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ON SOME FUNCTIONAL EQUATIONS FROM ADDITIVE AND NONADDITIVE MEASURES — IV

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INTRODUCTION

This paper deals with a functional equation connected with the *Shannon entropy*, the *entropy of degree* β and others. There are so many algebraic properties which are satisfied by them. Various systems of axioms were used, in literature, to characterize them.

Let $\Delta_n = \{P = (p_1, \ldots, p_n)/p_i \geqq 0, \sum_i p_i = 1\}$ be the set of all finite complete discrete probability distributions on a given partition of the sure event Ω into n events $E_1, \ldots, E_n$. In 1948 Shannon [11] introduced the measure of information

$$H_n(P) = - \sum_{i=1}^{n} p_i \log p_i \,, \quad P \in \Delta_n \tag{1}$$

known as Shannon's entropy. In 1967 Havrda and Charvát [5] proposed as a quantitative measure of the classification or an entropy of the experiment, the entropy of degree β

$$H_n^\beta(P) = \frac{\sum_{i=1}^{n} p_i^\beta - 1}{2^{1-\beta} - 1}\,, \quad P \in \Delta_n \quad (\beta \neq 1)\,. \tag{2}$$

Some of the algebraic properties satisfied by these measures are symmetry, branching or recurrence relation and expansibility. From these algebraic properties one obtains the sum representation [10], viz. $H_n(P) = \sum_i f(p_i)$, $H_n^\beta(P) = \sum_i g(p_i)$. It is evident that whereas the Shannon entropy is additive, the entropy of degree β is nonadditive. Thus, in the case of Shannon's entropy, the sum representation together with the property of additivity leads to the study of the functional equation

$$(3) \qquad \sum_{i=1}^{n} \sum_{j=1}^{m} f(x_i y_j) = \sum_{i=1}^{n} f(x_i) + \sum_{j=1}^{m} f(y_j)$$

$(x = (x_i) \in \Delta_n,\ y = (y_j) \in \Delta_m)$, while in the case of the entropy of degree β, the sum representation and the nonadditivity, lead to the study of the functional equation

$$(4) \qquad \sum_{i=1}^{n} \sum_{j=1}^{m} g(x_i y_j) = \sum_i g(x_i) + \sum_j g(y_j) + c \sum_i g(x_i) \sum_j g(y_j),$$

$(c = (2^{1-\beta} - 1)^{-1})$. So, a characterization of (1) or (2) can be achieved by solving (3) or (4). In this paper we solve the functional equation

$$(5) \qquad \sum_{i=1}^{n} \sum_{j=1}^{m} f_{ij}(x_i y_j) = \sum_{i=1}^{n} g_i(x_i) + \sum_{j=1}^{m} h_j(y_j) + \sum_{i=1}^{n} k_i(x_i) \sum_{j=1}^{m} l_j(y_j)$$

$(x = (x_i) \in \Delta_n,\ y = (y_j) \in \Delta_m)$, which includes (3) and (4) as special cases. Further, in the case of non-symmetric entropies, the sum representation together with the property of additivity leads to the study of the above equation (5) (refer to [4]). Usually (3) and (4) were solved [3, 1, 2, 6] under the hypothesis of continuity and the equations holding for all positive integers m, n. Recently (3) and (4) were studied in [7] for fixed m and n, under the condition of measurability of the functions involved. Along the same lines, we solve the functional equation (5) holding for some (arbitrary but) fixed pair (m, n) when the functions involved are all Lebesgue measurable, using simple methods adopted in [8] and show that the solutions indeed depend upon the pair m, n and these solutions may lead to the study of more information measures.

2. SOLUTION OF THE EQUATION (5)

In order to solve (5), we make use of the following two results [9, 12]. Let $I = [0, 1]$, $I_1 =]0, 1]$. We follow the convention $0 \log 0 = 0$, $0^\beta = 0$, $1^\beta = 1$.

Result 1. [9] Let $G_{ij} : I \times I \to \mathbb{R}$ (reals) $(i = 1, 2, \ldots, n,\ j = 1, 2, \ldots, m)$ be measurable in each variable and satisfy the functional equation

$$(6) \qquad \sum_{i=1}^{n} \sum_{j=1}^{m} G_{ij}(x_i, y_j) = 0$$

($x = (x_i) \in \Delta_n$, $y = (y_j) \in \Delta_m$) holding for some fixed $m, n\ (\geqq 3)$. Then G_{ij} are given by

$$\tag{7} \begin{aligned} G_{ij}(x, y) = {} & G_{ij}(x, 0) - \sum_{l=1}^{m} G_{il}(x, 0)\, y + G_{ij}(0, y) - \\ & - \sum_{k=1}^{n} G_{kj}(0, y)\, x + \sum_{k=1}^{n} G_{kj}(0, 0)\, x + \sum_{l=1}^{m} G_{il}(0, 0)\, y - \\ & - \sum_{k=1}^{n} \sum_{l=1}^{m} G_{kl}(0, 0)\, xy - G_{ij}(0, 0)\,, \\ & i = 1, 2, \ldots, n\,; \quad j = 1, 2, \ldots, m\,. \end{aligned}$$

Result 2. [12] Let $F, G, H, K, L: S \to C$ (complex numbers) satisfy

$$\tag{8} F(xy) = G(x) + H(y) + K(x)\, L(y)$$

where S is an arbitrary Abelian semigroup which has a fixed element 'a' such that $a \,.\, x = b$ is solvable for every $b \in S$. Then the general solutions of (8) are the following:

$$\text{(a)} \begin{cases} F(x) = \Phi(x) + \alpha_1\,, \quad G(x) = \Phi(x) - \alpha_3\, K(x) + \alpha_2 + \tfrac{1}{2}\alpha_1\,, \\ H(x) = \Phi(x) + (\tfrac{1}{2}\alpha_1) - \alpha_2\,,\ K, \text{ arbitrary}, \quad L(x) = \alpha_3\,; \end{cases}$$

$$\text{(b)} \begin{cases} F(x) = \alpha_1\, \Psi(x) + \Phi(x) + \alpha_2\,, \quad G(x) = \alpha_3\, \Psi(x) + \Phi(x) + \alpha_4\,, \\ H(x) = \alpha_5\, \Psi(x) + \Phi(x) + \alpha_6\,, \quad K(x) = \alpha_7\, \Psi(x) + \alpha_8\,, \\ L(x) = \alpha_9\, \Psi(x) + \alpha_{10}\,, \end{cases}$$

$$\text{with} \quad \alpha_1 = \alpha_7\alpha_9,\ \alpha_3 + \alpha_7\alpha_{10} = 0 = \alpha_5 + \alpha_8\alpha_9,\ \alpha_2 = \alpha_4 + \alpha_6 + \alpha_8\alpha_{10}\,;$$

$$\text{(c)} \begin{cases} F(x) = \alpha_1\, \Phi^2(x) + \alpha_2\, \Phi(x) + \Phi_1(x) + \alpha_3\,, \quad G(x) = \alpha_1\, \Phi^2(x) + \Phi_1(x) + \alpha_4\,, \\ H(x) = \alpha_1\, \Phi^2(x) + \alpha_5\, \Phi(x) + \Phi_1(x) + \alpha_6\,, \quad K(x) = 2\alpha_1\, \Phi(x) + \alpha_7\,, \\ L(x) = \Phi(x) + \alpha_8 \end{cases}$$

$$\text{with} \quad \alpha_2 = 2\alpha_1\alpha_8 = \alpha_5 + \alpha_7\,, \quad \alpha_3 = \alpha_4 + \alpha_6 + \alpha_8\alpha_7\,;$$

where Φ, Φ_1 and respectively Ψ satisfy

$$\tag{9} \Phi(xy) = \Phi(x) + \Phi(y)\,,$$

$$\tag{10} \Psi(xy) = \Psi(x)\, \Psi(y)\,, \quad x, y \in S;$$

and, (a′) which is obtained from (a) by interchanging $G \leftrightarrow H$ and $K \leftrightarrow L$, (the interchange of $G \leftrightarrow H$ and $K \leftrightarrow L$ in (b), (c) do not produce any new solution).

Now, we will determine all the measurable solutions of (5). Let f_{ij}, g_i, h_j, k_i, l_j $: I \to \mathbb{R}$ ($i = 1, 2, \ldots, n$; $j = 1, 2, \ldots, m$) be measurable and satisfy the functional equation (5) for a fixed pair $m, n\ (\geqq 3)$.

By setting

$$G_{ij}(x, y) = f_{ij}(xy) - y\, g_i(x) - x\, h_j(y) - k_i(x)\, l_j(y), \tag{11}$$

for $x, y \in I$, it is easy to see that (5) can be transformed into (6) and that G_{ij} is measurable in each variable. Hence, by Result 1, (7) holds with

$$G_{ij}(x, 0) = d_{ij} - c_j x - d_j\, k_i(x)$$

$$G_{ij}(0, y) = d_{ij} - b_i y - e_i\, l_j(y)$$

$$G_{ij}(0, 0) = d_{ij} - e_i d_j,$$

where

$$d_{ij} = f_{ij}(0), \quad b_i = g_i(0), \quad c_j = h_j(0), \quad e_i = k_i(0), \quad d_j = l_j(0). \tag{12}$$

Thus, from (7), (11) and (12) results

$$f_{ij}(xy) - d_{ij} + \Big(\sum_{k=1}^{n} \sum_{r=1}^{m} d_{kr} - \sum_{k=1}^{n} e_k \sum_{r=1}^{m} d_r\Big)\, xy =$$

$$= y\Big[g_i(x) - b_i + \sum_{k=1}^{n} b_k x + \sum_{r=1}^{m} d_r(k_i(x) - e_i)\Big] +$$

$$+ x\Big[h_j(y) - c_j + \sum_{r=1}^{m} c_r y + \sum_{k=1}^{n} e_k(l_j(y) - d_j)\Big] + (k_i(x) - e_i)\,(l_j(y) - d_j),$$

for $x, y \in I$, which by defining

$$\begin{cases} F_{ij}(x) = \dfrac{f_{ij}(x) - d_{ij}}{x} + \displaystyle\sum_{k=1}^{n} \sum_{r=1}^{m} d_{kr} - \sum_{k=1}^{n} e_k \sum_{r=1}^{m} d_r \\[2ex] G_i(x) = \dfrac{g_i(x) - b_i + \sum_{r=1}^{m} d_r(k_i(x) - e_i)}{x} + \displaystyle\sum_{k=1}^{n} b_k \\[2ex] H_j(x) = \dfrac{h_j(x) - c_j + \sum_{k=1}^{n} e_k(l_j(x) - d_j)}{x} + \displaystyle\sum_{r=1}^{m} c_r \\[2ex] K_i(x) = \dfrac{k_i(x) - e_i}{x}, \quad L_j(x) = \dfrac{l_j(x) - d_j}{x}, \end{cases} \tag{13}$$

for $x \in I_1$, $(i = 1, 2, \ldots, n;\ j = 1, 2, \ldots, m)$ can be rewritten in the form (8):

$$F_{ij}(xy) = G_i(x) + H_j(y) + K_i(x)\, L_j(y), \quad x, y \in I_1. \tag{14}$$

Thus Result 2 can be applied to determine the solutions of (5). Since f_{ij}, g_j, h_i, k_i, l_j are measurable, so are F_{ij}, G_i, H_j, K_i, L_j, which in turn implies the measurability of Φ, Φ_1 satisfying (9) and Ψ satisfying (10). So, Φ, Φ_1, Ψ occurring in (a), (b), (c), (a') are of the form

$$(15) \qquad \Phi(x) = a \log x\,, \quad \Phi_1(x) = b \log x\,, \quad \Psi(x) = x^{\beta-1} \quad \text{or} \quad = 0\,,$$

where a, b, β are real constants.

Thus, the solution of (14) corresponding to (a) has the form

$$F_{ij}(x) = a \log x + \alpha_1\,, \quad G_i(x) = a \log x - \alpha_3 K_i(x) + \alpha_2 + \tfrac{1}{2}\alpha_1$$
$$H_j(x) = a \log x + (\tfrac{1}{2}\alpha_1) - \alpha_2,\ K_i \text{ arbitrary}, \quad L_j(x) = \alpha_3$$

for $i = 1, 2, \ldots, n$, $j = 1, 2, \ldots, m$, $x \in I_1$. By fixing i and allowing j to vary, it is easy to see that α_3 and a are independent of j. But α_1, α_2 will be functions of i and j. Thus the solution of (14) corresponding to (a) has the form

$$(16) \qquad \begin{cases} F_{ij}(x) = a \log x + \beta_i + \gamma_j\,, \quad G_i(x) = a \log x - \alpha_3 k_i(x) + \beta_i \\ H_j(x) = a \log x + \gamma_j,\ K_i \text{ arbitrary}, \quad L_j(x) = \alpha_3 \end{cases}$$

for $i = 1, 2, \ldots, n$, $j = 1, 2, \ldots, m$.

Then the solution of (5), from (13), (14), (15) and (16) has the form

$$(\mathrm{a}_1) \qquad \begin{cases} f_{ij}(x) = ax \log x + (\beta_i + \gamma_j - \sum\limits_{k=1}^{n} \sum\limits_{r=1}^{m} d_{kr} + \sum\limits_{k=1}^{n} e_k \sum\limits_{r=1}^{m} d_r)\, x + d_{ij}\,, \\ g_i(x) = ax \log x + (\beta_i - \sum\limits_{k=1}^{n} b_k - \alpha_3 K_i(x) - \sum\limits_{r=1}^{m} d_r K_i(x))\, x + b_i\,, \\ h_j(x) = ax \log x + (\gamma_j - \alpha_3 \sum\limits_{k=1}^{n} e_k - \sum\limits_{r=1}^{m} c_r)\, x + c_j\,, \\ k_i(x) = x K_i(x) + e_i\,, \quad l_j(x) = \alpha_3 x + d_j\,, \end{cases}$$

for $x \in I_1$, $i = 1, 2, \ldots, n$, $j = 1, 2, \ldots, m$, where K_i is arbitrary. It is easy to see from (12) that (a_1) also holds for $x = 0$. Thus, (a_1) constitutes a solution of (5), where K_i is an arbitrary function and a, β_i, γ_j, α_3, d_{ij}'s, b_i's, c_j's, d_j's, e_i's are arbitrary constants.

Similarly, from the corresponding solution (a') of (14), (13) and (15), we obtain the following solution of (5):

$$(\mathrm{a}_1') \qquad \begin{cases} f_{ij}(x) = ax \log x + (\beta_i + \gamma_j - \sum\limits_{k=1}^{n} \sum\limits_{r=1}^{m} d_{kr} + \sum\limits_{k=1}^{n} e_k \sum\limits_{r=1}^{m} d_r)\, x + d_{ij} \\ g_i(x) = ax \log x + (\beta_i - \alpha_3 \sum\limits_{r=1}^{m} d_r - \sum\limits_{k=1}^{n} b_k)\, x + b_i \\ h_j(x) = ax \log x + (\gamma_j - \sum\limits_{r=1}^{m} c_r - \alpha_3 L_j(x) - \sum\limits_{k=1}^{n} e_k L_j(x))\, x + c_j \\ k_i(x) = \alpha_3 x + e_i\,, \quad l_j(x) = x L_j(x) + d_j \end{cases}$$

for $x \in I$, $i = 1, 2, \ldots, n$, $j = 1, 2, \ldots, m$, where L_j is an arbitrary function and a, β_i, γ_j, α_3, d_{ij}, b_i, c_i, d_j and e_i are arbitrary constants.

Similarly, from the corresponding solutions (b) and (c) of (14), (15), (12) and (13), the following solutions of (5) can be obtained:

$$(\mathrm{b}_1)\quad \begin{cases} f_{ij}(x) = \alpha_{ij}x^\beta + ax\log x + \left(\gamma_{ij} - \sum_{k=1}^{n}\sum_{r=1}^{m} d_{kr} + \sum_1^n e_k \sum_1^m d_r\right)x + d_{ij} \\ g_i(x) = \left(\gamma_i - D_i \sum_1^m d_r\right)x^\beta + ax\log x + \left(\delta_i - \sum_1^n b_k - \alpha_8 \sum_1^m d_r\right)x + b_i \\ h_j(x) = \left(A_j - E_j \sum_1^n e_k\right)x^\beta + ax\log x + \left(B_j - \alpha_{10}\sum_1^n e_k - \sum_1^m c_r\right)x + c_j \\ k_i(x) = D_i x^\beta + \alpha_8 x + e_i \\ l_j(x) = E_j x^\beta + \alpha_{10}x + d_j, \quad i = 1, 2, \ldots, n, \quad j = 1, 2, \ldots, m \end{cases}$$

$$\text{with}\quad \alpha_{ij} = D_i E_j,\ \gamma_i + D_i\alpha_{10} = 0 = A_j + \alpha_8 E_j,$$

$$\gamma_{ij} = \delta_i + B_j + \alpha_8\alpha_{10};$$

and

$$(\mathrm{c}_1)\quad \begin{cases} f_{ij}(x) = \alpha_1 A^2 x\log^2 x + (AA_j + b)x\log x + \left(\gamma_{ij} - \sum_1^n\sum_1^m d_{kr} + \sum_1^n e_k \sum_1^m d_r\right)x + d_{ij} \\ g_i(x) = \alpha_1 A^2 x\log^2 x + \left(b - 2\alpha_1 A\sum_1^m d_r\right)x\log x + \left(B_i - \sum_1^n b_k - \alpha_7\sum_1^m d_r\right)x + b_i \\ h_j(x) = \alpha_1 A^2 x\log^2 x + \left(A_5 D_j + b - A\sum_1^n e_k\right)x\log x + \left(D_j - \sum_1^m c_r - \beta_j \sum_1^n e_k\right)x + c_j \\ k_i(x) = 2\alpha_1 A x\log x + \alpha_7 x + e_i \\ l_j(x) = Ax\log x + \beta_j x + d_j, \quad i = 1, \ldots, n, \quad j = 1, 2, \ldots, m; \end{cases}$$

$$\text{with}\quad A_j = 2\alpha_1\beta_j = D_j + \alpha_7,\quad \gamma_{ij} = B_i + E_j + \beta_j\alpha_7.$$

How about the mixed solutions? Even though it is messy, it can be shown that, because of the linear independence of the functions $x\log x$, $x\log^2 x$, x, x^β ($\beta \neq 1, 0$), 1, the mixed solution cannot occur, unless $\beta = 1$ or 0 in which case $A = 0$ and the solutions are part of (b_1) (and (c_1)).

Thus, we have proved the following theorem.

Theorem. Let $f_{ij}, g_i, h_j, k_1, l_j\colon I \to \mathbb{R}$ ($i = 1, 2, \ldots, n$; $j = 1, 2, \ldots, m$) be measurable. Then, these functions satisfy the functional equation (5), for a fixed pair of integers m, n ($\geqq 3$) if, and only if, they are given either by (a_1) or by (a_1') or by (b_1) of by (c_1).

Remarks. 1. The summations $\sum_{k=1}^{n}\sum_{k=1}^{m} d_{kr}$, $\sum_{r=1}^{n} e_k$, $\sum_{r=1}^{m} d_r$ etc. appearing in (a_1), (a_1'),

(b_1) and (c_1) clearly establish the dependency of the solutions of (5) on m and n

2. For example, if $f_{ij} = f$, $g_i = g$, $h_j = h$, $k_i = k$, $l_j = l$ in (5), then the solution of (5) corresponding to (b_1) takes the form

$$f(x) = \alpha_1 x^\beta + ax \log x + (\alpha_2 + mned - mnd')\, x + d'$$

$$g(x) = (\alpha_3 - \alpha_7 md)\, x^\beta + ax \log x + (\alpha_4 - nb - \alpha_8 nd)\, x + b$$

$$h(x) = (\alpha_5 - \alpha_9 ne)\, x^\beta + ax \log x + (\alpha_6 - \alpha_{10} ne - mc)\, x + c$$

$$k(x) = \alpha_7 x^\beta + \alpha_8 x + e\,, \quad l(x) = \alpha_9 x^\beta + \alpha_{10} x + d$$

with $\alpha_1 = \alpha_7\alpha_9$, $\alpha_3 + \alpha_7\alpha_{10} = 0 = \alpha_5 + \alpha_8\alpha_9$, $\alpha_2 = \alpha_4 + \alpha_6 + \alpha_8\alpha_{10}$

a result found in [8], which clearly exhibits the dependency of the solution on m and n.

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