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ASYMPTOTIC BEHAVIOUR OF THE ROBUST TEST IN THE RIEDER'S MODEL OF CONTAMINATION

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Bounds of the asymptotic rate of convergence of the error probabilities of a robust test in the Rieder's model of contamination are found. At first the bound is derived for the case when the true distribution is assumed to be one of the pair of the least favourable ones, then under the hypothesis that data have not been contaminated (a superfluous use of the robust test). The requirement of the mutual consistency of the results is explained and proved. All results are illustrated by a numerical example.

1. INTRODUCTION

An occasional bad behaviour of classical statistical methods in cases, when data were even in a "slight" disagreement with an explanatory model, was a source of the activity which is referred to as robustness studies. In the stream of this activity one may easy recognize two main branches. One of them has yielded studies of the robustness of classical statistical methods, the robustness with respect to deviations from assumed statistical model, e.g. studies of the behaviour of the t-test for dependent observations. The other one has brought building up new models of the data, including heterogenity of them. Then a solution of questions under consideration has been searched in a classical statistical sense, e.g. finding minimax tests in a framework of a model of contamination [3], [4], [9], [5]. Having obtained such solution its properties should be studied in order to provide a solid base for applications.

This is a purpose of the present paper to give a hint for comparison of the effectiveness of the likelihood ratio test and a special robust version of it. As far as the author knows the only analogous comparison was published by Rieder [6], who examined a model with a family of local alternatives, and by Víšek [8] for model with the same contamination of the hypothesis and the alternative. In Rieder's approach the contiguity of the probability measures, the least favourable pair consists of, was implied. Moreover, in order to fulfil the requirement of the disjointness of hypothesis and alternatives in the model of local alternatives, it is necessary to admitt a decreasing, with increasing number of observations, influence of contamination upon data. On the contrary, in this paper a fixed part of data is considered to cause the contamination of them.

The minimax test problem for some model of contamination belongs to the latter branch (as mentioned above). It was shown ([3]) that the likelihood ratio test of the least favourable pair of probability distributions, if any, solves this problem. Huber and Strassen studied this problem in 1972 for a rather general model of contamination described by means of Choquet's capacity (see [2]). Unfortunately in their model the least favourable pairs of probability measures may be found only in an implicit form. Moreover, earlier Huber's ε-contamination model has appeared to be a special case of the Huber-Strassen's one only under assumption of compactness (compare [5]). Another considerably general model of contamination, affording an explicit description of the least favourable pairs of probability measures, was proposed by Rieder [5]. He has constructed the model of contamination which contains as special cases the ε-contamination and total variation contamination models. An interesting feature of Rieder's results is his having shown the possibility of existence of different least favourable pairs of probability measures.

As one of characteristics of the test may serve the asymptotic rate of convergence of the error probabilities. For the case of testing a simple hypothesis against a simple alternative Chernoff in 1952 showed this asymptotic rate to be equal to the logarithm of the minimal α-entropy. Therefore an estimation of the minimal α-entropy for the least favourable pair of distributions is the first task which is to be solved. On the other hand, being sure of the presence of the contamination, one should use the robust version of test, even if the discernibility of this test would be very bad. On the contrary, let us assume, for a while, to be in a situation when the presence of the contamination is questionable. Then the knowledge of the asymptotic rate of the error probabilities of the robust test under the assumption that the true distribution is not contaminated may prove very useful. If discernibility of robust test for non-contaminated distributions with respect to the most powerful one is only a little less, we may content ourselves with this less rate of convergence of the error probabilities. Then we use the robust version of test (to be ensured against contamination, if any) and do not waste the time and money to decide about presence of contamination. Therefore an estimation of the rate of convergence of the error probabilities of robust test is the second important task of robust testing which was attacted in the present paper. Finally the need of the consistency of derived estimations is explained and proved. To make easier the understanding of given results a numerical example is presented in the last paragraph.

2. NOTATIONS

Let $\mathscr{N}$ be the set of all positive integers, R – the real line, $\emptyset$ – the empty set.

Let $(\mathscr{X}, \mathscr{A})$ be a measurable space, $\mathscr{M}$ – the set of all probability measures defined on it. Let P and Q be a pair from $\mathscr{M}$ and let us denote p and q densities of P and Q with respect to a measure μ. Now denote $H_\alpha(p, q)$ $(\alpha \in (0, 1))$ and $H(p, q)$ the α-entropy and minimal α-entropy of the measures P and Q, respectively; i.e.

$$H_\alpha(p, q) = \int p^\alpha q^{1-\alpha} \, d\mu$$

and

$$H(p, q) = \inf_{0<\alpha<1} H_\alpha(p, q) .$$

For any $n \in \mathscr{N}$ put $\mathscr{X}_n = \bigtimes_{i=1}^{n} \mathscr{X}^{(i)}$, where $\mathscr{X}^{(i)} = \mathscr{X}$ for all i. Let the set $\{1, 2, 3, \ldots, n\}$ be denoted by N_n. Let us use $\|D\|$ for the cardinal number of $D(\subset N_n)$. For any $D \subset N_n$ and $\tau > 0$ let

$\mathscr{K}^n_{D,\tau}(p, q) = \{x \in \mathscr{X}_n : \prod_{i\in D} p(x_i) < \tau \prod_{i\in D} q(x_i)\}$. (For the case $D = N_n$ index D will be dropped out; analogously for τ, if $\tau = 1$.) Finally, define for $A \in \mathscr{A}_n$

$$e_n(p, A) = \int_A \prod_{i=1}^{n} p(x_i) \, d\mu$$

and

$$e_n(p, q, A) = e_n(p, A) + e_n(q, A^c) .$$

3. ASSERTION

To be able to present our result we must recall the Rieder's model of contamination and his result describing the least favourable pair.

Let for $i = 0, 1$, $P_i \in \mathscr{M}$, $\varepsilon_i \geqq 0$, $\delta_i \geqq 0$, $0 < \varepsilon_i + \delta_i < 1$ and μ be a measure which dominates simultaneously P_0 and P_1. Put for any $A \in \mathscr{A}$

$$v_i(A) = \min \{(1 - \varepsilon_i) P_i(A) + \varepsilon_i + \delta_i, 1\} \quad \text{if} \quad A \neq \emptyset$$
$$v_i(\emptyset) = 0$$

and

$$\mathscr{P}_i = \{Q \in \mathscr{M} : Q(A) \leqq v_i(A), \ \forall A \in \mathscr{A}\} .$$

Further let us assume that $\mathscr{P}_0 \cap \mathscr{P}_1 = \emptyset$.

Definition. The pair (Q_0, Q_1) $(Q_i \in \mathscr{M})$ is called a least favourable pair (LFP) for $(\mathscr{P}_0, \mathscr{P}_1)$ iff

$$Q_0(\pi > t) = \sup \{Q'(\pi > t) : Q' \in \mathscr{P}_0\}$$
$$Q_1(\pi > t) = \inf \{Q''(\pi > t) : Q'' \in \mathscr{P}_1\}$$

for all $t > 0$, where π is a suitable version of the Radon-Nikodym derivative dQ_1/dQ_0.

Assertion. (see [5]). Let Δ_0 and Δ_1 be a solution of the equations:

$$\Delta_0 P_0(\Delta < \Delta_0) - P_1(\Delta < \Delta_0) = \nu_1 + \omega_0 \Delta_0 , \tag{1}$$

$$P_1(\Delta > \Delta_1) - \Delta_1 P_0(\Delta > \Delta_1) = \nu_0 \Delta_1 + \omega_1 , \tag{2}$$

where

$$\nu_i = \frac{\varepsilon_i + \delta_i}{1 - \varepsilon_i}, \quad \omega_i = \frac{\delta_i}{1 - \varepsilon_i} \quad \text{and} \quad \Delta \in \mathrm{d}P_1/\mathrm{d}P_0 . \tag{3}$$

(Such solution always exists.) Then Q_0 and Q_1 is an LFP for $(\mathscr{P}_0, \mathscr{P}_1)$ iff:

(4) i) μ dominates Q_0, Q_1 – with densities q_0, q_1, say.

ii) $q_1/q_0 = (1 - \varepsilon_1)/(1 - \varepsilon_0) \max \{\Delta_0, \min \{\Delta, \Delta_1\}\}$ $(Q_0 + Q_1)$-a.e.

iii) $q_0 = (1 - \varepsilon_0) p_0$ μ-a.e. on $\{\Delta_0 \leqq \Delta \leqq \Delta_1\}$.

iv) $(1 - \varepsilon_0) p_1/\Delta_0 \leqq q_0 \leqq (1 - \varepsilon_0) p_0$ μ-a.e. on $\{\Delta < \Delta_0\}$.

v) $(1 - \varepsilon_0) p_0 \leqq q_0 \leqq (1 - \varepsilon_0) p_1/\Delta_1$ μ-a.e. on $\{\Delta_1 < \Delta\}$.

vi) $Q_0(\Delta < \Delta_0) = (1 - \varepsilon_0) P_0(\Delta < \Delta_0) - \delta_0$.

(From these relations follows the symmetric ones for q_1 and Q_1, e.g.

$$Q_1(\Delta_1 < \Delta) = (1 - \varepsilon_1) P_1(\Delta_1 < \Delta) - \delta_1 .) \tag{5}$$

4. RESULTS

Theorem 1. For minimal α-entropy of the LFP Q_0 and Q_1 holds:

$$H(q_0, q_1) \leqq \inf_{0<\alpha<1} (1 - \varepsilon_0)^\alpha (1 - \varepsilon_1)^{1-\alpha} \{H_\alpha(p_0, p_1) + \Delta_0^{1-\alpha}[P_0(\Delta < \Delta_0) - \omega_0] + \Delta_1^{-\alpha}[P_1(\Delta_1 < \Delta) - \omega_1]\} .$$

Proof. A straightforward computation gives: $(0 < \alpha < 1)$

$$H(q_0, q_1) \leqq \int q_0^\alpha q_1^{1-\alpha} \, \mathrm{d}\mu \leqq \int_{\{x \in \mathscr{X}; \Delta < \Delta_0, q_0(x) > 0\}} (q_1/q_0)^{1-\alpha} q_0 \, \mathrm{d}\mu + (1 - \varepsilon_0)^\alpha (1 - \varepsilon_1)^{1-\alpha} \int_{\{x \in \mathscr{X}; \Delta_0 \leqq \Delta \leqq \Delta_1\}} p_0^\alpha p_1^{1-\alpha} \, \mathrm{d}\mu +$$
$$+ \int_{\{x \in \mathscr{X}; \Delta_1 < \Delta(x), q_1(x) > 0\}} (q_0/q_1)^\alpha q_1 \, \mathrm{d}\mu \leqq \left(\frac{1 - \varepsilon_1}{1 - \varepsilon_0} \Delta_0\right)^{1-\alpha} Q_0(\Delta < \Delta_0) + (1 - \varepsilon_0)^\alpha (1 - \varepsilon_1)^{1-\alpha} H_\alpha(p_0, p_1) + \left(\frac{1 - \varepsilon_1}{1 - \varepsilon_0} \Delta_1\right)^{-\alpha} Q_1(\Delta > \Delta_1) .$$

Making use of (4) and (5) one obtains the assertion of the theorem. □

Theorem 2. Asymptotic rate of convergence of the error probabilities of the robust test with critical region $\mathscr{K}^n(q_0, q_1)$, when the true distribution is either P_0 or P_1, can be estimated as follows:

$$\limsup_{n\to\infty} n^{-1} \log e_n(p_0, p_1, \mathscr{K}^n(q_0, q_1)) \leqq$$

$$\leqq \inf_{0<\alpha<1} \log\left[\max\left\{\left(\frac{1-\varepsilon_1}{1-\varepsilon_0}\right)^{1-\alpha} [H_\alpha(p_0, p_1) + \Delta_0^{1-\alpha} P_0(\Delta < \Delta_0)]\,,\right.\right.$$

$$\left.\left.\left(\frac{1-\varepsilon_1}{1-\varepsilon_0}\right)^{\alpha} [H_\alpha(p_0, p_1) + \Delta_1^{-\alpha} P_1(\Delta > \Delta_1)]\right\}\right].$$

Proof. Using Lemma 1A of Appendix we may write

$$\limsup_{n\to\infty} n^{-1} \log e_n(p_0, p_1, \mathscr{K}^n(q_0, q_1)) = \tag{6}$$

$$= \max\{\limsup_{n\to\infty} n^{-1} \log e_n(p_0, \mathscr{K}^n(q_0, q_1))\,,$$

$$\limsup_{n\to\infty} n^{-1} \log e_n(p_1, [\mathscr{K}^n(q_0, q_1)]^c)\}\,.$$

Let us examine at first the first member of the right hand side of the last equality.

Put

$$D_{n0}(x) = \{i \in N_n : \Delta(x_i) < \Delta_0\}$$

and

$$D_{n1}(x) = \{i \in N_n : \Delta(x_i) > \Delta_1\}\,.$$

Let us have $x^0 \in \mathscr{K}^n(q_0, q_1)$. Then

$$\prod_{i=1}^n q_0(x_i^0) < \prod_{i=1}^n q_1(x_i^0)\,,$$

i.e.

$$\left(\frac{1-\varepsilon_0}{1-\varepsilon_1}\right)^n \prod_{i\notin D_{n0}(x^0)\cup D_{n1}(x^0)} p_0(x_i^0) < \prod_{i\notin D_{n0}(x^0)\cup D_{n1}(x^0)} p_1(x_i^0)\, \Delta_0^{\|D_{n0}(x^0)\|} \Delta_1^{\|D_{n1}(x^0)\|}\,.$$

For $i \in D_{n1}(x^0)$ we have $\Delta_1 < \Delta(x_i^0) = p_1(x_i^0)/p_0(x_i^0)$ and so the last inequality implies

$$\left(\frac{1-\varepsilon_0}{1-\varepsilon_1}\right)^n \prod_{i\notin D_{n0}(x^0)} p_0(x_i^0) < \prod_{i\notin D_{n0}(x^0)} p_1(x_i^0)\, \Delta_0^{\|D_{n0}(x^0)\|}\,. \tag{7}$$

Let us finally denote for $D \subset N_n$

$$A_0(D) = \{x \in \mathscr{X};\ \Delta(x_i) < \Delta_0 \text{ for } i \in D\}\,.$$

Then from (7) we may deduce that

$$x^0 \in \mathscr{K}^n_{D^c_{n0}(x^0),\tau}(p_0, p_1) \cap A_0(D_{n0}(x^0))\,,$$

where $\tau = [(1 - \varepsilon_1)/(1 - \varepsilon_0)]^n \Delta_0^{\|D_{n0}(x^0)\|}$. So we have found

(8) $$\mathscr{K}^n(q_0, q_1) \subset \bigcup_{B \subset N_n} \{\mathscr{K}^n_{B,\tau(B,n)}(p_0, p_1) \cap A_0(B^c)\},$$

where $\tau(B, n) = [(1 - \varepsilon_1)/(1 - \varepsilon_0)]^n \Delta_0^{n - \|B\|}$. Let us write $\tau(m, n)$ instead of $\tau(B, n)$ when $B = \{1, 2, \ldots, m\}$. Let $\alpha \in (0, 1)$. Making use of (8) and Lemma 2A of Appendix we obtain

(9) $$\limsup_{n\to\infty} \frac{1}{n} \log e_n(p_0, \mathscr{K}^n(q_0, q_1)) \leqq$$

$$\leqq \limsup_{n\to\infty} \frac{1}{n} \log \sum_{m=1}^{n} \binom{n}{m} \int_{\mathscr{K}^n_{m,\tau(m,n)}(p_0,p_1)} \prod_{i=1}^{m} p_0(x_i)\, \mathrm{d}\mu [P_0(\Delta < \Delta_0)]^{n-m} \leqq$$

$$\leqq \limsup_{n\to\infty} \frac{1}{n} \log \sum_{m=1}^{n} \binom{n}{m} \left[\left(\frac{1 - \varepsilon_1}{1 - \varepsilon_0}\right)^n \Delta_0^{n-m}\right]^{1-\alpha} H_\alpha^m(p_0, p_1)\, [P_0(\Delta < \Delta_0)]^{n-m} \leqq$$

$$\leqq \limsup_{n\to\infty} \frac{1}{n} \log \left(\frac{1 - \varepsilon_1}{1 - \varepsilon_0}\right)^{(1-\alpha)n} [H_\alpha(p_0, p_1) + \Delta_0^{1-\alpha} P_0(\Delta < \Delta_0)]^n =$$

$$= \log \left\{\left(\frac{1 - \varepsilon_1}{1 - \varepsilon_0}\right)^{1-\alpha} [H_\alpha(p_0, p_1) + \Delta_0^{1-\alpha} P_0(\Delta < \Delta_0)]\right\}.$$

To estimate the second member of the right hand side of (6) let us derive analogously as above

$$[\mathscr{K}^n(q_0, q_1)]^c \subset \bigcup_{B \subset N_n} \{\mathscr{K}^n_{B,\nu(B,n)}(p_1, p_0) \cap A_1(B^c)\},$$

where $\nu(B, n) = [(1 - \varepsilon_0)/(1 - \varepsilon_1)]^n \Delta_1^{\|B\| - n}$ (note that p_0 and p_1 are interchanged). Applying Lemma 2A once again we finally find

(10) $$\limsup_{n\to\infty} \frac{1}{n} \log e_n(p_1, [\mathscr{K}^n(q_0, q_1)]^c) \leqq$$

$$\leqq \limsup_{n\to\infty} \frac{1}{n} \log \sum_{m=1}^{n} \binom{n}{m} \int_{\mathscr{K}^n_{m,\nu(B,n)}(p_1,p_0)} \prod_{i=1}^{m} p_1(x_i)\, \mathrm{d}\mu [P_1(\Delta > \Delta_1)]^{n-m} \leqq$$

$$\leqq \limsup_{n\to\infty} \frac{1}{n} \log \sum_{m=1}^{n} \binom{n}{m} \left[\left(\frac{1 - \varepsilon_0}{1 - \varepsilon_1}\right)^n \Delta_1^{m-n}\right]^\alpha H_{1-\alpha}^m(p_1, p_0)\, [P_1(\Delta > \Delta_1)]^{n-m} \leqq$$

$$\leqq \limsup_{n\to\infty} \frac{1}{n} \log \left(\frac{1 - \varepsilon_0}{1 - \varepsilon_1}\right)^{\alpha n} [H_\alpha(p_0, p_1) + \Delta_1^{-\alpha} P_1(\Delta > \Delta_1)]^n =$$

$$= \log \left\{\left(\frac{1 - \varepsilon_0}{1 - \varepsilon_1}\right)^\alpha [H_\alpha(p_0, p_1) + \Delta_1^{-\alpha} P_1(\Delta > \Delta_1)]\right\}.$$

Keeping in mind symmetry of the cases $\varepsilon_0 \geqq \varepsilon_1$ and $\varepsilon_1 \geqq \varepsilon_0$ and combining (6), (9) and (10) we may easy verify the assertion of theorem. □

Remark 1. As the logarithm of the bound of minimal α-entropy $H(q_0, q_1)$ estimates the worst possible rate of convergence of the error probabilities when robust test is used, the rate of this convergence cannot be worse in the case when the true distribution is any other from $\mathscr{P}_0$ or $\mathscr{P}_1$. In the Theorem 2 the bound of the such rate (when data are distributed either by P_0 or by P_1) is given. On the other hand the fact that $P_0 \in \mathscr{P}_0$ and $P_1 \in \mathscr{P}_1$ is a straightforward consequence of the definition of $\mathscr{P}_0$ and $\mathscr{P}_1$. Therefore to have any sense, the bound given in the Theorem 2 should not be greater than that one in the Theorem 1. The following lemma shows that it is the case. (It is necessary to realize, that in the case when the bound in Theorem 1 is greater than one, it should be replaced by one, because $H(p, q) \in [0, 1]$; analogously if the bound in Theorem 2 is positive, it should be replaced by zero.)

Lemma 1. Let $\alpha \in (0, 1)$. Then

$$\begin{aligned}(11) \qquad & \min\{(1-\varepsilon_0)^\alpha (1-\varepsilon_1)^{1-\alpha} [H_\alpha(p_0, p_1) + \Delta_0^{1-\alpha}(P_0(\Delta < \Delta_0) - \omega_0) + \\ & + \Delta_1^{-\alpha}(P_1(\Delta_1 < \Delta) - \omega_1)], 1\} \geqq \\ & \geqq \min\left\{\max\left\{\left(\frac{1-\varepsilon_1}{1-\varepsilon_0}\right)^{1-\alpha} [H_\alpha(p_0, p_1) + \Delta_0^{1-\alpha} P_0(\Delta < \Delta_0)],\right.\right. \\ & \left.\left.\left(\frac{1-\varepsilon_0}{1-\varepsilon_1}\right)^\alpha [H_\alpha(p_0, p_1) + \Delta_1^{-\alpha} P_1(\Delta_1 < \Delta)^3\right\}, 1\right\}.\end{aligned}$$

Proof. Let the right hand side of this inequality be equal to one. Then either

$$(12) \qquad \left(\frac{1-\varepsilon_1}{1-\varepsilon_0}\right)^{1-\alpha} [H_\alpha(p_0, p_1) + \Delta_0^{1-\alpha} P_0(\Delta < \Delta_0)] > 1$$

or

$$(13) \qquad \left(\frac{1-\varepsilon_0}{1-\varepsilon_1}\right)^\alpha [H_\alpha(p_0, p_1) + \Delta_1^{-\alpha} P_1(\Delta_1 < \Delta)] > 1\,.$$

Let (12) be true. Then

$$(14) \qquad (1-\varepsilon_0)^\alpha (1-\varepsilon_1)^{1-\alpha} [H_\alpha(p_0, p_1) + \Delta_0^{1-\alpha} P_0(\Delta < \Delta_0)] > 1 - \varepsilon_0\,.$$

The definition of Δ_0 and Δ_1 together with the assumption $\mathscr{P}_0 \cap \mathscr{P}_1 = \emptyset$ (see [5], Lemma 4.3) yield inequalities

$$\Delta_0 < \frac{1-\varepsilon_0}{1-\varepsilon_1} < \Delta_1$$

(compare 5.27 in [5]). Now from (2) follows:

$$P_1(\Delta_1 < \Delta) \geqq \Delta_1 \nu_0 + \omega_1\,,$$

i.e.

$$\Delta_1^{-\alpha}[P_1(\Delta_1 < \Delta) - \omega_1] - \omega_0\Delta_1^{1-\alpha} \geqq \frac{\varepsilon_0}{1 - \varepsilon_0}\,\Delta_1^{1-\alpha} \geqq \frac{\varepsilon_0}{1 - \varepsilon_0}\left(\frac{1 - \varepsilon_0}{1 - \varepsilon_1}\right)^{1-\alpha}$$

and finally

$$(15) \qquad (1 - \varepsilon_0)^\alpha (1 - \varepsilon_1)^{1-\alpha} \{\Delta_1^{-\alpha}[P_1(\Delta_1 < \Delta) - \omega_1] - \omega_0\Delta_0^{1-\alpha}\} \geqq \varepsilon_0 .$$

Summing (14) and (15) one obtains that the minimum at the left hand side of inequality (11) is equal to one.

Analogously from (13) and (1) the same conclusion may be derived.

To complete the proof it is necessary to show that under assumption

$$(16) \qquad \left(\frac{1 - \varepsilon_1}{1 - \varepsilon_0}\right)^{1-\alpha} [H_\alpha(p_0, p_1) + \Delta_0^{1-\alpha}P_0(\Delta < \Delta_0)] < 1$$

and

$$(17) \qquad \left(\frac{1 - \varepsilon_1}{1 - \varepsilon_0}\right)^{-\alpha} [H_\alpha(p_0, p_1) + \Delta_1^{-\alpha}P_1(\Delta_1 < \Delta)] < 1$$

the following inequality

$$(1 - \varepsilon_0)^\alpha (1 - \varepsilon_1)^{1-\alpha} \{H_\alpha(p_0, p_1) + \Delta_0^{1-\alpha}[P_0(\Delta < \Delta_0) - \omega_0] +$$
$$+ \Delta_1^{-\alpha}[P_1(\Delta_1 < \Delta) - \omega_1]\} \geqq$$
$$\geqq \max\left\{\left(\frac{1 - \varepsilon_1}{1 - \varepsilon_0}\right)^{1-\alpha} {}^2H_\alpha(p_0, p_1) + \Delta_0^{1-\alpha}P_0(\Delta < \Delta_0)] ,\right.$$
$$\left.\left(\frac{1 - \varepsilon_0}{1 - \varepsilon_1}\right)^\alpha [H_\alpha(p_0, p_1) + \Delta_1^{-\alpha}P_1(\Delta_1 < \Delta)]\right\}$$

holds. But from (16) and (15) follows

$$\varepsilon_0 \left(\frac{1 - \varepsilon_1}{1 - \varepsilon_0}\right)^{1-\alpha} [H_\alpha(p_0, p_1) + \Delta_0^{1-\alpha}P_0(\Delta < \Delta_0)] < \varepsilon_0 \leqq$$
$$\leqq (1 - \varepsilon_0)^\alpha (1 - \varepsilon_1)^{1-\alpha} \{\Delta_1^{-\alpha}[P_1(\Delta_1 < \Delta) - \omega_1] - \omega_0\Delta_0^{1-\alpha}\} .$$

Adding $(1 - \varepsilon_0)((1 - \varepsilon_1)/(1 - \varepsilon_0))^{1-\alpha}[H_\alpha(p_0, p_1) + \Delta_0^{1-\alpha}P_0(\Delta < \Delta_0)]$ to the both sides of the this inequality we obtain

$$\left(\frac{1 - \varepsilon_1}{1 - \varepsilon_0}\right)^{1-\alpha} [H_\alpha(p_0, p_1) + \Delta_0^{1-\alpha}P_0(\Delta < \Delta_0)] <$$
$$< (1 - \varepsilon_0)^\alpha (1 - \varepsilon_1)^{1-\alpha} \{H_\alpha(p_0, p_1) + \Delta_0^{1-\alpha}[P_0(\Delta < \Delta_0) - \omega_0] +$$
$$+ \Delta_1^{-\alpha}[P_1(\Delta_1 < \Delta) - \omega_1]\} .$$

Similarly from (17) and a relation analogous to (15) the inequality

$$\left(\frac{1-\varepsilon_0}{1-\varepsilon_1}\right)^\alpha [H_\alpha(p_0, p_1) + \Delta_1^{-\alpha} P_1(\Delta_1 < \Delta)] <$$
$$< (1-\varepsilon_0)^\alpha (1-\varepsilon_1)^{1-\alpha} \{H_\alpha(p_0, p_1) + \Delta_0^{1-\alpha}[P_0(\Delta < \Delta_0) - \omega_0] +$$
$$+ \Delta_1^{-\alpha}[P_1(\Delta_1 < \Delta) - \omega_1]\},$$

follows; it concludes the proof. □

5. NUMERICAL EXAMPLE

Remark 2. The simplicity of the following example allows to calculate the value of $H(q_0, q_1)$ by the formula

$$H(q_0, q_1) = (1-\varepsilon_0)\left\{H(p_0, p_1)\left[1 - \frac{2}{\sqrt{(2\pi)}}\int_{\{\Delta < \Delta_0\}} \exp\left\{-\frac{(2x-\mu)^2}{8}\right\} dx\right] + \right.$$
$$\left. + 2\Delta_0^{\frac{1}{2}}[P_0(\Delta < \Delta_0) - \omega_0]\right\}.$$

Example 1. Let $P_0 = N(0, 1)$ and $P_1 = N(\mu, 1)$. Let $\hat{H}(q_0, q_1)$ and Re_n denote the bounds given in Theorem 1 and 2 respectively.

$\varepsilon_0 = \varepsilon_1 = \delta_0 = \delta_1 = \cdot 05$

μ	$H(p_0, p_1)$	$H(q_0, q_1)$	$\hat{H}(q_0, q_1)$	Re_n
1	·8825	·9917	1	1
2	·6065	·8654	1	·9316
3	·3246	·7291	·9053	·6514
4	·1353	·6471	·7291	·4665
5	·0439	·6134	·6427	·3769
10	3·73E-6	·5999	·5999	·3333

$\varepsilon_0 = \varepsilon_1 = \delta_0 = \delta_1 = \cdot 005$

μ	$H(p_0, p_1)$	$H(q_0, q_1)$	$\hat{H}(q_0, q_1)$	Re_n
1	·8825	·9059	1	·9507
2	·6065	·6653	·7522	·6809
3	·3247	·4310	·4969	·4120
4	·1353	·2866	·3248	·2312
5	·0439	·2244	·2406	·1433
10	3·73E-6	·1990	·1990	·1005

6. APPENDIX

Lemma 1A. Let $\{a_n\}_{n=1}^{\infty}$ and $\{b_n\}_{n=1}^{\infty}$ be sequences of nonnegative numbers and let for every $n \in \mathcal{N}$ we have $\max\{a_n, b_n\} > 0$. Then

$$\limsup_{n\to\infty} n^{-1} \log(a_n + b_n) = \limsup_{n\to\infty} n^{-1} \log \max\{a_n, b_n\}$$

and

$$\liminf_{n\to\infty} n^{-1} \log(a_n + b_n) = \liminf_{n\to\infty} n^{-1} \log \max\{a_n, b_n\}\,.$$

Proof. The proof of this well-known lemma is only a technical matter. The complete proof can be found e.g. in [7]. □

Lemma 2A. Let f, g be two densities of probability measures (with respect to a measure μ) and $\tau > 0$. Let $\alpha \in (0, 1)$. Then for any $n \in \mathcal{N}$

$$e_n(f, \mathscr{K}^n_{n,\tau}(f, g)) < \tau^{1-\alpha} H^n_\alpha(f, g)\,.$$

Proof.

$$e_n(f, \mathscr{K}^n_{n,\tau}(f, g)) = \int_{\mathscr{K}^n_{n,\tau}(f,g)} \prod_{i=1}^{n} f(x_i)\, \mathrm{d}\mu \leqq$$

$$\leqq \tau^{1-\alpha} \int_{\mathscr{K}^n_{n,\tau}(f,g)} \prod_{i=1}^{n} f^\alpha(x_i)\, g^{1-\alpha}(x_i)\, \mathrm{d}\mu \leqq \tau^{1-\alpha} H^n_\alpha(f, g)\,.$$ □

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