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DIFFUSION APPROXIMATION FOR A CONTROLLED SERVICE SYSTEM

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The aim of the paper is to suggest a procedure how to control a service system under the possibility of the system's ruin, when the additions of the capital are random variables with a given distribution function. Using diffusion approximation for the capital the original problem is converted into the problem of controlling continuous Markov processes. A procedure how to compute the optimal control policy is presented.

1. INTRODUCTION

A service system $M/M/1$ without possibility of queueing is considered. Its input is composed of n mutually independent Poisson processes with arrival rates $a_i q > 0$, $i = 1, \ldots, n$. The service time of the i-th type of customer is a random variable having exponential distribution with a service rate $b_i q > 0$, $i = 1, \ldots, n$. (The parameter q is sufficiently large). The behaviour of the system is described by a random process $\{i_t, t \in [0, T]\}$ with a finite set of states $\{0, 1, \ldots, n\}$. $i_t = 0$ means that the system is vacant at time t, $i_t = j, j = 1, \ldots, n$, means that the system serves a customer of j-th type. Further, we assume that the functioning of the system depends on a quantity varying in time according to its performance. The quantity is called capital, it is denoted by $\{V_t, t \in [0, T]\}$, and it has a positive initial value $V_0 > 0$. During the service of a customer of type i, $i = 1, \ldots, n$, the capital increases by a random variable X per unit time with a given distribution function $F_i(x)$, $i = 1, \ldots, n$. The yield X' of the next served customer is independent with the distribution function $F_k(x)$, where k denotes his type, etc. The distribution function has the following properties:

$$(1) \qquad \int_{-\infty}^{\infty} x \, \mathrm{d}F_i(x) = c_i > 0 \,, \quad \int_{-\infty}^{\infty} x^2 \, \mathrm{d}F_i(x) = d_i q > 0 \,,$$
$$\int_{-\infty}^{\infty} x^4 \, \mathrm{d}F_i(x) = O(q^2) \,, \quad i = 1, \ldots, n \,.$$

Let x_t denote the rate of increase of the capital at time t. Thus, x_t is piecewise constant. Following the state change a new value of x_t is selected.

If $i_t = 0$, the capital has a constant decrease $c_0 < 0$ per unit time. If the capital reaches zero, the ruin occurs, and the system ceases to work. The ruin moment is a random variable $\tau = \inf\{t, V_t \leqq 0\}$. To measure the utility of the system's performance, we introduce the optimality criterion

$$\mathsf{E}_y\left\{\int_0^\tau e^{-\lambda t}\,\mathrm{d}V_t - N\,e^{-\lambda\tau}\right\},$$

where E_y is the mathematical expectation under the condition that the initial capital equals y, the discount factor λ is a chosen positive number. $N > 0$ denotes the penalty for the ruin. With regard to the danger of the ruin the strategy has to depend on the actual capital. It is given by a vector function $\boldsymbol{u}(y) = (u_1(y), \ldots, u_n(y))$, where $0 \leqq u_i(y) \leqq 1$ has the following meaning: if the system is vacant and if the capital equals y and the i-th customer arrives, then $u_i(y)$ denotes the probability of his accepting. The strategies with a bounded derivative are admissible and their totality is denoted by $\mathscr{U}$.

The system with $d_i = c_i^2$, $i = 1, \ldots, n$ was investigated in [2]. No diffusion approximation was used; the system of Bellman's equations was derived directly for the expected discounted criterion.

2. LIMIT DISTRIBUTION OF THE CAPITAL

We shall prove a limit theorem for $\{V_t, t \in [0, T]\}$ under the assumption that the parameter q tends to infinity and $\boldsymbol{u}(y) \in \mathscr{U}$ is a stationary control policy.

Let us define $\theta(\boldsymbol{u}(y))$, $\sigma^2(\boldsymbol{u}(y))$ (further the abbreviated denotation $\theta(y)$, $\sigma^2(y)$ will be used) together with $w(i, x, y)$ and $w_2(i, x, y)$, $i = 1, \ldots, n$, as a solution of the following system of equations

$$x - b_i q\, w(i, x, y) - \theta(y) = 0\,, \quad i = 1, \ldots, n\,, \tag{2}$$

$$c_0 + \sum_{k=1}^n a_k q\, u_k(y) \int_{-\infty}^{\infty} w(k, x, y)\,\mathrm{d}F_k(x) - \theta(y) = 0\,,$$

$$w(i, x, y)^2 - b_i q\, w_2(i, x, y) - \sigma_q^2(y) = 0\,, \quad i = 1, \ldots, n\,, \tag{3}$$

$$\sum_{k=1}^n a_k q\, u_k(y) \int_{-\infty}^{\infty} [w(k, x, y)^2 + w_2(k, x, y)]\,\mathrm{d}F_k(x) - \sigma_q^2(y) = 0\,.$$

(We set $w(0, x, y)$ and $w_2(0, x, y)$ zero.)

Letting q to infinity we obtain

$$\theta(y) = \frac{c_0 + \sum_{k=1}^n \frac{a_k c_k}{b_k} u_k(y)}{1 + \sum_{k=1}^n \frac{a_k}{b_k} u_k(y)}\,, \qquad \sigma^2(y) = \frac{2\sum_{k=1}^n \frac{a_k d_k}{b_k^2} u_k(y)}{1 + \sum_{k=1}^n \frac{a_k}{b_k} u_k(y)}\,. \tag{4}$$

We are going to show that the evolution of V_t will be sufficiently closely described by the stochastic differential equation

$$\mathrm{d}v_t = \theta(v_t)\,\mathrm{d}t + \sigma(v_t)\,\mathrm{d}W_t\,, \quad v_0 = V_0\,, \quad t \in [0, T]\,, \tag{5}$$

where $\{W_t, t \in [0, T]\}$ is a standardized Wiener process.

Let C_T be the space of all continuous functions on $[0, T]$ with the uniform metric. Further, for $t \in [0, T]$, let $\mathscr{C}_t$ be the σ-algebra on C_T generated by the sets

$$\{f \in C_T; f(s) \leqq x\}\,, \quad s \in [0, t]\,, \quad x \in (-\infty, \infty)\,.$$

The random function $\{y_t, t \in [0, T]\}$ is defined on $(C_T, \mathscr{C}_T)$ by the relation $y_t(f) = f(t)$, $t \in [0, T]$, $f \in C_T$. The probability distribution of $\{V_t, t \in [0, T]\}$ is the probability measure $\mathscr{P}_T^q$ induced on $(C_T, \mathscr{C}_T)$ by $\{V_t, t \in [0, T]\}$.

Theorem. Let the stationary control $u(y)$ have a bounded derivative on $(-\infty, \infty)$. Then $\mathscr{P}_T^q$ converges, as $q \to \infty$, weakly to the probability distribution $\mathscr{P}_T$ of a random process $\{v_t, t \in [0, T]\}$ such that

$$\mathrm{d}v_t = \theta(v_t)\,\mathrm{d}t + \sigma(v_t)\,\mathrm{d}W_t\,, \quad t \in [0, T]\,, \tag{6}$$

$$\mathscr{P}_T(v_0 = V_0) = 1\,,$$

where $\{W_t, t \in [0, T]\}$ is a standardized Wiener process.

According to the result of [5] $\mathscr{P}_T$ is unique. The proof of the theorem will be decomposed into a sequence of lemmas. Let $\mathscr{F}_t$ be the σ-algebra of random events defined by the history of the service system up to time t.

Lemma 1. Let

$$M_t = V_t - V_0 - \int_0^t \theta(V_s)\,\mathrm{d}s + \int_0^t (w(s) - w(s-))\,\mathrm{d}N_s,$$

where $w(s) = w(i_s, x_s, V_s)$ and $N_t = \sum_i \chi\{v_i \leqq t\}$, v_i are the moments of state changes. Then $\{M_t, t \in [0, T]\}$ is a martingale with respect to $\{\mathscr{F}_t, t \in [0, T]\}$.

Proof. Let Δ be arbitrarily small.

$$\mathsf{E}[M_{t+\Delta} - M_t \mid (i_t, x_t, V_t) = (i \neq 0, x, y)] =$$
$$= (1 - b_i q\Delta)(x\Delta - \theta(y)\,\Delta) - b_i q\Delta\, w(i, x, y) + o(\Delta) = o(\Delta)\,.$$

The last equality holds in virtue of (2). The same is valid for $i = 0$. Thus $\{M_t, t \in [0, T]\}$ is the martingale. □

The above mentioned martingale has the following property.

Lemma 2. For $0 \leqq t_1 \leqq t_2 \leqq T$

$$\mathsf{E}(M_{t_2} - M_{t_1})^4 \leqq B(t_2 - t_1)^2 + A(t_2 - t_1),$$

where $A \to 0$ as $q \to \infty$ and B is a constant with respect to q.

Proof. Let $\Delta = (t_2 - t_1)\, n^{-1}$, $Y_k = M_{(k+1)\Delta} - M_{k\Delta}$.
Then

$$\mathsf{E}(M_{t_2} - M_{t_1})^4 = \mathsf{E}(\sum_{k=0}^{n-1} Y_k)^4 = \mathsf{E}(\sum_{k=0}^{n-1} Y_k^4 + 4\sum_{m=0}^{n-1}(\sum_{k<m} Y_k)\, Y_m^3 + + 6\sum_{m=0}^{n-1}(\sum_{k<m} Y_k)^2\, Y_m^2).$$

Letting $n \to \infty$ we get

$$\mathsf{E}(M_{t_2} - M_{t_1})^4 = \mathsf{E}\left(\int_{t_1}^{t_2} (w(s) - w(s-))^4 \,\mathrm{d}N_s + + 4\int_{t_1}^{t_2} (M_{s-} - M_{t_1})\,(w(s) - w(s-))^3 \,\mathrm{d}N_s + + 6\int_{t_1}^{t_2} (M_{s-} - M_{t_1})^2\,(w(s) - w(s-))^2 \,\mathrm{d}N_s\right).$$

Let us denote by ${}^iN_t$ the counting process of transitions into state i and by ${}^iQ_t$ the corresponding transition rate. If we define

$$\bar{w}_k(i, y) = \int_{-\infty}^{\infty} w(i, x, y)^k \,\mathrm{d}F_i(x),$$

we have

$$\mathsf{E}(M_{t_2} - M_{t_1})^4 = \mathsf{E}\left[\int_{t_1}^{t_2} w(s-)^4\, {}^0Q_s \,\mathrm{d}s - 4\int_{t_1}^{t_2} (M_{s-} - M_{t_1})\,. w(s-)^3\, {}^0Q_s \,\mathrm{d}s + 6\int_{t_1}^{t_2} (M_{s-} - M_{t_2})^2\, w(s-)^2\, {}^0Q_s \,\mathrm{d}s + + \sum_{i=1}^{n}\left(\int_{t_1}^{t_2} \bar{w}_4(i, V_s)\, {}^iQ_s \,\mathrm{d}s + 4\int_{t_1}^{t_2} (M_{s-} - M_{t_1})\, \bar{w}_3(i, V_s)\, {}^iQ_s \,\mathrm{d}s + + 6\int_{t_1}^{t_2} (M_{s-} - M_{t_1})^2\, \bar{w}_2(i, V_s)\, {}^iQ_s \,\mathrm{d}s\right)\right].$$

Using Hölder inequality and the fact that $\mathsf{E}(M_{t_2} - M_{t_1})^4 = f(t_2)$ is a non-decreasing function in t_2, we obtain

$$f(t_2) \leqq A_1(t_2 - t_1) + A_2(t_2 - t_1) f(t_2)^{1/4} + B_1(t_2 - t_1) f(t_2)^{1/2},$$

where $A_1, A_2 \to 0$ as $q \to \infty$ and B_1 is independent of q. The statement of the lemma follows from the above inequality. □

Lemma 3. For $t \in [0, T]$

$$\int_0^t (w(s) - w(s-))\, dN_s = w(t) - w(0) - \int_0^t x_s\, w'(s)\, ds\,,$$

where $w'(s) = (\partial/\partial y)\, w(i_s, x_s, V_s)$.

Proof. Let (v_a, v_{a+1}) be the interval between two transitions and $v_a < t < v_{a+1}$. Then $x_s = x$, $i_s = i$, $V_s = y + x(s - v_a)$ for $s \in (v_a, t)$ and

$$w(t) - w(v_a) = w(i, x, y + x(t - v_a)) - w(i, x, y) = \int_{v_a}^t x\, w'(i, x, V_s)\, ds\,.$$

Composing all such intervals we obtain the assertion of the lemma. □

Let us denote

$$Y_t = V_t - V_0 - \int_0^t \theta(V_s)\, ds\,, \quad t \in [0, T]\,, \tag{7}$$

and let $\mathscr{R}_T^q$ be the probability distribution of $\{Y_t, t \in [0, T]\}$.

Lemma 4. The family of $\mathscr{R}_T^q$ is tight.

Proof. According to [1] it is sufficient to prove that

$$\lim_{\delta \to 0} \overline{\lim_{q \to \infty}}\, \mathscr{R}_T^q \,(\sup_{|t-s|<\delta} |y_t - y_s| > \varepsilon) = 0\,, \quad \varepsilon > 0\,. \tag{8}$$

By Lemma 3

$$Y_t = M_t - w(t) + w(0) + \int_0^t x_s\, w'(s)\, ds\,.$$

Take $\varepsilon > 0$, $\delta > 0$ and $q > 0$. Then

$$\begin{aligned}
&\mathscr{R}_T^q(\sup_{|t-s|<\delta} |y_s - y_t| > \varepsilon) = P(\sup_{|t-s|<\delta} |Y_s - Y_t| > \varepsilon) \leqq \\
&\leqq P\left(\sup_{|t-s|<\delta} |M_t - M_s| > \frac{\varepsilon}{3}\right) + P\left(\sup_{|t-s|<\delta} |w(t) - w(s)| > \frac{\varepsilon}{3}\right) + \\
&+ P\left(\sup_{|t-s|<\delta} \left|\int_s^t x_u\, w'(u)\, du\right| > \frac{\varepsilon}{3}\right) \leqq \sum_{j=0}^{[T/\delta]} P\left(\sup_{j\delta \leqq s \leqq (j+1)\delta} |M_s - M_{j\delta}| > \frac{\varepsilon}{9}\right) + \\
&+ P\left(\sup_{0 \leqq s \leqq T} 2|w(s)| > \frac{\varepsilon}{3}\right) + P\left(\int_0^T |x_s\, w'(s)|\, ds > \frac{\varepsilon}{3}\right) \leqq \\
&\leqq \sum_{j=0}^{[T/\delta]} \left(\frac{9}{\varepsilon}\right)^4 \mathsf{E}(M_{(j+1)\delta} - M_{j\delta})^4 + \left(\frac{6}{\varepsilon}\right)^4 \mathsf{E} \sup_{0 \leqq s \leqq T} |w(s)|^4 + \frac{3}{\varepsilon}\, \mathsf{E} \int_0^T |x_s\, w'(s)|\, ds\,.
\end{aligned}$$

In the last step submartingale inequality was used. From Lemma 2

$$\mathscr{R}_T^q\,(\sup_{|t-s|<\delta} |y_s - y_t| > \varepsilon) \leqq \left(\frac{T}{\delta} + 1\right)\left(\frac{9}{\varepsilon}\right)^4 (B\delta^2 + A\delta) + \left(\frac{6}{\varepsilon}\right)^4 C + \frac{3}{\varepsilon}\, D\,,$$

where each of A, C, D tends to zero as $q \to \infty$. Thus (8) is immediately obtained. □

The weak limit of any convergent subsequence $\mathscr{R}_T^{q_j}$, $q_j \to \infty$, is denoted by $\mathscr{R}_T$. Its existence is guaranteed by Lemma 4.

Lemma 5. $\{y_t, t \in [0, T]\}$ is on $(C_T, \mathscr{C}_T, \mathscr{R}_T)$ a quadratically integrable martingale with respect to $\{\mathscr{C}_t, t \in [0, T]\}$.

Proof. Let $0 \leqq s_1 < s_2 < \ldots < s_k < s < t \leqq T$ and let $f(x_1, \ldots, x_k)$ be a bounded continuous function on $\mathbb{R}^k$. From the martingale property of $\{M_t, t \in [0, T]\}$ follows

$$\mathsf{E}(M_t - M_s) f(M_{s_1}, \ldots, M_{s_k}) = 0 .$$

From the proof of Lemma 4 results

$$\mathsf{E} \sup_{0 \leqq t \leqq T} |M_t - Y_t|^4 \to 0 \quad \text{as} \quad q \to \infty . \tag{9}$$

Using (9)

$$\mathsf{E}(Y_s - Y_t) f(Y_{s_1}, \ldots, Y_{s_k}) - \mathsf{E}(M_t - M_s) f(M_{s_1}, \ldots, M_{s_k}) \to 0$$

as $q \to \infty$.
This gives the martingale property of $\{y_t, t \in [0, T]\}$. The integrability of its square follows from (9) and Lemma 2. □

Lemma 6. On $(C_T, \mathscr{C}_T, \mathscr{R}_T)$

$$\mathscr{E}_T\{(y_t - y_s)^2 \mid \mathscr{C}_s\} = \mathscr{E}_T\left\{\int_s^t \sigma^2(v_u)\, \mathrm{d}u \mid \mathscr{C}_s\right\} \tag{10}$$

holds for $0 \leqq s < t \leqq T$, where $\{v_t, t \in [0, T]\}$ is the solution of

$$v_t = V_0 + y_t + \int_0^t \theta(v_s)\, \mathrm{d}s , \quad t \in [0, T] . \tag{11}$$

($\mathscr{E}_T$ denotes the mathematical expectation with respect to $\mathscr{R}_T$).

Proof. Note that $\theta(y)$ is Lipschitz continuous, and hence (11) has the unique solution. As in preceding proof, to establish (10) it suffices to show that

$$\int (y_t - y_s)^2 f(y_{s_1}, \ldots, y_{s_k})\, \mathrm{d}\mathscr{R}_T = \int \left(\left(\int_s^t \sigma^2(v_u)\, \mathrm{d}u\, f(y_{s_1}, \ldots, y_{s_k})\, \mathrm{d}\mathscr{R}_T\right),\right.$$

when $s_1, \ldots, s_k$ and $(f(x_1, \ldots, x_k)$ are the same as in Lemma 5. From (9)

$$\mathsf{E}(M_t - M_s)^2 f(s_1, \ldots, Y_{s_k}) - \mathsf{E}(Y_t - Y_s)^2 f(Y_{s_1}, \ldots, Y_{s_k}) \to 0 \quad \text{as} \quad q \to \infty . \tag{12}$$

Let $t < s$, then

$$\mathsf{E}\{(M_t - M_s)^2 \mid \mathscr{F}_s\} = \mathsf{E}\left\{\int_s^t \sigma_q^2(V_u)\, \mathrm{d}u + w_2(s) - w_2(t) + \int_s^t x_u\, w_2'(u)\, \mathrm{d}u \mid \mathscr{F}_s\right\}. \tag{13}$$

The relation (13) is proved by proving martingale property for

$$\overline{M}_t = \int_0^t (w(s) - w(s-))^2\, \mathrm{d}N_s - \int_0^t \sigma_q^2(V_s)\, \mathrm{d}s + \int_0^t (w_2(s) - w_2(s-))\, \mathrm{d}N_s .$$

The same method as in Lemma 1 is used with respect to (3). Lemma 3 also holds for w_2.
From (13)

$$\text{(14)} \qquad \mathsf{E}(M_t - M_s)^2 f(Y_{s_1}, \ldots, Y_{s_k}) - \mathsf{E} \int_0^t \sigma_q^2(V_u)\, \mathrm{d}u\, f(Y_{s_1}, \ldots, Y_{s_k}) \to 0 \quad \text{as} \quad q \to \infty\,.$$

(12) and (14) together give the assertion of this lemma. □

Corollary.

$$y_t = y_0 + \int_0^t \sigma(v_s)\, \mathrm{d}W_s\,, \quad t \in [0, T]\,,$$

where $\{W_t, t \in [0, T]\}$ is a Wiener process on $(C_T, \mathscr{C}_T, \mathscr{R}_T)$.

Proof. $\{W_t = \int_0^t \sigma(v_s)^{-1}\, \mathrm{d}y_s, t \in [0, T]\}$ is a martingale, which satisfies

$$\mathscr{E}_T\{(W_t - W_s)^2 \mid \mathscr{C}_s\} = t - s \quad \text{for} \quad 0 \leqq s < t \leqq T\,.$$

This relation is a characteristic property of a Wiener process. □

The proof of the Theorem follows with regard to (7) and (11) from the fact that (6) holds on $(C_T, \mathscr{C}_T, \mathscr{R}_T)$.

3. OPTIMALITY

Let us make a slight change in denotation.

$$\theta(u) = \theta(\boldsymbol{u}(y))\,, \quad \varkappa(u) = \tfrac{1}{2}\sigma^2(\boldsymbol{u}(y)) \quad \text{for} \quad \boldsymbol{u}(y) = u \in U = [0, 1]^n$$

For the limiting diffusion the optimality criterion has the form

$$\mathsf{E}_y \left\{ \int_0^\tau \mathrm{e}^{-\lambda t}\, \mathrm{d}v_t - N\, \mathrm{e}^{-\lambda \tau} \right\}, \quad \lambda > 0\,,$$

or

$$\text{(15)} \qquad v(y) = \mathsf{E}_y \left\{ \int_0^\tau \mathrm{e}^{-\lambda t}\, \theta(v_t)\, \mathrm{d}t - N\, \mathrm{e}^{-\lambda \tau} \right\},$$

where $\{v_t, t \in [0, T]\}$ satisfies (5) with $v_0 = y$ and $\tau = \inf\{t > 0, v_t \leqq 0\}$.

The problem of maximization of $v(y)$ is the problem of controlling the one-dimensional Markov process with differential generator

$$\theta(u) \frac{\mathrm{d}}{\mathrm{d}y} + \varkappa(u) \frac{\mathrm{d}^2}{\mathrm{d}y^2}$$

in such a way that (15) is maximal.

We define

$$(16)\qquad \hat{v}(y) = \sup_{u\in\mathscr{U}} v(y)\,.$$

$\hat{v}(y)$ fulfils the Bellman equation

$$(17)\qquad \max_{u\in U}\{\varkappa(u)\,\hat{v}''(y) + \theta(u)\,(\hat{v}'(y) + 1) - \lambda\,\hat{v}(y)\} = 0$$

(see for example [3]) with initial condition $\hat{v}(0) = -N$ and $\hat{v}(\infty) < \infty$. The primes denote the derivatives with respect to y. The optimal process has the generator

$$\theta(\hat{u}(y))\frac{\mathrm{d}}{\mathrm{d}y} + \varkappa(\hat{u}(y))\frac{\mathrm{d}^2}{\mathrm{d}y^2}\,,$$

where $\hat{\boldsymbol{u}}(y)$ is the maximizer of the curly bracket in (17). $\hat{\boldsymbol{u}}(y)$ is not necessarily an element of $\mathscr{U}$.

Now, we shall construct the optimal strategy $\hat{\boldsymbol{u}}(y)$. The whole construction is divided into four parts.

1. Let us put $y = +\infty$. Then

$$\hat{v}(\infty) = \lambda^{-1}\max_{u\in U}\theta(u) = \lambda^{-1}\,\theta(\hat{u}(\infty))\,.$$

2. Further, we solve for $y \geqq 0$

$$(18)\qquad \varkappa(\hat{u}(\infty))\,v''(y) + \theta(\hat{u}(\infty))\,(v'(y) + 1) - \lambda\,v(y) = 0\,.$$

The only bounded solution has the form

$$v(y) = K\,\mathrm{e}^{py} + \lambda^{-1}\,\theta(\hat{u}(\infty))\,,$$

where p is the only negative root of the quadratic equation corresponding to (18) and K is an unknown constant. We shall assume $K < 0$. In such case v is the increasing function. When chossing K two cases can occur:

(i) There exists K so that $v(0) = -N$ and simultaneously

$$(19)\qquad (\varkappa(\hat{u}(\infty)) - \varkappa(u))\,v''(0) + (\theta(\hat{u}(\infty)) - \theta(u))\,(v'(0) + 1) \geqq 0 \quad \text{for all} \quad u\in U\,.$$

Then the construction is finished and the optimal strategy $\hat{u}(y) = \hat{u}(\infty)$ for all $y \geqq 0$.

(ii) Case (i) does not hold. Then we choose K so that

$$\min_{u\in U}\left[(\varkappa(\hat{u}(\infty)) - \varkappa(u))\,v''(0) + (\theta(\hat{u}(\infty)) - \theta(u))\,(v'(0) + 1)\right] = 0\,.$$

3. Let (ii) occur. The strategy by which the minimum is achieved is denoted by ${}^0u$ and we solve the following equation for $y \leqq 0$

$$\varkappa({}^0u)\,v''(y) + \theta({}^0u)\,(v'(y) + 1) - \lambda\,v(y) = 0$$

with terminal conditions

$$v(0) = K + \lambda^{-1}\,\theta(\hat{u}(\infty))\,,\quad v'(0) = Kp\,.$$

Either, for $y_N < 0$ such that $v(y_N) = -N$ the following inequality holds

$$(20) \qquad \min_{u \in U \setminus \{\hat{u}(\infty)\}} \{(\varkappa(^0u) - \varkappa(u))\, v''(y_N) + (\theta(^0u) - \theta(u))\,(v'(y_N) + 1)\} \geqq 0\,,$$

and the construction is completed. Or, there exists $0 > y_a > y_N$, such that

$$\min_{u \in U - \{\hat{u}(\infty)\}} \{(\varkappa(^0u) - \varkappa(u))\, v''(y_a) + (\theta(^0u) - \theta(u))\,(v'(y_a) + 1)\} = 0$$

The minimizing strategy is denoted by ^1u and the whole procedure is repeated for $y \leqq y_a$ so many times till we obtain such y_N that $v(y_N) = -N$ and corresponding inequality (20) holds.

4. The last step of the construction is the shifting of the end point y_N into zero. The resulting strategy is thus

$$\hat{u}(y) = {}^ju(y + y_N) \quad \text{for} \quad y_j \leqq y + y_N < y_{j-1}\,.$$

Its optimality follows from the construction.

4. EXAMPLE

In this section a numerical example is given. Only two types of customers are considered. The following parameters are chosen:

$$\begin{array}{lll} a_1 = 1 & a_2 = 0{,}1 & c_0 = -1 \\ b_1 = 1 & b_2 = 0{,}5 & \lambda = 1 \\ c_1 = 1 & c_2 = 50 & \\ d_1 = 1 & d_2 = 100 & \end{array}$$

1. According to the foregoing section the optimal strategy for $y = +\infty$ equals
$\hat{u}(\infty) = (\hat{u}_1(\infty),\ \hat{u}_2(\infty)) = (0, 1)$ and
$\theta(\hat{u}(\infty)) = 7, 5,\ \ \varkappa(\hat{u}(\infty)) = 33, 33.$

2. The solution of (18) is

$$v(y) = K\, e^{-0{,}32y} + 7{\cdot}5\,.$$

3. When choosing K, case (ii) occurs. We get $^0u = (1, 1)$,
$\theta(^0u) = 4{,}55$, $\varkappa(^0u) = 18{,}64$ and the parameter $K = -5{,}57$. For $y \leqq 0$ the following equation is solved

$$18{,}64v''(y) + 4{,}55(v'(y) + 1) - v(y) = 0$$

with terminal conditions $v(0) = 1{,}93$ and $v'(0) = 9{,}28$. The solution has the form

$$v(y) = -18{,}54\, e^{-0{,}38y} + 15{,}92\, e^{0{,}14y} + 4{,}55\,.$$

For fixed N we find $y_N < 0$ such that $v(y_N) = -N$.

The following table gives several mutually corresponding values.

N	1	5	10	100
y_N	-0.32	-0.75	-1.29	-10.98

4. The optimal strategy $\hat{u}(y)$ equals

$$\hat{u}(y) = \begin{cases} (1, 1) & \text{for} \quad 0 \leqq y < -y_N \\ (0, 1) & \text{for} \quad -y_N \leqq y . \end{cases}$$

Remark. If $d_2 \leqq 15{,}00$ the optimal strategy would equal $(0,1)$ for all $y \geqq 0$.

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