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SOME REMARKS ON THE STABILITY PROBLEM FOR LINEAR SPACE AUTOMATA AND SEMICONTINUITY OF CUT POINT LANGUAGES*

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The stability problem for linear space automata is formulated for state spaces that are completely metrizable topological vector spaces. It is shown that a well known characterization of stable automata carries over to the general situation, provided the state space is normed; an example due to Dubinsky shows that the latter condition must not be relaxed. Moreover a uniformity on the set of all such automata is defined which allows a topological characterization of stability. This uniformity is used, too, for an investigation of the semicontinuity of cut point languages, and it is shown that there exists a stochastic system which produces any cut point language with probability 1.

INTRODUCTION

Let E be a real linear space, X a nonvoid set. $\mathscr{A} := (X, E; H)$ is said to be a linear space automaton with input alphabet X, and state space E if $H : X \times E \to E$ is a map such that $H(x, \cdot)$ is linear for any $x \in X$. $\mathscr{A}$ is said to be stable if small perturbations for input letters cause only small perturbations for arbitrary input words. In order to define this more precisely, in [12] E is assumed to be an Euclidean space, i.e. $E = \mathbb{R}^n$ for some $n \in \mathbb{N}$, and X to be finite. Let $\|\cdot\|$ be a norm on E, and denote by $\|\cdot\|$, too, a consistent matrix norm for $n \times n$-matrices (i.e. $\|fA\| \leqq \|f\| \, \|A\|$ holds for any $f \in E$, and for any $n \times n$-matrix A). Then $\mathscr{A}$ is said to be stable (strongly stable in [12]) if given another linear space automaton $\mathscr{A}' := (X, E; H')$, $\varepsilon > 0$, there exists $\delta = \delta(\varepsilon) > 0$ such that

$$\sup \{\|H(x, \cdot) - H'(x, \cdot)\| \; ; \; x \in X\} < \delta$$

implies

$$\sup \{\|H(v, \cdot) - H'(v, \cdot)\| \; ; \; v \in X^*\} < \varepsilon ,$$

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where δ does not depend upon $\mathscr{A}'$. The main result in [12] then is that stability of $\mathscr{A}$ is equivalent to $\|H(v, \cdot)\| \to 0$, as $|v| \to \infty$.

In this paper an attempt is made to have a look at the problem from a more general point of view. X is no longer assumed to be finite, and E is assumed to be in a class of topological linear spaces which includes Banach spaces as well as Fréchet spaces. An additional requirement is that $H(x, \cdot)$ is continuous for any $x \in X$. In absence of norms, stability has to be reformulated, and it is shown that stability implies the convergence result above, when the norm is replaced by an arbitrary continuous seminorm. If moreover E is normed, one gets the equivalence cited above. Since any topological linear space automaton generates an automaton which works on the topological dual E' of E, the question arises whether stability carries over to the dual automaton. After characterizing those automata on the dual space which arise as dual automata, it is shown that stability of the dual automaton implies stability of the given, and that the converse is true, too, provided E' has a special property (which is shared by normed spaces). An elegant counterexample provided by Ed. Dubinsky demonstrates that the equivalence cited above is indeed a speciality of normed spaces: he shows that $H(v, \cdot) \to 0$, as $|v| \to \infty$ does not imply stability in case the space is a Fréchet space which does not admit a norm. The concept of nearness of two states carries over to the concept of nearness of two automata. This gives rise to define for every language $L \subset X^*$ a uniformity $\mathscr{U}_L$ on the set of all automata under consideration, and stability of an automaton turns out to be describable by a comparison of the local neighbourhood bases of the topologies generated by $\mathscr{U}_X$, and $\mathscr{U}_{X^*}$, respectively.

The latter uniformity is helpful, too, in the investigation of the set valued mapping which assigns to any automaton $\mathscr{A}$ and to any cut point θ, $0 \leqq \theta \leqq 1$, the cut point language $S(\mathscr{A}, \theta)$. For this, X is assumed to be a compact topological space, and the automata considered are assumed to be acts, i.e. jointly continuous in both arguments. This yields that $\mathscr{A} \mapsto S(\mathscr{A}, \theta)$ is both complementary lower and upper semicontinuous with respect to the topology generated by $\mathscr{U}_{X^*}$.

Using this, it is shown that if X is metrizable the production of $S(\mathscr{A}, \theta)$ may be represented in a stochastic manner, i.e. that there exists a stochastic system Q_θ such that if Q_θ is endowed with the data of $\mathscr{A}$, $S(\mathscr{A}, \theta)$ is produced with probability one by Q_θ, and that the latter is in some sense the smallest set with this property.

STABILITY OF TOPOLOGICAL LINEAR SPACE AUTOMATA

Let E be a locally convex Hausdorff topological vector space as the set of states, X an arbitrary set as the set of inputs.

$$\mathscr{A} = (X, E; H)$$

is said to be a *topological linear space automaton* (abbreviated by tlsa) if

$$H : X \times E \to E$$

is a map such that $H(x, \cdot)$ is continuous and linear for any $x \in X$ with the additional property that, given $f \in E$,

$$\{H(x, f);\ x \in X\}$$

is bounded in E. If $\mathscr{A}$ is in state f, $\mathscr{A}$ will be in state $H(x, f)$ after input of $x \in X$; informally the continuity condition means that if two states are close (i.e. represent similar information), an arbitrary input does not disturb this closeness. The condition that, given $f \in E$, $\{H(x, f);\ x \in X\}$ is bounded can be reformulated: given a continuous seminorm p on E,

$$\sup \{p(H(x, f));\ x \in X\} < \infty$$

holds; less formally it means that the set of possible new states is geometrically not too large. H is extended in the usual way to

$$H : X^* \times E \to E$$

by defining

$$H(e, f) := f\,,$$

and

$$H(vx, f) := H(x, H(v, f))\,,$$

thus $H(v, \cdot)$ is linear and continuous for every $v \in X^*$. This is not difficult to establish.

1. Examples.

a) Let $(Z, \mathfrak{Z})$ be a measurable space in the sense of Probability Theory, and let given $x \in X$, $K(x, z)$ be a measure on $\mathfrak{Z}$ such that

$$z \mapsto K(x, z)\,(C)$$

is a bounded $\mathfrak{Z}$-measurable function on Z for every $C \in \mathfrak{Z}$ (in particular, $\gamma_x := \sup\{K(x, z)\,(Z)\,;\ z \in Z\} < \infty$). Denote by $\mathscr{F}(Z, \mathfrak{Z})$ the linear space of all bounded $\mathfrak{Z}$-measurable functions on Z, and endow this space with the usual supremum norm p_∞ (with $p_\infty(f) = \sup\{|f(z)|;\ z \in Z\}$). Defining

$$H(x, f) : z \mapsto \int f \,\mathrm{d}K(x, z)\,,$$

we have

$$H(x, f) \in \mathscr{F}(Z, \mathfrak{Z})\,,$$

if $f \in \mathscr{F}(z, \mathfrak{Z})$, and

$$p_\infty(H(x, f)) \leqq p_\infty(f)\,\gamma_x\,.$$

Thus, if

$$\gamma := \sup \{\gamma_x;\ x \in X\} < \infty ,$$

then $(X, \mathscr{F}(Z, \mathfrak{Z}); H)$ is a tlsa. In particular this is true if $(X, (Z, \mathfrak{Z}); K)$ is a stochastic automaton, i.e. if

$$K(x, z)(Z) = 1$$

for any input letter $x \in X$ and any state $z \in Z$.

b) Now assume that X and Z are topological spaces such that X is compact. Z is endowed with the Borel sets, i.e. the smallest σ-field on Z that contains the open sets. Let K be as above, and assume that K has the following property: if (x_α), and (z_β) are nets in X resp. Z such that

$$x_\alpha \to x,\ z_\beta \to z ,$$

this implies

$$\int f \,\mathrm{d}K(x_\alpha, z_\beta) \to \int f \,\mathrm{d}K(x, z) ,$$

whenever $f \in \mathscr{C}(Z) := \{g;\ g: Z \to \mathbb{R} \text{ is bounded and continuous}\}$. Define H as above, then $(X, \mathscr{C}(Z); H)$ is a tlsa, since $\gamma < \infty$ in this case, when $\mathscr{C}(Z)$ is endowed with the norm topology, or with the topology of pointwise convergence.

c) (cp. [7]) Let X, and Z be finite, say $Z = \{1, \dots, n\}$, and assume $(X, Z; K)$ to be a stochastic automaton. Since $\mathscr{C}(Z)$ equals $\mathbb{R}^n$ in this case, $H(x, \cdot)$ is represented by a stochastic $n \times n$ matrix $H(x)$; moreover, if $f \in \mathbb{R}^n$, the equality

$$H(x, f) = f\, H(x)$$

holds. □

Let $\mathscr{A} = (X, E; H)$ be a tlsa, then $\mathscr{A}$ will be identified with H, when no confusion arises; define the linear operator $H(v)$ by

$$H(v)(f) := H(v, f) .$$

In order to make life easier, a technical assumption on E is imposed: we assume E to be barreled; an equivalent formulation is that every seminorm on E which is upper semicontinuous is continuous. The class of barreled spaces includes the complete normed spaces as well as the complete metrizable locally convex spaces ([11], II.7.1). Denote by Σ_E the set of continuous seminorms on E, and by $\Lambda(E)$ the linear space of all continuous linear mappings from E to E. $\Lambda(E)$ is topologized as follows: given $p \in \Sigma_E$, and a bounded set $B \subset E$, define for $L \in \Lambda(E)$

$$q_{B,p}(L) := \sup \{p(L(f));\ f \in B\} ,$$

and take as a base for the neighbourhoods of $0 \in \Lambda(E)$ sets of the form

$$\{L;\ q_{B_i,p_i}(L) < \varepsilon_i,\ 1 \leqq i \leqq n\} ,$$

where $B_i \subset E$ are bounded, $\varepsilon_i > 0$, $p_i \in \Sigma_E$.

(In terms of topological linear spaces, this topology is the γ-topology on $\Lambda(E)$, where γ is the set of all bounded sets in E, cp. [11], § III.3). Let $\Sigma_{\Lambda(E)}$ be the set of continuous seminorms on $\Lambda(E)$ with respect to this topology. Note that if E is a normed space, i.e. if

$$\Sigma_E = \{p\} ,$$

then $\Lambda(E)$ is a normed space, too, where

$$\Sigma_{\Lambda(E)} = \{q\} ,$$

and

$$q(L) = \sup \{p(L(f));\ p(f) \leqq 1\} .$$

After these preparations, stability of a tlsa can be defined. Roughly, a tlsa is stable if small perturbations for input letters cause only small perturbations for arbitrary input words or, equivalently, if H is approximated by another automaton H' on input letters, this approximation holds for input words, too. If E is a normed space, the formulation due to [12], Definition 6, quoted in the Introduction above can be used. In absence of norms, this leads to

2. Definition. The tlsa H is said to be *stable* if the following condition holds: given $\varepsilon > 0$ and $q \in \Sigma_{\Lambda(E)}$, there exist $\tilde{q} \in \Sigma_{\Lambda(E)}$ and $\delta > 0$ such that for an arbitrary tlsa H'

$$\sup \{\tilde{q}(H(x) - H'(x));\ x \in X\} < \delta$$

implies

$$\sup \{q(H(v) - H'(v));\ v \in X^*\} < \varepsilon .$$

An equivalent formulation for the latter implication is that, given a bounded set $B \subset E$, and a neighbourhood U of 0 in E, there exists a bounded set $C \subset E$ and a neighbourhood V of 0 in E such that

$$\{H(x,f) - H'(x,f);\ x \in X,\ f \in C\} \subset V \Rightarrow$$
$$\Rightarrow \{H(v,f) - H'(v,f);\ v \in X^*,\ f \in B\} \subset U .$$

This yields as an immediate consequence

3. Proposition. If H is stable, then $|v| \to \infty$ implies $q(H(v)) \to 0$ for every continuous seminorm q on $\Lambda(E)$.

Proof. 1. Since, given $f \in E$, $\{H(x,f);\ x \in X\}$ is bounded, and E is a barreled space, $\{H(x);\ x \in X\}$ is equicontinuous. This means that given $q \in \Sigma_{\Lambda(E)}$, $\varepsilon > 0$, there exists $\hat{q} \in \Sigma_{\Lambda(E)}$, $\delta > 0$ such that

$$q(H(x,f)) < \varepsilon$$

for any $x \in X$, provided

$$\hat{q}(f) < \delta ,$$

hence $\{H(x);\ x \in X\}$ is bounded in $A(E)$ ([11], III.4.1, 4.2), or equivalently,

$$\sup\{q(H(x));\ x \in X\} < \infty$$

holds for any $q \in \Sigma_{A(E)}$.

2. (cp. [12], proof of Theorem 6) Define for r, $0 < r < 1$,

$$H_r(x, f) := r \,.\, H(x, f)\,,$$

then $H_r(v)$ equals $r^{|v|} \,.\, H(v)$, and

$$\hat{q}(H(v) - H_r(v)) = (1 - r^{|v|}) \,.\, \hat{q}(H(v))\,,$$

where $\hat{q} \in \Sigma_{A(E)}$ is an arbitrary seminorm. Now fix $\varepsilon > 0$, $q \in \Sigma_{A(E)}$, and choose $\hat{q}$, $\delta > 0$ for H according to the definition of stability above for q and $\varepsilon/2$. Since $H_r(x) \to H(x)$ uniformly in x, as $r \to 1$, there is r such that

$$\hat{q}(H(x) - H_r(x)) < \delta$$

for any $x \in X$, thus

$$q(H(v) - H_r(v)) < \varepsilon/2$$

for any $v \in X^*$. But this implies that there exists $k \in \mathbb{N}$ such that for all $v \in X^*$ with $|v| \geqq k$

$$q(H(v) < \varepsilon$$

holds. □

In a special case, the converse of Proposition 3 holds:

4. Proposition. If E is a normed space with adjoint norm q on $A(E)$, and if $q(H(v)) \to 0$, as $|v| \to \infty$, then H is stable.

Proof. The arguments follow rather close the argument for the proof of the if-part of Theorem 6 in [12] and are given in detail for the reader's convenience.

Since E is a Banach space, $A(E)$ is a Banach algebra, and consequently the norm $\|\cdot\|$ on $A(E)$ has the property that

$$\|L_1 L_2\| \leqq \|L_1\| \,.\, \|L_2\|$$

holds. This is the crucial property since

$$H(v_1 v_2) = H(v_1)\, H(v_2)$$

holds for all $v_i \in X^*$. Note also that since $\{H(x);\ x \in X\}$ is bounded,

$$R := \sup_{x \in X} \|H(x)\|$$

is finite. We may and do assume that $R \leqq 1$.

Now let $\varepsilon > 0$ be given, and choose $\varepsilon_0 > 0$ with

$$k\varepsilon_0^{k-1} < \varepsilon$$

for any $k \in \mathbb{N}$. This is possible since $k \mapsto kr^{k-1}$ is a decreasing function, provided $0 < r < 1$. Now choose $\varepsilon_1 > 0$ such that $2\varepsilon_1 < \varepsilon_0$. For ε_1 there exists $N_0 \in \mathbb{N}$ such that

$$\|H(v)\| < \tfrac{1}{2}\varepsilon_1$$

for any $v \in X^*$, $|v| \geqq N_0$. Since semigroup multiplication is uniformly continuous in $\Lambda(E)^{N_0}$, there exists $\delta > 0$ such that

$$\sup_{x \in X^*} \|H(x) - H'(x)\| < \delta \quad \text{implies} \quad \sup_{|v| = N_0} \|H(v) - H'(v)\| < \tfrac{1}{2}\varepsilon_1 \, .$$

In particular, this implies that $\|H(v)\| < \varepsilon_0$ and $\|H'(v)\| < \varepsilon_0$ hold for any v of length N_0.

Now let

$$\sup_{x \in X} \|H(x) - H'(x)\| < \delta$$

and assume that $w \in X^*$ is given with $|w| > N_0$. Then w is split into $v_0, \ldots, v_k$ such that $|v_i| = N_0$ for $i = 1, \ldots, k$, and $|v_0| \leqq N_0$. Consequently, we have

$$\|H(w) - H'(w)\| \leqq$$

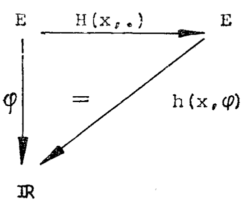
$$\leqq \sum_{i=0}^{k} \|H(v_0)\| \cdot \ldots \cdot \|H(v_{i-1})\| \cdot \|H(v_i) - H'(v_i)\| \cdot \|H'(v_{i+1})\| \cdot \ldots \cdot \|H'(v_k)\| <$$

$$< \tfrac{1}{2}\varepsilon_1 \, \varepsilon_0^k + \sum_{i=1}^{k} \varepsilon_0^{i-1} \, \tfrac{1}{2}\varepsilon_1 \varepsilon_0^{k-1-i} < \tfrac{1}{2}\varepsilon_1 \, \tfrac{1}{2}\varepsilon_0^k + \tfrac{1}{2}\varepsilon_1 \cdot k \cdot \varepsilon_0^{k-1} < \varepsilon \, . \qquad \square$$

Any topological linear automaton H with state space E yields an automaton on $E := \{\varphi : E \to \mathbb{R};\ \varphi$ is linear and continuous$\}$ by a canonical construction in the following manner: if $\varphi \in E'$, $x \in X$ and $f \in E$, define

$$h(x, \varphi)(f) := \varphi(H(x, f)) \, ,$$

hence $h(x, \varphi)$ makes the following diagram commutative:



It is then easily seen that $h(x, \varphi) \in E'$ for any $x \in X, \varphi \in E'$, and that $h(x)$ is always linear. In order to investigate the continuity of

$$h(x) : \varphi \mapsto h(x, \varphi) \, ,$$

E' must be topologized. This is done by the strong topology on E', which is generated by the seminorms

$$q^p(\varphi) := \sup \{|\varphi(f)|;\ p(f) = 1\}$$

($p \in \Sigma(E)$). Now from a well-known Theorem for topological vector spaces ([9], 21.6(i)) it is derived that $h(x)$ is continuous. Thus it remains to demonstrate that $\{h(x, \varphi);\ x \in X\}$ is bounded in E' for any $\varphi \in E'$. Given $\varepsilon > 0$,

$$U := \{f \in E;\ |\varphi(f)| < \varepsilon\}$$

is open in E, and

$$V := \cap \{H(x)^{-1}[U];\ x \in X\}$$

is open in E, too, since $\{H(x);\ x \in X\}$ is equicontinuous ([11], III.4.1 (a)). It is easily seen that

$$|h(x, \varphi)(f)| < \varepsilon$$

for any $x \in X$, provided $f \in U$, thus $\{h(x, \varphi);\ x \in X\}$ is equicontinuous, hence bounded ([11], III.4.3).

Consequently, $\mathscr{A}^d := (X, E'; h)$ is a topological linear automaton and is called the *dual* of $\mathscr{A} := (X, E; H)$. In [7], the dual of a linear automaton is introduced in case the space E of states is $\mathbb{R}^n$ for some $n \in \mathbb{N}$. Note that in that paper no additional topological considerations have been necessary. Those automata with state space E' which arise as dual automata, and stability properties of dual automata are characterized by the following Theorems.

5. Theorem. Let $\mathscr{D} = (X, E'; h)$ be a tlsa. Then there exists a tlsa $\mathscr{A} = (X, E; H)$ such that $\mathscr{D} = \mathscr{A}^d$ if and only if $\{h(x);\ x \in X\} \subset \Lambda(E')$ is equicontinuous.

Proof. 1. Assume $\mathscr{D} = \mathscr{A}^d$, then given a neighbourhood U of 0 in E', it has to be shown that there exists a 0-neighbourhood V in E' such that

$$h(x, \varphi) \in U$$

for any $x \in X$ whenever

$$\varphi \in V.$$

Without loss of generality, U can be assumed to be the polar B^0 of a bounded set $B \subset E$, i.e.

$$U = \{\varphi;\ \forall f \in B : \varphi(f) \leqq 1\}$$

([1], Proposition 23.14). Since $\{H(x);\ x \in X\}$ is equicontinuous,

$$A := \cup \{H(x)(B);\ x \in X\}$$

is bounded in E ([11], III.3.3,c), thus

$$V := A^0$$

is the looked for 0-neighbourhood in E'.

2. Fix for the moment $x \in X$. Since

$$h(x) : E' \to E'$$

is strongly continuous, it is continuous when both copies of E' are endowed with the weak topology $\sigma(E', E)$ (which is generated by the seminorms

$$q_f(\varphi) := (|\varphi(f)|, f \in E)$$

see [6], Proposition 3.12.6. From [9], 21.5, we infer that there exists a linear map

$$H(x) : E \to E$$

such that

$$h(x, \varphi)(f) = \varphi(H(x, f))$$

holds for any φ and for any f, and $H(x)$ is continuous when both copies of E are endowed with the weak topology $\sigma(E, E')$ (which is generated by the seminorms

$$q_\varphi(f) := q_f(\varphi),$$

$\varphi \in E'$). From [6], p. 258, Proposition 3.6.8, it is seen that $H(x)$ is continuous with respect to the given topology on E, since E is barreled. In order to complete the proof,

$$\{H(x, f);\ x \in X\}$$

has shown to be bounded in E for any fixed f. From the equicontinuity of $\{h(x);\ x \in X\}$ we infer that, given $B \subset E$ bounded there exists $C \subset E$ bounded such that

$$h(x, \varphi) \in B^0$$

for all $x \in X$ whenever

$$\varphi \in C^0 .$$

B can assumed to be closed, convex, and circled. Thus the equicontinuity of $\{h(x);\ x \in X\}$ implies

$$B^0 \subset \{H(x, f);\ f \in C,\ x \in X\}^0 ,$$

hence

$$\{H(x, f);\ f \in C,\ x \in X\} \subset \{\ldots\}^0 \subset B^{00} = B ,$$

since B^{00} is convex and $\sigma(E, E')$-closed, hence coincides with B by [2], Cor. 1 for Proposition 3.3.2.1. Thus in particular

$$\{H(x, f);\ x \in X\}$$

is bounded, hence $\mathscr{A} = (X, E; H)$ is a tlsa, and obviously $\mathscr{D} = \mathscr{A}^d$. □

If $\mathscr{A}$ is stable, the question arises whether $\mathscr{A}^d$ is stable, and conversely. For this, $A(E')$ has to be topologized; this is done exactly in the same manner as it has been done for $A(E)$ above by the seminorms

$$\{q^p;\ p \in \Sigma_E\} .$$

Thus a base for the neighbourhoods of $0 \in \Lambda(E')$ is formed by the sets

$$\{M(U^0, B^0);\ U \text{ is a 0-neighbourhood in } E,\ B \subset E \text{ is bounded}\},$$

where

$$M(P, Q) := \{F \in \Lambda(E');\ F(P) \subset Q\},$$

is defined for $P, Q \subset E'$.

6. Theorem. Let $\mathscr{A} = (X, E; H)$ be a tlsa with dual $\mathscr{A}^d = (X, E'; h)$. Then the following holds:

a) If $\mathscr{A}$ is stable, and E' is barreled, then $\mathscr{A}^d$ is stable;

b) If $\mathscr{A}^d$ is stable, then $\mathscr{A}$ is stable.

Proof. a) Given a bounded $S \subset E'$ and a 0-neighbourhood $W \subset E'$, it suffices to demonstrate that there exist $T \subset E'$ bounded and a 0-neighbourhood $V \subset E'$ such that

$$\{h(x) - h'(x);\ x \in X\} \subset M(T, V)$$

implies

$$\{h(v) - h'(v);\ v \in X^*\} \subset M(S, V)$$

for any tlsa $\mathscr{B} = (X, E'; h')$. Since, given $\varphi \in E'$, $\{h'(x, \varphi);\ x \in X\}$ is bounded in E', and E' is barelled, $\{h'(x);\ x \in X\}$ is equicontinuous ([11], III.4.2), hence $\mathscr{B}$ is the dual of a tlsa $\mathscr{C} = (X, E; H')$ by Theorem 5. Since S is bounded, there exists a closed convex 0-neighbourhood $G \subset E$ such that

$$S \subset G^0,$$

and there exists a bounded set $B \subset E$ such that

$$B^0 \subset G.$$

$\mathscr{A}$ is stable, hence there are $C \subset E$ bounded, and a 0-neighbourhood $F \subset E$ such that

$$\{H(x, f) - H'(x, f);\ f \in C,\ x \in X\} \subset F$$

implies that

$$\{H(v, f) - H'(v, f);\ v \in X^*,\ f \in B\} \subset G.$$

Consequently,

$$\{h(v) - h'(v);\ v \in X^*\} \subset M(G^0, B^0),$$

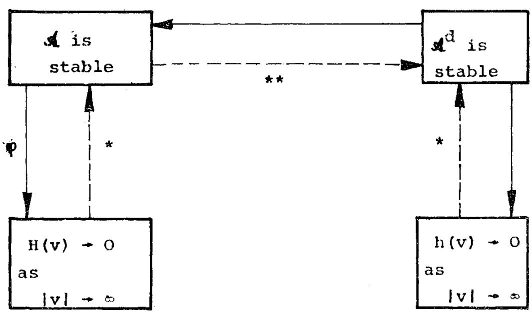
provided

$$\{h(x) - h'(x);\ x \in X\} \subset M(F^0, C^0).$$

b) Stability of $\mathscr{A}$ can be proved by a similar argument, having in mind the only-if part of Theorem 5. □

Note that E' is barreled in case E is a Fréchet space, i.e. a locally convex space which is metrizable by a complete metric.

Summarizing, the following diagram shows in what a manner the stability of a topological linear automaton can be related to the stability of its dual; for this let $\mathscr{A} = (X, E; H)$ be the automaton with its dual $\mathscr{A}^d = (X, E'; h)$.



* holds in case E is a Banach space

** holds in case E' is barreled

We will now deal with an example, kindly provided by Ed. Dubinsky [5], that shows that the statements connected with the simply starred arrows do not necessarily hold in case E lacks the Banach space property.

Let $X := \{x\}$ be a one letter alphabet, and let E the Fréchet space consisting of all real sequences $r = (r_n)$, $n \in \mathbb{N}$ with

$$p_k(r) := \sup \{|r_n| \exp(-n/(k+1));\ n \in \mathbb{N}\} < \infty ,$$

the topology of which is generated by the set $\{p_k;\ k \in \mathbb{N}\}$ of seminorms. This Fréchet space is nuclear, and is isomorphic to the complex functions analytic on the open unit disk with the compact open topology. A subbase for the neighbourhoods at $0 \in E$ is given by the sets $\{U_k;\ k \geqq 0\}$, where

$$U_k := \{r \in E;\ p_k(r) < 1/(k+1)\} .$$

From the nuclearity of E it is deduced that any bounded set is contained in a set of the form B_t, where $t \in E$ has only positive components, and B_t is defined by $B_t := \{s \in E;\ |s_n| < t_n \text{ for every } n \in \mathbb{N}\}$.

Let $m \in E$ be arbitrary such that $m_n > 0$ for any $n \in \mathbb{N}$, then it will be shown that one may find a continuous linear operator $T: E \to E$ with the property that the statement

$$T(B_a) \subset U_k \text{ implies } T_g(B_m) \subset U_1 \text{ for any } g \in \mathbb{N}$$

is false for any $a \in E$, $k \in \mathbb{N}$. This will give the required counterexample upon defining

$$H(x) := 0,$$
$$H'(x) := T.$$

This suffices since X has only one letter.

Now let $a \in E$ be fixed with $a_n > 0$ for every $n \in \mathbb{N}$, and let $k \in \mathbb{N}$ be arbitrary. Since $a \in E$, we know that

$$\lim_{n \to \infty} a_n \,.\, \exp\left(-(n+1)/(k+1)\right) = 0,$$

thus there exists $n_0 \in \mathbb{N}$ such that $b_n = e$ for $n > n_0$, where b_n is defined by

$$b_n := \min\{e, 1/[(k+1)\,.\,a_n]\,.\,\exp\left((n+1)/(l+1)\right)\}.$$

Let $g' \in \mathbb{N}$ be so large that $g > n_0$, and that

$$b_1 \,.\, \ldots \,.\, b_{n_0} \,.\, \exp\left(-(n_0 + 1/2) + g/2\right) > 1/m_1.$$

Define now the operator $T: E \to E$ in the following way:

$$(Tr)_n := \begin{cases} b_{n-1} r_{n-1}, & \text{if } 2 \leqq n \leqq g+2 \\ 0, & \text{otherwise.} \end{cases}$$

It is obvious that T is linear and continuous. Now let $r \in B_a$, then $|r_n| \leqq a_n$ holds always, and if $2 \leqq n \leqq g + 2$, then

$$|(Tr)_n| = |r_{n-1}| \,.\, b_{n-1} \leqq |r_{n-1}| \,.\, 1/[(k+1)\,.\,a_n] \exp\left(-n/(k+1)\right) \leqq$$
$$\leqq 1/(k+1)\,.\,\exp\left(n/(k+1)\right),$$

hence $Tr \in U_k$. On the other hand let $r_1 := m_1$, and let $r_n := 0$ if $n > 1$, then r is a member of B_m, and

$$(T^g r)_{g+1} = b_1 \,.\, \ldots \,.\, b_g \,.\, m_1 = m_1 b_1 \,.\, \ldots \,.\, b_{n_0} \,.\, \exp\left(g - n_0\right) >$$
$$> \exp\left(n_0 + 1/2 - g/2 + p - n_0\right) = \exp\left((g+1)/2\right).$$

This shows $T^g r \notin U_1$.

Thus $H(v) \to 0$, as $|v| \to \infty$ does not imply stability in the sense defined here, hence the corresponding question posed on p. 32 of [4] has to be answered in the negative. On the other hand, Dubinsky's example demonstrates convincingly that the definition of stability has implications that are counterintuitive – since the first example of a stable automaton of course should be that of one with constant state transitions.

It will now be investigated how stability may be characterized using a uniformity on the set of all tlsa, where an extra condition, viz., joint continuity, will be added. This characterization may be considered as generalization of Rabin's notion of stability, which has originally been formulated in terms of cut point languages.

Let in what follows now X be a compact Hausdorff space, and let E be a locally convex linear space over $\mathbb{R}$ with topological dual E'. Endowed with the topological sum from $(X^n)_{n \geqq 0}$, X^* is locally as well as σ-compact.

We need a specialization of topological linear automata, namely state linear acts, in which an initial state will be fixed.

7. Definition. The tlsa $(X, E; H)$ is said to be a *state linear act* iff the following conditions hold: $H : X \times E \to E$ is continuous, $H(x, \cdot)$ is linear for any input letter $x \in X$, and $f_0 \in E$ is a fixed initial state.

Note that any state linear act constitutes a tlsa upon neglecting the initial state, since compact subsets in topological vector spaces are bounded. As above extend H to a map $H : X^* \times E \to E$, then H is readily seen to be continuous such that $H(v, \cdot)$ is linear for any $v \in X^*$. Now let $\mathscr{H}$ be the set of all state linear acts (with fixed X, E, and f_0), and identify $(X, E; H)$ with H for the sake of simplicity. Consider a continuous and linear $\Psi : E \to \mathbb{R}$ with $\Psi \neq 0$. This linear form can be regarded as a mathematical model of measuring: think of H as a mechanical or biological system, which is given by a linear space of continuous functions over the time axis and which – as time passes – changes continuously and is linearly dependent upon external stimuli. Fix some epochs t_i, $1 \leqq i \leqq n$, such that the average value of $H(v, f_0)$ at these moments will be considered as the measured value after input of $v \in X^*$, hence the value in question is

$$\frac{1}{n} \sum_{i=1}^{n} H(v, f_0)(t_i) =: \Psi(H(v, f_0)),$$

where $\Psi := 1/n \sum_{i=1}^{n} \varepsilon(t_i)$, and $\varepsilon(t)(f) := f(t)$ is the evaluation map at t. Now fix $\theta \in \mathbb{R}$, then

$$S(H, \theta) := \{v \in V^*;\ \Psi(H^*(v, f_0)) \leqq \theta\}$$

is sais to be the *language at the cut point* θ. These languages have been introduced by M. Rabin in his classical paper [10] in which he discusses (finite) stochastic automata for the first time; in that paper, too, a discussion of the stability of stochastic automata is found. Roughly speaking, Rabin calls a stochastic automaton stable if the cut point language is not affected when instead of the given automaton another with only slightly different behavior is considered. This looks like a topological characterization of stability. Before working this out, bibliographical accuracy demands to mention that the cut point languages discussed by Rabin are actually the complements of the languages considered here. This change to the complement has been undertaken in order to get rid of some tedious formalities which would arise in this case below.

Define a uniformity on $\mathscr{H}$ in the following way: H_1 and H_2 are thought to be close iff they behave in a similar manner for all those states which come from a bounded set in E. Formally, let $B \subset E$ be bounded, and U an open neighbourhood of 0, then upon defining

$$P(B, U) := \{(H_1, H_2);\ \forall f \in B\ \forall v \in X^* : H_1(v, f) - H_2(v, f) \in U\},$$

the sets

$$\{P(B, U);\ B \subset E \quad \text{bounded},\ U \text{ open 0-neighbourhood}\}$$

evidently constitute a base for a separated uniformity on $\mathscr{H}$. Assume for the moment that for the topological linear automata, rather than for the acts H_i, $i = 1, 2$, $(H_1, H_2) \in P_L(B, U)$ holds whenever $H_1(v, f) - H_2(v, f) \in U$ for all $f \in B$ and all $v \in L$, then this constitutes the base for a uniformity $\mathscr{U}_L$ on the space of all topological linear automata. This uniformity is separated in case $X \subset L$ holds. Now denoting by $\mathscr{U}_L(H)$ the neighbourhood filter for H induced by $\mathscr{U}_L$, the foretold topological characterization of stability is now evident: H is stable iff $\mathscr{U}_X(H)$ coincides with $\mathscr{U}_{X^*}(H)$. But now again let us restrict our attention to state linear acts (which has some technical advantages).

Denote by Γ the topology generated by this uniformity, then Γ has some pleasant natural properties in the following sense.

9. Lemma. a) $H \mapsto H(v, f_0)$ is continuous for fixed v;

b) if E is endowed with the weak topology $\sigma(E, E')$, then

$$\begin{cases} X^* \times \mathscr{H} \to E \\ (v, H) \mapsto H(v, f_0) \end{cases}$$

is (jointly) continuous.

Proof. a) is evident from the boundedness of $\{f_0\}$.

b) Consider the space $\mathscr{C}(X^*, E)$ of continuous maps from X^* to E, and endow $\mathscr{C}(X^*, E)$ with the compact open topology, which has as a subbase the sets

$$\{W(K, U);\ K \subset X^* \text{ is compact, } U \text{ is a } \sigma(E, E')\text{-open set in } E\},$$

where $g \in W(K, U)$ iff $g[K] \subset U$, see [8], p. 221f. Since X^* is localy compact, the compact open topology is the smallest topology on $\mathscr{C}(X^*, E)$ which makes the evaluation map

$$e : \begin{cases} X^* \times \mathscr{C}(X^*, E) \to \mathscr{C}(X^*, E) \\ (v, g) \qquad\qquad \mapsto (v \mapsto H^*(v, f_0)), \end{cases}$$

continuous. Now define

$$F:\begin{cases}\mathscr{H} \to \mathscr{C}(X^*, E)\\ H \mapsto (v \mapsto H^*(v, f_0)),\end{cases}$$

then it is enough to show that, given $H \in \mathscr{H}, K \subset X^*$ compact, and a $\sigma(E, E')$-open set U, there exists a bounded set $B \subset E$ and a 0-neighbourhood U_0 in the given topology of E such that

$$(H_1, H) \in P(B, U_0) \Rightarrow F(H_1) \in W(K, U)$$

holds, provided $F(H) \in W(K, U)$.

From the definition of $\sigma(E, E')$ we see that it is no loss of generality to assume that there exists $h \in E$, $\varphi \in E'$, $\varepsilon > 0$ such that

$$U = h + \{f \in E;\ |\varphi(f)| < \varepsilon\},$$

hence

$$\gamma := \max_{v \in K} |\varphi(H(v, f_0) - h)| < \varepsilon$$

holds. Putting

$$B := \{f_0\},$$
$$U_0 := \{f \in E;\ |\varphi(f)| < \varepsilon - \gamma\},$$

the desired implication holds. □

This in mind semicontinuity of S can be investigated. Remember that a set valued map $F : A \to \mathscr{P}(B) := \{B_0; B_0 \subset B\}$ is *upper (lower) semicontinuous* in a iff $\{x; F(x) \subset U\}$ [resp. $\{x; F(x) \cap U \neq \emptyset\}$] is a neighbourhood for a, provided $U \subset B$ is open. Call a set valued map $F : A \to \mathscr{P}(B)$ *complementary lower semicontinuous* in a iff $B - F : x \mapsto B - F(x)$ is lower semicontinuous in a.

10. Proposition Let $H \in \mathscr{H}$ be a state linear act such that $\emptyset \neq S(H, \theta) \neq X^*$ holds. Then $S(\cdot, \theta)$ is upper, and complementary lower semicontinuous in H.

Proof. "lower": this part is an immediate consequence of part a) in the lemma above, since

$$\{H;\ \Psi(H(v, f_0)) > \theta\}$$

is open in $\mathscr{H}$.

"upper": Because of part b) in 9,

$$W := \{(H, v);\ v \in S(H, \theta)\}$$

is closed in $\mathscr{H} \times X^*$. Now let M be a closed subset of X^* such that

$$S(H, \theta) \cap M = \emptyset$$

holds, then $M_n := M \cap X^n$ is compact, and

$$W \cap (\{H\} \times M_n) = \emptyset$$

for any $n \in \mathbb{N}$. Since $\{H\} \times M_n$ is compact, the Theorem of Gottschalk - Hedlund ([8], Theorem 5.12) implies that there exist open neighbourhoods $K_n \subset X^n$ of M_n and $G_n \subset \mathscr{H}$ of H such that

$$W \cap (G_n \times K_n) = \emptyset$$

for any $n \in \mathbb{N}$. Consequently,

$$V := \cup \{G_n \times K_n;\ n \in \mathbb{N}\}$$

is open in $\mathscr{H} \times X^*$, and

$$U := \{H_1;\ (H_1, v) \in V \text{ for some } v \in X^*\}$$

is an open neighbourhood of H with the property that

$$S(H_1, \theta) \cap M = \emptyset$$

holds, whenever $H_1 \in U$. □

Now we have gathered enough information in order to demonstrate that $S(\cdot, \theta)$ is weakly measurable on

$$\mathscr{H}_\theta := \{H \in \mathscr{H};\ S(H, \theta) \neq \emptyset\}\ .$$

This means that

$$\{H \in \mathscr{H}_\theta;\ S(H, \theta) \cap U \neq \emptyset\}$$

is measurable, provided $U \subset X^*$ is open. But this requires that $\mathscr{H}_\theta$ has shown to be a Borel set with respect to (the Borel σ-field generated by) Γ. The latter is accomplished in the following manner:

$$K_0 := \{H;\ \Psi(H(v, f_0)) = \theta \text{ for some } v \in X^n\}$$

is a measurable subset of $\mathscr{H}$, since

$$\{H;\ \Psi(H(v, f_0)) = \theta \text{ for some } v \in X^n\}$$

is closed in $\mathscr{H}$ because of the compactness of X, and the continuity statted in Lemma 9, b). The construction of Γ now yields that

$$K' := \{H;\ \Psi(H(v, f_0)) \geqq \theta \text{ for all } v \in X^*\}$$

is closed, too, consequently

$$\{H;\ S(H, \theta) = \emptyset\} = K' - K_0$$

is measurable, hence $\mathscr{H}_\theta$ is, as the complement of the latter set. Now endow $\mathscr{H}_\theta$ with the trace of the Borel σ-field defined by Γ, then

$$S(\cdot, \theta) : \mathscr{H}_\theta \to \{F \subset X^*;\ F \text{ is closed}\}$$

is weakly measurable, if X is in addition a metric space. Hence one gets from Corollary 4.2 in [3]:

11. Theorem. If X is a compact metric space, there exists a transition probability Q_0 from $\mathscr{H}_\theta$ to X^* such that $S(H, \theta)$ is the smallest closed subset C of X^* such that $Q_\theta(H)(C) = 1$ holds for any $H \in \mathscr{H}_\theta$.

Now it becomes clear why we have restricted our attention to $\mathscr{H}_\theta$, rather than to $\mathscr{H}$, since the statement on $S(H, \theta)$ implies the contradiction $0 = Q_\theta(H)(S(H, \theta)) = 1$ in case $H \notin \mathscr{H}_\theta$.

Let us interpret Q_θ as a stochastic system that, if endowed with the data of H, accepts the language $B \subset X^*$ with probability $Q_\theta(H)(B)$ – strictly speaking we must restrict our attention to Borel languages B. Then the Theorem above states that $S(H, \theta)$ is the smallest closed language which is accepted surely, i.e. with probability 1. In this sense cut point languages have a kind of threshold character.

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