$ has an interior maximum at $\bar{\lambda}(0 < \bar{\lambda} < \mu < \mu k)$ which is the solution of $F'(\lambda, \mu, k, C) = 0$. Then by (3), one can get

$$R'(\bar{\lambda}) = C\left[\frac{k+1}{2k}\left\{\frac{2\bar{\lambda}\mu - \bar{\lambda}^2}{\mu(\mu-\bar{\lambda})^2}\right\} + \frac{1}{\mu}\right]. \tag{18}$$

The optimal value of the objective function is given by

$$\bar{F}(\lambda, \mu, k, C) = R(\bar{\lambda}) - C\left[\frac{k+1}{2k}\left\{\frac{\bar{\lambda}^2}{\mu(\mu-\bar{\lambda})}\right\} + \frac{\bar{\lambda}}{\mu}\right]. \tag{19}$$

In what follows, I am particularly interested in what way $\bar{\lambda}$ respond to the change in μ and C.

Theorem 3. For given R and C, $\bar{\lambda}$ is a continuous in μ on $(0, \mu^{(1)})$.

Proof. To prove this theorem, I shall find out how $\bar{\lambda}$ is affected by a small change $d\mu$ for a μ satisfying Condition I, i.e., $\mu < \mu^{(1)}$. For $\mu < \mu^0$, $d\bar{\lambda}/d\mu = 0$. If $\mu > \mu^0$, $d\bar{\lambda}/d\mu$ can be found out by implicit differentiation, i.e.,

$$\frac{d\bar{\lambda}}{d\mu} = \frac{-[C(k+1)\{2\lambda\mu(\mu-\lambda) + (2\lambda\mu - \lambda^2)(\lambda+\mu)\} - C(\mu-\lambda)^3]}{\mu^2[R''(\lambda)\,k\mu(\mu-\lambda)^3 - C(k+1)\{(\mu-\lambda)^2 + (2\lambda\mu - \lambda^2)\}]}. \tag{20}$$

It is concluded that $\bar{\lambda}$ is continuous and differentiable in μ at least on the close interval $[\mu^0, \mu^{(1)}]$.

Similar arguments can be given for $\bar{\lambda}$ in terms of C on the close interval $[C_j^{(1)}, C^*]$, where $C^* \equiv \mu R'(0)$.

Now I shall prove the continuity of $\bar{\lambda}$ on $(0, \mu^{(1)})$. It is clear that if $\mu^{(1)} \leqq \mu^0$, then $\bar{\lambda}(\mu) = 0$ for all $\mu < \mu^{(1)}$. So $\bar{\lambda}$ is continuous in μ on the open interval $(0, \mu^{(1)})$.

Now assume that $\mu^{(1)} > \mu^0$. Since $\bar{\lambda}(\mu) = 0$ for $\mu \leqq \mu^0$, therefore, $\bar{\lambda}$ is continuous in μ on $(0, \mu^0)$. Hence, it remains to prove that $\bar{\lambda}$ is continuous on $[\mu^0, \mu^{(1)})$. First of all it will be proved that $\bar{\lambda}$ is continuous on the open interval $(\mu^0, \mu^{(1)})$. Let $\mu' \in \in (\mu^0, \mu^{(1)})$ such that $\lambda(\mu')$ is optimal value.

Let us choose arbitrary small number $\varepsilon > 0$ so that $\bar{\lambda}(\mu') - \varepsilon > 0$ and $\bar{\lambda}(\mu') + \varepsilon < \mu'$. Now it is to be shown that

$$|\mu - \mu'| < \delta \Rightarrow |\bar{\lambda}(\mu) - \bar{\lambda}(\mu')| < \varepsilon .$$

Clearly, Condition I is met for $\mu' < \mu^{(1)}$. Thus $F(\lambda, \mu', k, C)$ is strictly concave in $\bar{\lambda}$ on $[0, \mu')$. Therefore,

$$F(\bar{\lambda}(\mu'), \mu', k, C) > F(\bar{\lambda}(\mu') - \varepsilon , \mu', k, C) \tag{21}$$

$$F(\bar{\lambda}(\mu'), \mu', k, C) > F(\bar{\lambda}(\mu') + \varepsilon , \mu', k, C) . \tag{22}$$

Further, $F(\lambda, \mu, k, C)$ is continuous function in μ for some fixed $\lambda < \mu < \mu k$. Hence, there exists a positive δ such that

$$F(\bar{\lambda}(\mu'), \mu, k, C) > F(\bar{\lambda}(\mu') - \varepsilon, \mu, k, C) \tag{23}$$

$$F(\bar{\lambda}(\mu'), \mu, k, C) > F(\bar{\lambda}(\mu') + \varepsilon, \mu, k, C) \tag{24}$$

for all $\mu \in (\mu' - \delta, \mu' + \delta) \subset (\mu^0, \mu^{(1)})$.

Again assume that Condition I is met for all $\mu \in (\mu' - \delta, \mu' + \delta)$. Therefore, $F(\lambda, \mu, k, C)$ is strictly concave in λ on $[0, \mu)$. With the help of (23) and (24) $F(\lambda, \mu, k, C)$ will have maximum at $(\lambda(\mu') - \varepsilon, \bar{\lambda}(\mu') + \varepsilon)$. Thus it is concluded that

$$\mu \in (\mu' - \delta, \mu' + \delta) \ \Rightarrow \ \bar{\lambda}(\mu) \in (\bar{\lambda}(\mu') - \varepsilon, \bar{\lambda}(\mu') + \varepsilon) .$$

Equivalently, $|\mu - \mu'| < \delta \Rightarrow |\bar{\lambda}(\mu) - \bar{\lambda}(\mu')| < \varepsilon$ which means that $\bar{\lambda}$ is continuous in μ on $(\mu^0, \mu^{(1)})$. In a similar fashion the continuity can be proved at μ^0. Thus $\bar{\lambda}$ is continuous in μ on $[\mu^0, \mu^{(1)})$. Hence, $\bar{\lambda}$ is continuous in μ on $(0, \mu^{(1)})$. □

ACKNOWLEDGEMENTS

Gracious thanks to my external supervisor Prof. Kanti Swarup of I. I. P. A., New Delhi, for his guidance. Thanks are also due to Dr. P. K. Gupta of J. V. Jain College, Saharanpur, for his guidance and encouragement.

(Received December 18, 1978.)

REFERENCES

[1] R. K. Gupta, V. Srinivasan, P. L. Yu: Optimal State-Dependent Pricing Policies for a Class of Stochastic Multi - Unit Rental System. Graduate School of Business, Research Report No. 280, Stanford University 1975.

[2] S. A. Lippman: Applying a new device in the optimization of exponential queueing systems. Oper. Res. *23* (1975), 687—710.

[3] F. T. Man: Optimal control of time varying queueing systems. Management Sci. *19* (1973), 1249—1256.

[4] B. Tilt: Queueing Process with Optimization by One or More Decision Makers. Ph. D. Thesis, G. I. T. (1977).

[5] A. J. Rolf: The Control of a Multiple Facility Multiple Channel Queueing System with Parallel Input Streams. Technical Report No. 22, Graduate School of Business, Stanford University 1965.

[6] J. Young: Administrative control of multiple channel queueing systems with parallel input streams. Oper. Res. *14* (1966), 145—156.

[7] N. Hadidi: A queueing model with variable arrival rates. J.Periodica Mathematica Hungarica *6* (1975), 39—47.

Dr. Rakesh Kumar Verma, Department of Mathematics, J. V. Jain College, Pradumn Nagar, Saharanpur — 247 001. India.