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SOME GENERAL PROPERTIES OF ELLIPTICALLY SYMMETRIC AND SOME RELATED RANDOM PROCESSES

OLDŘICH KROPÁČ

Elliptically symmetric random processes form a special class of generally non-Gaussian random processes the analytical properties of which are very similar to or in some important characteristics even identical with the corresponding properties of normal processes. Elliptically symmetric distributions may be generated by means of a joint normal distribution assuming for the standard deviation to be random variable with given distribution function.

1. INTRODUCTION

The first knowledge about random processes has been derived using some very limiting assumptions, viz that the processes are stationary and normal (Gaussian). Advanced problems of applied practice require, however, for processes which do not possess the above mentioned advantageous (from the analytical point of view) properties, to be accessible to analytical treatment in a similar way as it is with stationary and normal processes. In this paper, we shall be interested in processes which are non-Gaussian.

Among the generally non-Gaussian random processes there is a special class of so-called elliptically symmetric processes the analytical properties of which are very similar to the properties of normal processes. Their denotation stems from the fact that the geometric loci of constant probability densities of the joint distribution describing such a process are concentrated ellipses [5]. It may be shown [1], [2] that an elliptically symmetric distribution may be generated by means of a joint normal distribution with the standard deviation being random variable with given distribution function.

2. GENERATION OF ELLIPTICALLY SYMMETRIC RANDOM VARIABLES

First, we shall derive some properties of two jointly distributed random variables X_1, X_2. Without detriment of generality we may assume for the primary normal distribution to have zero expected values of both components, i.e. $\mathsf{E}_{x1} = \mathsf{E}_{x2} = 0$. Thus, we write

$$(1) \qquad f_{12}(x_1, x_2) = (2\pi\sigma_1\sigma_2)^{-1} (1 - \varrho^2)^{-1/2} \,. \\ . \exp\left[-\tfrac{1}{2}(1 - \varrho^2)^{-1} (x_1^2/\sigma_1^2 - 2\varrho x_1 x_2/(\sigma_1\sigma_2) + x_2^2/\sigma_2^2)\right]$$

where σ_1, σ_2 will be considered to be particular realizations of continuous, mutually independent random variables Σ_1, Σ_2 having probability densities $g_1(\sigma_1), g_2(\sigma_2)$, respectively. When Σ_1, Σ_2 take particular values σ_1, σ_2, then $f_{12}(x_1, x_2 \mid \sigma_1, \sigma_2)$ is to be considered to be conditional with respect to σ_1, σ_2. Assuming random changes of Σ_1, Σ_2 according to $g_1(\sigma_1), g_2(\sigma_2)$ we are interested in the unconditioned probability density $f_{12}^*(x_1, x_2)$ given by the relation for the total probability density

$$(2) \qquad f_{12}^*(x_1, x_2) = \int_0^\infty \int_0^\infty f_{12}(x_1, x_2 \mid \sigma_1, \sigma_2)\, g_1(\sigma_1)\, g_2(\sigma_2)\, d\sigma_1\, d\sigma_2 \,.$$

The joint probability density $f_{12}^*(x_1, x_2)$ provided the primary conditional distribution is normal according to equation (1) may thus be expressed as follows:

$$(3) \qquad f_{12}^*(x_1, x_2) = \int_0^\infty \int_0^\infty (2\pi\sigma_1\sigma_2)^{-1} (1 - \varrho^2)^{-1/2} \,. \\ . \exp\left[-\tfrac{1}{2}(1 - \varrho^2)^{-1} (x_1^2/\sigma_1^2 - 2\varrho x_1 x_2/(\sigma_1\sigma_2) + x_2^2/\sigma_2^2)\right] g_1(\sigma_1)\, g_2(\sigma_2)\, d\sigma_1\, d\sigma_2 \,.$$

First, we shall prove the general properties of the marginal distributions of components $X_1 X_2$. According to the definition

$$f_1^*(x_1) = \int_{-\infty}^\infty f_{12}^*(x_1, x_2)\, dx_2 .$$

After inserting for $f_{12}^*(x_1, x_2)$ from (3) and interchanging the order of integration

$$f_1^*(x_1) = \int_0^\infty \int_0^\infty \left\{\int_{-\infty}^\infty f_{12}(x_1, x_2 \mid \sigma_1, \sigma_2)\, dx_2\right\} g_1(\sigma_1)\, g_2(\sigma_2)\, d\sigma_1\, d\sigma_2 = \\ = \int_0^\infty (2\pi)^{-1/2} \sigma_1^{-1} \exp\left[-\tfrac{1}{2}(x_1/\sigma_1)^2\right] g_1(\sigma_1)\, d\sigma_1 \int_0^\infty g_2(\sigma_2)\, d\sigma_2$$

where of the marginal probability density $f_1^*(x_1)$ of the component X_1^* may be deduced to be

$$(4) \qquad f_1^*(x_1) = \int_0^\infty f_1(x_1 \mid \sigma_1)\, g_1(\sigma_1)\, d\sigma_1$$

and a similar expression holds for $f_2^*(x_2)$.

The same result may be obtained when assuming for Σ_1, Σ_2 to be correlated.

3. MOMENTS OF ELLIPTICALLY SYMMETRIC VARIABLES AND PROCESSES

Now, we shall derive the moments of the marginal distribution $f_1^*(x_1)$ and for the sake of brevity the indices 1 will be omitted. According to the assumption $\mathsf{E}_x = 0$ the centred moments are by definition

$$\mu_x^{*(r)} = \int_{-\infty}^{\infty} x^r f^*(x)\,\mathrm{d}x$$

and after inserting for $f^*(x)$ from (4) and interchanging the order of integration we have

$$\mu_x^{*(r)} = \int_0^{\infty} \left\{ \int_{-\infty}^{\infty} (2\pi)^{-1/2}\,\sigma^{-1} x^r \exp\left[-\tfrac{1}{2}(x/\sigma)^2\right] \mathrm{d}x \right\} g(\sigma)\,\mathrm{d}\sigma\,.$$

The expression in braces is the r-th central moment of the normal distribution. Its odd moments are zero and even moments are equal to $(2k-1)!!\,\sigma^{2k}$ where $2k = r$ and $(2k-1)!! = 1\,.\,3\,.\,\ldots\,.\,(2k-1)$. Thus, we obtain for the marginal moments of an elliptically symmetric distribution

$$(5)\qquad \mu_x^{*(2k+1)} = 0\,,$$

$$\mu_x^{*(2k)} = (2k-1)!! \int_0^{\infty} \sigma^{2k} g(\sigma)\,\mathrm{d}\sigma = (2k-1)!!\, m_\sigma^{(2k)}\,.$$

Consider again the joint probability density according to (3) but with $\varrho = 0$. Then

$$(6)\qquad f_{12}^*(x_1, x_2) = \int_0^{\infty}\int_0^{\infty} f_{12}(x_1, x_2 \mid \sigma_1, \sigma_2)\, g_1(\sigma_1)\, g_2(\sigma_2)\,\mathrm{d}\sigma_1\,\mathrm{d}\sigma_2 =$$

$$= \int_0^{\infty} f_1(x_1 \mid \sigma_1)\, g_1(\sigma_1)\,\mathrm{d}\sigma_1 \int_0^{\infty} f_1(x_2 \mid \sigma_2)\, g_2(\sigma_2)\,\mathrm{d}\sigma_2 = f_1^*(x_1) f_2^*(x_2)$$

i.e. the components X_1^*, X_2^* are statistically independent.

We shall further consider the joint probability density $f_{12}^*(x_1, x_2)$ of a stationary random process $X^*(t)$. Then in equation (3) we put $\sigma_1 = \sigma_2 = \sigma$ with the probability density $g(\sigma)$. The mixed moment $\mu_x^{*(1,1)}$ will then be expressed following the definition

$$\mu_x^{*(1,1)} = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x_1 x_2 f_{12}^*(x_1, x_2)\,\mathrm{d}x_1\,\mathrm{d}x_2$$

yielding gradually

$$(7)\quad \mu_x^{*(1,1)} = \int_0^{\infty} \left\{ \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x_1 x_2 f_{12}(x_1, x_2 \mid \sigma)\,\mathrm{d}x_1\,\mathrm{d}x_2 \right\} g(\sigma)\,\mathrm{d}\sigma = \varrho m_\sigma^{(2)} = \varrho \mathsf{D}_x^*\,.$$

It follows from (7) that the correlation coefficient ϱ of the conditional normal process has the same meaning for any arbitrary elliptically symmetric random process derived from the primary conditional normal process.

It may be shown in a similar way that the remaining mixed moments $\mu_x^{*(r,s)}$ are zero for $r + s$ odd, and for $r + s$ even they may be expressed in the form

$$\mu_x^{*(r,s)} = \psi_x^{(r,s)}(\varrho) \int_0^\infty \sigma^{r+s} g(\sigma) \, d\sigma = \psi_x^{(r,s)}(\varrho) \, m_\sigma^{(r+s)}$$

where $\psi_x^{(r,s)}(\varrho)$ is written for a particular function of ϱ depending on (r, s).

4. PROPERTIES OF THE ENVELOPE AND THE PHASE

For some engineering applications it is advantageous to express the normal stationary process $X(t)$ in the form

$$X(t) = A(t) \cos [\omega_0 t + \Theta(t)]$$

where $A(t)$ and $\Theta(t)$ are mutually independent stationary random processes having the meaning of the envelope and the phase, respectively, of the process $X(t)$. Further properties of $A(t)$ and $\Theta(t)$ will be derived as follows: In order to be able to express $A(t)$ and $\Theta(t)$ uniquely it is necessary to define another appropriate function using them. It is useful to introduce a process $Y(t) = A(t) \sin [\omega_0 t + \Theta(t)]$ uncorrelated for a fixed t with $X(t)$. In this way, a transformation of the $[X(t), Y(t)]$ vector process given in Cartesian coordinates should be realized into the vector process $[A(t), \Theta(t)]$ given in polar coordinates. The equivalence of probability elements expressed in both systems, i.e.

$$f_{xy}(x, y) \, dx \, dy = f_{a\vartheta}(a, \vartheta) \, da \, d\vartheta$$

may be expressed, for $X(t)$, $Y(t)$ normal, with the same correlation coefficients and uncorrelated in the same time instant, in the form [8]

$$(8) \qquad (2\pi)^{-1} \sigma^{-2} \exp [-\tfrac{1}{2}\sigma^{-2}(x^2 + y^2)] \, dx \, dy = \\ = a\sigma^{-2} \exp [-\tfrac{1}{2}(a/\sigma)^2] \, da (2\pi)^{-1} \, d\vartheta \, .$$

This relation expresses the known fact that the envelope and the phase of a normal stationary random process are mutually independent, the distribution of the envelope follows the Rayleigh law and the distribution of the phase is uniform over $\langle 0, 2\pi)$. Multiplying both sides of (8) by $g(\sigma)$ and integrating over σ we obtain

$$(9) \qquad \int_0^\infty (2\pi)^{-1} \sigma^{-2} \exp [-\tfrac{1}{2}\sigma^{-2}(x^2 + y^2)] \, g(\sigma) \, d\sigma \, dx \, dy = \\ \int_0^\infty a\sigma^{-2} \exp [-\tfrac{1}{2}(a/\sigma)^2] \, g(\sigma) \, d\sigma \, da (2\pi)^{-1} \, d\vartheta$$

which may be considered to express the equivalence for the probability elements for the elliptically symmetric process $X^*(t)$; i.e.

$$f^*_{xy}(x, y)\,\mathrm{d}x\,\mathrm{d}y = f^*_a(a)\,\mathrm{d}a\,f^*_\vartheta(\vartheta)\,\mathrm{d}\vartheta\,. \tag{10}$$

Comparing (9) and (10) we may state that for an elliptically symmetric random process generated by means of $g(\sigma)$, it holds that the envelope and the phase are mutually independent, the distribution of the phase is uniform over $\langle 0, 2\pi)$ and the distribution of the envelope is given as follows

$$f^*_a(a) = \int_0^\infty a\sigma^{-2} \exp\left[-\tfrac{1}{2}(a/\sigma)^2\right] g(\sigma)\,\mathrm{d}\sigma\,. \tag{11}$$

5. CHARACTERISTICS OF LEVEL CROSSINGS

For solving strength and reliability problems where loading or environmental processes are of random nature the characteristics connected with the level crossing problem are of fundamental importance. For solving such problems it is necessary to know the joint distribution of the random process $X(t)$ and of its first derivative $\dot{X}(t)$ defined on the basis of mean-square convergence. In may be easy verified [7] that processes $X(t), \dot{X}(t)$ are uncorrelated at the same time instant. For a normal process we thus express the joint probability density of $(X, \dot{X})$ in the form

$$f_{x\dot{x}}(x, \dot{x}) = (2\pi)^{-1/2}\,\sigma_x^{-1} \exp\left[-\tfrac{1}{2}(x/\sigma_x)^2\right] (2\pi)^{-1/2}\,\sigma_{\dot{x}}^{-1} \exp\left[-\tfrac{1}{2}(\dot{x}/\sigma_{\dot{x}})^2\right].$$

The standard deviation σ_x is assumed to be known. In order to express $\sigma_{\dot{x}}$ the correlation coefficient $\varrho_x(\tau)$ of $X(t)$ must be known, or better, the Fourier transform of it, i.e. the spectral density of unit power $S^0_x(\omega)$ with the property $\int_0^\infty S^0_x(\omega)\,\mathrm{d}\omega = 1$. Then $\sigma^2_{\dot{x}}$ may be expressed as

$$\sigma^2_{\dot{x}} = \sigma^2_x \int_0^\infty \omega^2\, S^0_x(\omega)\,\mathrm{d}\omega = \sigma^2_x\omega^2_2 \tag{12}$$

where $\omega^2_2 = \int_0^\infty \omega^2\, S^0_x(\omega)\,\mathrm{d}\omega$ must be finite which is the necessary and sufficient condition for the existence of $\dot{X}(t)$.

The joint probability density of normally distributed $(X, \dot{X})$ may thus be expressed as follows

$$f_{x\dot{x}}(x, \dot{x}) = (2\pi\sigma^2_x\omega_2)^{-1} \exp\left[-\tfrac{1}{2}\sigma_x^{-2}(x^2 + (\dot{x}/\omega_2)^2)\right]. \tag{13}$$

Consider now for σ_x to be random with probability density $g(\sigma_x)$. Then multiplying (13) by $g(\sigma_x)$ and integrating over σ_x we get the joint probability distribution function of an elliptically symmetric process $X^*(t)$ and of its first derivative $\dot{X}^*(t)$ in the form

$$f^*_{x\dot{x}}(x, \dot{x}) = \int_0^\infty (2\pi\sigma^2_x\omega_2)^{-1} \exp\left[-\tfrac{1}{2}\sigma_x^{-2}(x^2 + (\dot{x}/\omega_2)^2)\right] g(\sigma_x)\,\mathrm{d}\sigma_x\,.$$

Integrating over x we obtain the probability density function of $\dot{X}^*(t)$

$$(14) \qquad f_{\dot{x}}^*(\dot{x}) = \int_0^\infty (2\pi)^{-1/2} (\sigma_x \omega_2)^{-1} \exp\left[-\tfrac{1}{2}(\dot{x}/(\sigma_x \omega_2))^2\right] g(\sigma_x)\, d\sigma_x$$

which is of the same type as has $X^*(t)$, see equation (4), from which it may be deduced by a simple substitution $x \to \dot{x}/\omega_2$, i.e. when $f_x^*(x)$ is known, then $f_{\dot{x}}^*(\dot{x}) = \omega_2^{-1} f_x^*(\dot{x}/\omega_2)$.

The number of crossings in a unit time interval of the level C (with positive slope) by the process $X(t)$ is given by the relation [7]

$$N_C = \int_0^\infty \dot{x} f_{x\dot{x}}(C, \dot{x})\, d\dot{x}\,.$$

For a normal process it may be easy derived [7], [9] that the number of zero crossings ($C = 0$) of a centred process is

$$N_0 = (2\pi)^{-1} \sigma_{\dot{x}}/\sigma_x = \omega_2/(2\pi)\,.$$

For an eliptically symmetric process generated by means of $g(\sigma_x)$ we obtain after a detailed specification

$$(15) \qquad N_0 = \int_0^\infty (2\pi)^{-1/2} \sigma_x^{-1} \underbrace{\int_0^\infty \dot{x}(2\pi)^{-1/2} (\sigma_x \omega_2)^{-1} \exp\left[-\tfrac{1}{2}\dot{x}^2(\sigma_x\omega_2)^{-2}\right] dx}_{= (2\pi)^{-1/2}\, \sigma_x \omega_2} \,.\, g(\sigma_x)\, d\sigma_x = \omega_2/(2\pi)$$

i.e. the number of zero crossings is not dependent on the probability density $g(\sigma_x)$ generating the elliptically symmetric random process and is the same as for the normal process. In a similar way the number of maxima of the process $X(t)$ in a unit time interval may be expressed as the number of zero crossings (with negative slope) of the process $\dot{X}(t)$, thus

$$(16) \qquad N_E = \omega_4/(2\pi\omega_2)$$

where $\omega_4^2 = \int_0^\infty \omega^4 S_x^0(\omega)\, d\omega < \infty$.

It is useful to introduce the ratio

$$(17) \qquad \nu = N_0/N_E = \omega_2^2/\omega_4 = \varphi[S^0(\omega)]$$

having the meaning of a numerical characteristics of the structural complexity of the process deduced from its power spectral density $S(\omega)$. The parameter ν is of great importance for expressing distribution functions of extremes, amplitudes and instantaneous mean values which are necessary for strength and reliability calculations of structural and machine components loaded by random environmental processes. It follows from the presented deduction that the parameter ν does not depend on the form of $g(\sigma_x)$ and is identical with that for the primary normal process.

For the characterization of random loading processes a further parameter has been introduced as a result of a rather empirical approach, viz the correlation coefficient r_{Mm} between the maximum and the next minimum of the random process $X(t)$. It may be easy shown [3] that for a normal process this parameter may be expressed as

$$r_{Mm} = \varrho_x(\tau = (2N_E)^{-1}) = \varrho_x(\pi\omega_2/\omega_4) \tag{18}$$

where $\varrho_x(\tau)$ is again the correlation coefficient of the process $X(t)$. However, we have shown that the correlation coefficient $\varrho_x(\tau)$ of the normal process $X(t)$ remains unchanged to be also correlation coefficient for all elliptically symmetric processes $X^*(t)$ derived from $X(t)$. The same holds for N_E. Thus the relation (18) holds also for all types of elliptically symmetric random processes.

Summing up, the parameters v and r_{Mm} of an elliptically symmetric random process are independent on the particular form of the distribution function. The relations between v and r_{Mm} deduced in detail for a normal process in [3] may thus be used for all elliptically symmetric random processes, as well.

6. DISTRIBUTIONS OF EXTREMES AND SOME ADJOINT VARIABLES

For the strength and reliability calculations the characteristics of local extremes of the random process are very important. For the probability density of maxima X_M of a normal stationary process the following expression has been deduced [7], [9]

$$\begin{aligned} f_M(x_M) = {} & (1 - v^2)^{1/2} (2\pi)^{-1/2} \sigma_x^{-1} \exp\left[-\tfrac{1}{2}(1 - v^2)^{-1} (x_M/\sigma_x)^2\right] + \\ & + v x_M \sigma_x^{-2} \exp\left[-\tfrac{1}{2}(x_M/\sigma_x)^2\right] \Phi\left[v(1 - v^2)^{-1/2} (x_M/\sigma_x)\right] \end{aligned} \tag{19}$$

with the parameter v according to (17) where $\Phi(z)$ is the error integral $\Phi(z) = \int_{-\infty}^{z} (2\pi)^{-1/2} \exp(-\frac{1}{2}t^2)\,\mathrm{d}t$.

For an elliptically symmetric random process generated by means of $g(\sigma_x)$

$$f_M^*(x_M) = \int_0^\infty f_M(x_M \mid \sigma_x)\, g(\sigma_x)\, \mathrm{d}\sigma_x\,. \tag{20}$$

The evaluation of this expression leads to complex transcendental functions even for simple forms of $g(\sigma_x)$. Thus, we shall consider only the both limit cases, viz for $v = 0$ and $v = 1$.

The case $v = 1$ occurs very frequently describing the so-called narrow-band random vibrating processes which have their power expressed by the dispersion D_x concentrated into a small frequency band around ω_0. Then in the expression (19) the first term vanishes so that

$$f_M^*(x_M, v = 1) = \int_0^\infty x_M \sigma_x^{-2} \exp\left[-\tfrac{1}{2}(x_M/\sigma_x)^2\right] g(\sigma_x)\, \mathrm{d}\sigma_x\,, \quad x_M \geqq 0\,. \tag{21}$$

This means that the distribution of maxima of a narrow-band random process (being by its nature a point process) is identical with the distribution of the continuous envelope process (compare with equation (11)).

In the second limit case, $\nu = 0$, the second term in (19) vanishes and

$$f_M^*(x_M,\ \nu = 0) = \int_0^\infty (2\pi)^{-1/2}\, \sigma_x^{-1} \exp\left[-\tfrac{1}{2}(x_M/\sigma_x)^2\right] g(\sigma_x)\, \mathrm{d}\sigma_x \tag{22}$$

i.e. the distribution of maxima as a point process is identical with that of the continuous process $X^*(t)$. Let us state on this occasion that a process with $\nu = 0$ may be imagined as a low-frequency random process modulated with a high-frequency low-power random process. It must be explicitly mentioned that the value $\nu = 0$ does not characterize the so-called broad-band processes with the power spectral density being nearly constant over a given frequency interval (frequency limited white noise).

For calculations for fatigue life two further quantities derived from the extremes of the random process $X(t)$ has been proved to be important, viz the instantaneous mean value X_S, as the arithmetic mean of the maximum and next minimum, i.e.

$$X_S = \tfrac{1}{2}(X_M + X_m) \tag{23}$$

and the amplitude X_R being the half of the difference between the maximum and next minimum

$$X_R = \tfrac{1}{2}(X_M - X_m)\,. \tag{24}$$

For the calculation of the joint probability density $f_{SR}(x_S, x_R)$ it is formally possible to write the transformation using the joint probability density $f_{Mm}(x_M, x_m)$ but even when considering normal distribution of $X(t)$ very complicated hypergeometric functions appear in the course of evaluation which could be hardly usable in current technical practice. Therefore, some approximate relations have been proposed for $f_S(x_S)$ and $f_R(x_R)$ from the requirement that for the limit values $\nu = 0$ and $\nu = 1$ the approximate expressions should coincide with the analytically precise forms and in the intermediate region of ν the relation between ν and r_{Mm} which has been deduced analytically should be approximately maintained [3].

For X_S the approximation using the normal distribution

$$f_S(x_S) = \left[2\pi(1 - \nu^2)\right]^{-1/2} \sigma_x^{-1} \exp\left[-\tfrac{1}{2}(1 - \nu^2)^{-1} (x_S/\sigma_x)^2\right] \tag{25}$$

has proved to be suitable and for X_R the approximation using the Weibull distribution has been found adequate

$$f_R(x_R) = (\nu^2 + 1)\,(\sqrt{(2)}\,\nu\sigma_x)^{-1}\,(x_R/(\sqrt{(2)}\,\nu\sigma_x))^{\nu^2}\,. \\ .\exp\left[-(x_R/(\sqrt{(2)}\,\nu\sigma_x))^{\nu^2+1}\right] \tag{26}$$

which for $\nu = 1$ turns into the Rayleigh one in correspondence with the other characteristics of this process.

For an analytical description of mentioned quantities for elliptically symmetric random processes the relations expressioning the unconditioned probability densities with considering the random character of the standard deviation σ_x may be written, i.e. particularly

$$f_S^*(x_S) = \int_0^\infty f_S(x_S \mid \sigma_x)\, g(\sigma_x)\, \mathrm{d}\sigma_x \,, \tag{27}$$

$$f_R^*(x_R) = \int_0^\infty f_R(x_R \mid \sigma_x)\, g(\sigma_x)\, \mathrm{d}\sigma_x \,. \tag{28}$$

The evaluation of $f_S^*(x_S)$ is identical with the evaluation of $f_1^*(x)$ or of $f_{12}^*(x_1, x_2)$ of the process $X^*(t)$ but the evaluation of $f_R^*(x_R)$ is much more complicated but for some simple forms of $g(\sigma_x)$ it is also realizable in an analytically closed form.

A survey of some elliptically symmetric random processes the probability density functions of which may be easy described using rather elementary analytical tools is given in [2]. A concise review of this survey is reproduced in Table 1. The function

Type	$f_1(x)$	$f_{12}(R,\varrho)$	$f_a(a)$
normal	$\frac{1}{\sqrt{2\pi}\, b} \exp\left(-\frac{x^2}{2b^2}\right)$ $b>0,\ x\in(-\infty,\infty)$	$\frac{1}{2\pi b^2 \sqrt{1-\varrho^2}} \exp\left(-\frac{R^2}{2b^2}\right)$ $R \geqq 0$	$\frac{a}{b^2} \exp\left(-\frac{a^2}{2b^2}\right)$ $a \geqq 0$
generalized Laplace	$\frac{2^{-k+1/2}}{\sqrt{\pi}\, b\, \Gamma(k)} \left(\frac{\lvert x\rvert}{b}\right)^{k-1/2} K_{k-1/2}\left(\frac{\lvert x\rvert}{b}\right)$ $b>0,\ k>0,\ x\in(-\infty,\infty)$	$\frac{2^{-k}\, b^{-2}}{\pi\, \Gamma(k) \sqrt{1-\varrho^2}} \left(\frac{R}{b}\right)^{k-1} K_{k-1}\left(\frac{R}{b}\right)$ $R \geqq 0$	$\frac{2^{1-k}}{b\, \Gamma(k)} \left(\frac{a}{b}\right)^{k} K_{k-1}\left(\frac{a}{b}\right)$ $a \geqq 0$
Pearson VII	$\frac{b^{2k}\, \Gamma(k+1/2)}{\sqrt{\pi}\, \Gamma(k)} (x^2+b^2)^{-k-1/2}$ $b>0,\ k>0,\ x\in(-\infty,\infty)$	$\frac{b^{2k} k}{\pi \sqrt{1-\varrho^2}} (R^2+b^2)^{-k-1}$ $R \geqq 0$	$2k b^{2k} a (a^2+b^2)^{-k-1}$ $a \geqq 0$
Pearson II	$\frac{\Gamma(k+1)}{\sqrt{\pi}\, b^{2k}\, \Gamma(k+1/2)} (b^2-x^2)^{k-1/2}$ $b>0,\ k>0,\ x\in\langle -b,b\rangle$	$\frac{k}{\pi b^{2k} \sqrt{1-\varrho^2}} (b^2-R^2)^{k-1}$ $R\in\langle 0,b\rangle$	$\frac{2k}{b^{2k}} a (b^2-a^2)^{k-1}$ $a\in\langle 0,b\rangle$

Table 1. Some families of elliptically symmetric random processes $X(t)$

$f_1(x)$ — marginal probability density of the process $X(t)$

$f_{12}(R, \varrho)$ — joint probability density of the process $X(t)$

$R = (x_1^2 + x_2^2 - 2\varrho x_1 x_2)^{1/2} (1 - \varrho^2)^{-1/2}$

$f_a(a)$ — marginal probability density of the adjoint envelope process

$A(t) = [X^2(t) + (\dot{X}(t)/\omega_0)^2]^{1/2}$

$K_k(x)$ — modified Bessel function of the second kind (MacDonald function) of the k-th order

$K_k(x)$ entering in the generalized Laplace distribution is the modified Bessel function of the second kind (also called MacDonald function) [4].

7. SOME RELATED PROCESSES

In practical applications, random processes often occur the distribution functions of which are distinctly non-symmetric. One posible way, how to introduce such a distribution consists in using a suitable nonlinear transformation applied to normal distribution. Nevertheless, we must have in mind that some of the advantageous properties of the elliptically symmetric distributions will be lost.

Consider one of the most simple transformation, viz $U(t) = |X(t)|$ where $X(t)$ is $N(0, \sigma^2)$. Thus the process $U(t)$ will be described within the frame of the correlation theory by means of probability density functions

$$f_1(u) = (2/\pi)^{1/2}\, \sigma^{-1} \exp\left[-\tfrac{1}{2}(u/\sigma)^2\right], \quad u \geqq 0, \tag{29}$$

$$f_{12}(u_1, u_2) = (2/\pi)\, \sigma^{-2}(1 - \varrho^2)^{-1/2} \,. \tag{30}$$
$$. \exp\left[-\tfrac{1}{2}(1 - \varrho^2)^{-1}\, \sigma^{-2}(u_1^2 - 2\varrho u_1 u_2 + u_2^2)\right], \quad u_1, u_2 \geqq 0,$$

where σ has the meaning of the scale parameter,
ϱ has the meaning of the parameter of correlation coupling.

General moments derived from $f_1(u)$ are as follows:

$$m_u^{(1)} \equiv \mathsf{E}_u = (2/\pi)^{1/2}\, \sigma,$$
$$m_u^{(2)} = \sigma^2,$$
$$m_u^{(3)} = (8/\pi)^{1/2}\, \sigma^3,$$
$$m_u^{(4)} = 3\sigma^2.$$

Thus, the variance $\mathsf{D}_u = \sigma^2(1 - 2/\pi)$ and the standard deviation $\sigma_u = \sigma(1 - 2/\pi)^{1/2}$, i.e. the scale parameter and the standard deviation do not coincide.

The covariance function may be easy deduced using the Price's theorem [6] yielding

$$m_u^{(1,1)} \equiv \mathsf{B}_u = (2\sigma^2/\pi)\left[\varrho \arcsin \varrho + (1 - \varrho^2)^{1/2}\right] \tag{31}$$

where of the correlation coefficient has the form

$$r = \left[2/(\pi - 2)\right]\left[\varrho \arcsin \varrho + (1 - \varrho^2)^{1/2} - 1\right]. \tag{32}$$

In a similar way as with the normal process, we shall consider for the scale parameter to be random variable Σ with the probability density $g(\sigma)$. Then, for the probability densities of the unconditioned process $U^*(t)$, similar expressions may be written as for those derived from the normal process $X(t)$.

For general moments, the following expressions are valid:

$$m_u^{*(r)} = \psi_u^{(r)} \int_0^\infty \sigma^r\, g(\sigma)\, d\sigma = \psi_u^{(r)} m_\sigma^{(r)} \tag{33}$$

where $\psi_u^{(r)}$ stands for some constants independent on σ. From the general moments, using commonly known relations, centred moments or normalized invariants may be easy deduced.

The correlation coefficient of the unconditioned process $U^*(t)$ is the same as for the process $U(t)$ and is independent on the form of the probability density $g(\sigma)$.

In this way some non-symmetrical random processes which are closely related to the elliptically symmetric processes may be easy generated. For their analytical description Table 1 may be used considering the changed definition interval for $u \geqq 0$ and some minor changes of the normalizing constants in $f_1(u)$ and $f_{12}(u_1, u_2)$ as it has been done for the one-sided Gaussian distribution in (29) and (30). It follows from the definition of the envelope (Chapter 3) that its distribution for the process $U^*(t)$ is the same as for the corresponding process $X^*(t)$.

It holds generally that for any memoryless transformation the correlation coefficient of the resulting process depends only on the marginal distribution and the correlation coefficient of the primary process and the transformation considered. When introducing the randomly time-dependent scale parameter of thus transformed primary random process, arbitrary forms of marginal distributions may be generated without affecting the correlation coefficient of the conditional distribution. The expressions for the joint distributions will be generally possible only using functional series. The remaining characteristics considered in this paper will also have more involved properties than it has been deduced for elliptically symmetric distributions.

8. CONCLUSIONS

When concluding the knowledge derived in this paper we may state that elliptically symmetric random processes form a special but fairly comprehensive class of non-Gaussian random processes the properties of which are very close to or in some characteristics important for engineering applications even identical with the corresponding properties of normal random processes. When considering elliptically symmetric random processes as being generated by means of a conditional normal process the standard deviation of which is a random variable with given probability density, a very suitable analytical tool is defined for introducing not only marginal and joint distribution functions of an elliptically symmetric process but also some further characteristics important for engineering applications.

The correlation coefficient for all types of elliptically symmetric random processes is identical with the correlation coeficient of the primary conditional normal process. An elliptically symmetric random process may be described by means of its envelope and its phase which are mutually statistically independent, the distribution of the phase being uniform over $\langle 0, 2\pi)$ and the distribution of the envelope being defined by means of the conditional Rayleigh distribution and the same probability density function $g(\sigma)$ of the scale parameter as it is defined for the distribution of the state variable itself.

For elliptically symmetric random processes of the narrow-band type the parameter v of which approaches one, the distribution function of the envelope and the distribution function of maxima coincide. Important characteristics of random processes necessary for strength and reliability calculations as are the correlation coefficient of maximum and next minimum and the parameter of the structural complexity are defined only by means of the power spectral density of the process and are independent on the distribution of the state variable. For expression of extreme values, instantaneous mean values and amplitudes, the same analytical procedure may be used as for the expression of the state variable distribution.

In concluding this short presentation we may express the conviction that elliptically symmetric random processes form a special but considerably large class of non-Gaussian processes which for their closeness to normal processes and their accessibility to required analytical operations are very suitable for a broad area of engineering applications.

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Ing. Oldřich Kropáč, CSc., Výzkumný a zkušební letecký ústav (Aeronautical Research and Test Institute), 199 05 Praha 9 - Letňany. Czechoslovakia.