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BASIC TERMS OF THE THEORY OF COMPARTMENTAL SYSTEMS

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The terms of the theory of compartmental systems are defined and explained. A language with the set-theoretical base is used. The biological theory of types is accepted. The language, enlarged by time terms is plausible for description of large time-variable systems with hierarchical structure.

In the epistemic process over large systems we often have to classify the elements of the universe into classes and then we do not consider the properties of individual elements and the relations among them, but the properties of classes and relations among these classes. An example of this sort are the so called compartmental systems, which may be roughly described as follows:

1. They are built up of elements belonging to different classes. All elements belonging to one class are considered as equivalent and by classification criterion undistinguishable.
2. All elements of individual classes are aggregated so that the properties of these classes and their relations may be different from the properties and relations of classes without aggregation.
3. The number of elements aggregated in one class need not be steady in time.
4. The elements aggregated in one class may be transformed into elements aggregated in another class.
5. The objects of the environment of the system may be transformed into elements of the system and vice versa.
6. These systems are characterised further by an important role of entities of various type-levels. At the same time they are dynamic, time-variable systems.

The terms "compartment" and "compartmental system" have been used in some scientific branches, e.g. in biology and in medical sciences, for a long time. The concept of compartment was implicitely used in the pioneer work of Schönheimer and Rittenberg [1, 2] on the tracer method in the study of metabolism and also in Theorell's papers [3, 4]. An explicit definition of this term, applicable to metabolic studies, was formulated by Sheppard [5] and was taken over by Atkins [6] and Jacques [7]: "Compartment is a quantity of a substance which has uniform kinetics of transformation or transport" [6]. In pharmacokinetics, i.e. in the study of distribution, transport and transformation of drugs [8] and in ecology [9] the term "compartment" has the same meaning. In a similar meaning was this term introduced into the research of the kinetics of cell populations by Lajtha [10]. A simulation language respecting the Sheppard's meaning of "compartment" was constructed by Kindler [11]. A compartment modelling method with an approximative distribution of particles among compartments was described by Kotva [13]. A notion of compartmental system, where different states of particles were partitioned into compartments, was discussed in terms of the Markovian process by Ličko [12].

The research of subcellular structures has given to the term "compartment" a rather different meaning: The cell is partitioned into compartments, where various sorts of enzymes are bounded, e.g. Giese [14]. Thus the compartment does not concern substances with uniform kinetics, but processes catalysed by different enzymes. "Compartment" has implicitely got a functional meaning in pharmacodynamics, i.e. in the study of effects of drugs, e.g. Ariens [15]. A special functional compartmentalisation is used in the network thermodynamics of Katchalsky and co. [16, 17]. In the domain of physiological regulations, e.g. regulation of body fluids, the compartmentalisation is used to be applied not only from the view of kinetics of substances, but from other views too, e.g. Gamble [18]. In our days the terms "compartment" and "compartmental system" are deep-rooted, but they are often used without exact definition. (The foregoing passage was written not to review the matter, but only to show the development of various scientific uses of the term "compartment".)

In this paper the authors attempt to formulate exact definitions and explanations of basic terms of the theory of compartmental systems. The formalism introduced in a paper of Bek and Růžička [19] will be used. In that paper basic terms of the theory of higher level systems have been defined: The definitions are formulated in an exact language with the set-theoretical base and the logical theory of types is accepted. The language is enlarged by time-terms: time-variables or time-constants appear in the formulae. Such a language is plausible for description of large, time-variable systems. All introduced terms are time-relative.

Next we define the term "compartment".

D 1. Let a system $\mathcal{S} = \langle \boldsymbol{U}, \mathcal{R} \rangle$ in the interval Δt exist, abbr $\langle \mathcal{S}, \Delta t \rangle \in \mathcal{S}\mathit{yst}$, and at the same time let the following conditions be satisfied:

1. There is an equivalence relation $R_e^{(2)} \in \mathcal{R}$, $R_e^{(2)} \subset \boldsymbol{U}^{\langle 2 \rangle} \times \Delta t$ which is symmetric, reflexive and transitive on $\boldsymbol{U}$ at an instant $t_i \in \Delta t$.

2. A relation R_e corresponds univocally to the relation $R_e^{(2)}$ so that $\langle x_1, x_2 \rangle \in R_e \leftrightarrow$

$\leftrightarrow \exists t_i \in \Delta t'$, $\langle x_1, x_2, t_i \rangle \in R_e^{(2)}$, $\Delta t' \subseteq \Delta t$. By the relation R_e the set U is partioned into equivalence classes $R_{e_1}, \ldots, R_{e_n}$. By the rule: $x_j \in R_{e_j} \leftrightarrow \exists t_i \in \Delta t'$, $\langle x_j, t_i \rangle \in R_{e_j}^{(1)}$ a class $R_{e_j}^{(1)}$ univocally corresponds to every class R_{e_j}, $j \in J$, $1 \leqq j \leqq n$ of them.

3. For every element x_j, for which $\langle x_j, t_i \rangle \in R_{e_j}^{(1)}$ holds, there is a class P_{j,t_i} of all properties of x_j (which correspond to values of pertinent magnitudes) at the instant $t_i \in \Delta t'$.

4. Let P_j be the union of classes P_{j,t_i} of all properties of elements x_j belonging in the interval $\Delta t'$ to the equivalence class R_{e_j}, $P_j = \bigcup_{t_i \in \Delta t'} P_{j,t_i}$.

5. Let $\text{card}_{t_i}(R_{e_j})$ be the number of elements of the class R_{e_j} at the instant $t_i \in \Delta t'$.

The equivalence class R_{e_j} will be called "*compartment A_j of the system $\mathcal{S}$ in the interval $\Delta t'$*", abbr $\langle A_j, \mathcal{S}, \Delta t' \rangle \in \mathit{comp}$ if R_{e_j} is in the field of an operation ϱ_{A_j}, ϱ_{A_j} being an univocal correspondence of the triplet $\langle P_{j,t_i}, \text{card}_{t_i}(R_{e_j}), t_i \rangle$ to the class Q_{j,t_i}. The class Q_{j,t_i} is the class of properties (distinct from $\text{card}_{t_i}(R_{e_j})$ and corresponding to values of pertinent magnitudes) of the compartment A_j at the instant t_i. $\varrho_{A_j} \subset P_j \times \text{card}(R_{e_j}) \times \Delta t' \times Q_j$ where: $\text{card}(R_{e_j})$ is the cardinality of the equivalence class R_{e_j}, i.e. the set of $\text{card}_{t_i}(R_{e_j})$ for all t_i, $t_i \in \Delta t' \subseteq \Delta t$, $\langle \mathcal{S}, \Delta t \rangle \in \mathcal{S}\mathit{yst}$, Q_j is the union of classes Q_{j,t_i} for all $t_i \in \Delta t'$, i.e. $Q_j = \bigcup_{t_i \in \Delta t'} Q_{j,t_i}$. The operation ϱ_{A_j} will be called "*aggregation on the class R_{e_j} in the interval $\Delta t'$*".

The class of properties of the compartment A_j in the interval $\Delta t'$ will be represented by the union $U_j = \text{card}(R_{e_j}) \cup Q_j$ and *the class of properties of the compartment A_j at the instant $t_i \in \Delta t'$* will be represented by the union $U_{j,t_i} = \{\text{card}_{t_i}(R_{e_j})\} \cup Q_{j,t_i}$, $U_j \bigcup_{t_i \in \Delta t'} U_{j,t_i}$.

D 2. The property V_{j,t_i}^k will be called "*the k-th input property of the compartment A_j at the instant t_i*", iff following conditions are satisfied:

1. $\langle A_j, \mathcal{S}, \Delta t' \rangle \in \mathit{comp}$, $\langle U, \mathcal{R} \rangle = \mathcal{S}$, $\langle \mathcal{S}, \Delta t \rangle \in \mathcal{S}\mathit{yst}$, $t_i \in \Delta t' \subseteq \Delta t$;

2. There is an interval $\Delta t^{(2)}$ such that $t_i - |\Delta t^{(2)}|$ is the instant, which precedes the instant t_i just by the length of the interval $\Delta t^{(2)}$;

3. There is an object y, distinct from A_j and being neither an element, nor a property, nor a relation among properties of the compartment A_j at the instant $t_i - (\Delta t^{(2)}|$;

4. There is a property Y such that for some instant $t_i \in \Delta t'$ the proposition holds: if y has at the instant $t_i - |\Delta t^{(2)}|$ a property Y, then the compartment A_j lawfully*) has at the instant t_i the property V_{j,t_i}^k, $V_{j,t_i}^k \in V_{j,t_i} \subseteq U_{j,t_i}$, where V_{j,t_i} is the set of input properties at the instant t_i.

*) The term "lowfully" here has the usual meaning: a relation $\mathcal{R}$ among the entities Y, V and the time interval Δt and with respect to conditions P is lowfull, if for every instant t_i the proposition holds: the existence of the entity Y at the instant t_i and under the conditions P determines the existence of the entity V at the instant $t_i + |\Delta t|$.

D 3. The property W^l_{j,t_i} will be called "*the l-th output property of* the compartment A_j at the instant t_i, iff following conditions are satisfied:

1. $\langle A_j, \mathscr{S}, \Delta t' \rangle \in \mathscr{comp}$, $\langle U, \mathscr{R} \rangle = \mathscr{S}$, $\langle \mathscr{S}, \Delta t \rangle \in \mathscr{Syst}$, $t_i \in \Delta t' \subseteq \Delta t$,

2. There is an interval $\Delta t^{(2)}$ such that $t_i + |\Delta t^{(2)}|$ is the instant, which follows the instant t_i just after interval $\Delta t^{(2)}$ elapsed;

3. There is an object y', distinct from A_j and being neither an element, nor a property nor a relation among properties of the compartment A_j at the instant $t_i + |\Delta t^{(2)}|$;

4. There is a property Y' such that for some instant $t_i \in \Delta t'$ the proposition holds: if the compartment A_j has at the instant t_i a property W^l_{j,t_i}, $W^l_{j,t_i} \in W_{j,t_i} \subseteq U_{j,t_i}$ then y' lawfully has the property Y'. W_{j,t_i} here is the set of output properties at the instant t_i.

D 4. The subset Z_{j,t_i}, $Z_{j,t_i} \subseteq U_{j,t_i}$, of properties of the compartment A_j at the instant t_i will be called "*the value of the state of the compartment A_j at the instant t_i*", the sets V_{j,t_i}, W_{j,t_i} "***the input and the output values of the compartment** A_j at the instants t_i, t_k respectively*", iff relations R_ϑ, R_λ exist, such that for every instant t_i, $t_i \in \Delta t'$, $\langle A_j, \mathscr{S}, \Delta t' \rangle \in \mathscr{comp}$,$\langle \mathscr{S}, \Delta t \rangle \in \mathscr{Syst}$ following propositions hold:

1. There is an interval $\Delta t^{(2)}$ such that

$$\langle\langle V_{j,t_i}, Z_{j,t_i}\rangle, W_{j,t_k}\rangle \in R_\lambda,\ t_k = t_i + |\Delta t^{(2)}|, \quad t_k \in \Delta t' ;$$

2. There is an interval $\Delta t^{(3)}$ such that

$$\langle\langle V_{j,t_i}, Z_{j,t_i}\rangle, Z_{j,t_l}\rangle \in R_\vartheta,\ t_l = t_i + |\Delta t^{(3)}|,\ t_l \in \Delta t'$$

D 5. By the term "*behaviour of the compartment A_j in the interval $\Delta t'$*" is denoted the couple R_ξ of the relations R_ϑ, R_λ, satisfying the conditions of the definition D 4.

Remark. If $Z_{j,t_i} = Z_{j,t_l}$ for every $t_i, t_l \in \Delta t^{(3)}$, then $\langle V_{j,t_i}, W_{j,t_k}\rangle \in R_\lambda$ and R_ϑ is an identity and $R_\xi = R_\lambda$. The output value depends only on its input value and the behaviour of the compartment is called "*combinatory in the interval $\Delta t'$*". In an opposite case, for $Z_{j,t_i} \neq Z_{j,t_l}$ for at least one couple $t_i, t_l \in \Delta t^{(3)}$, the behaviour of the compartment is called "*sequentional in the interval $\Delta t'$*".

D 6. We say that "*the compartment A_j exchanges its element x_j with its environment in the interval $\Delta t^{(2)}$*, iff at the same time following conditions are satisfied:

1. $\langle A_j, \mathscr{S}, \Delta t' \rangle \in \mathscr{comp}$, $\langle \mathscr{S}, \Delta t \rangle \in \mathscr{Syst}$, $\quad \Delta t^{(2)} \subseteq \Delta t' \subseteq \Delta t$;

2. There is $t_i \in \Delta t'$, $t_i + |\Delta t^{(2)}| \in \Delta t'$ so that

$$\langle x_j, t_i \rangle \notin R^{(1)}_{e_j},\ \langle x_j, t_i + |\Delta t^{(2)}| \rangle \in R^{(1)}_{ej}\ ,$$

or

$$\langle x_j, t_i \rangle \in R^{(1)}_{e_j},\ \langle x_j, t_i + |\Delta t^{(2)}| \rangle \notin R^{(1)}_{e_j}$$

and according to D 1: $R^{(1)}_{e_j} \subseteq U \times \Delta t'$.

In the first case the term "*assimilation of the element* x_j *by the compartment* A_j *in the interval* $\Delta t^{(2)}$" and in the second case the term "*dissimilation of the element* x_j *by the compartment* A_j *in the interval* $\Delta t^{(2)}$" may be used. Designate:

$\underset{\Delta t^{(2)}}{\Delta}$ in card (R_{e_j}) ... the number of elements assimilated by A_j in $\Delta t^{(2)}$

$\underset{\Delta t^{(2)}}{\Delta}$ ex card (R_{e_j}) ... the number of elements dissimilated by A_j in $\Delta t^{(2)}$

$\text{card}_{t_i}(R_{e_j}) = \text{card}\{x_j \mid \langle x_j, t_i\rangle \in R^{(1)}_{e_j}\}$ ... the number of elements of the compartment A_j at the instant t_i.

The following proposition (balance condition) is evident:
For every compartment A_j:

$$\text{card}_{t_a}(R_{e_j}) + \underset{\Delta t^{(2)}}{\Delta} \text{in card}\,(R_{e_j}) = \underset{\Delta t^{(2)}}{\Delta} \text{ex card}\,(R_{e_j}) + \text{card}_{t_b}(R_{e_j}),$$

where $\Delta t^{(2)} = \langle t_a, \ldots, t_b\rangle$, i.e. t_a, t_b are the first and the last instants of $\Delta t^{(2)}$.

D 7. As "*a coupling between the compartments* A_i, A_j *in the interval* $\Delta t'$" we shall denote the relation $G_{i,j}$ satisfying these conditions:

1. $\langle A_i, \mathcal{S}, \Delta t'\rangle \in \mathcal{comp}$, $\langle A_j, \mathcal{S}, \Delta t'\rangle \in \mathcal{comp}$, $\langle \mathcal{S}, \Delta t\rangle + \mathcal{Syst}$, $\Delta t' \subseteq \Delta t$;
2. There is an interval $\Delta t^{(2)}$ such that for some instants $t_i \in \Delta t'$

$$t_i + |\Delta t^{(2)}| \in \Delta t', \langle\langle W_{i,t_i}, t_i\rangle, \langle V_{j,t_i+|\Delta t^{(2)}|}, t_i + |\Delta t^{(2)}|\rangle\rangle \in G_{i,j}$$

holds, where W_{i,t_i} is the value of the output of the compartment A_i at t_i, $V_{j,t_i+|\Delta t^{(2)}|}$ is the value of the input of the compartment A_j at $t_i + |\Delta t^{(2)}|$.

D 8. By "*the compartmental system* $\mathcal{A}$ *of the system* $\mathcal{S}$ *in the interval* $\Delta t'$" we understand the pair $\mathcal{A} = \langle A, \Gamma\rangle$, where

1. $A = \{A_1, A_2, \ldots, A_n\}$, $n \geqq 2$, $\langle A_1, \mathcal{S}, t_{o_1}\rangle, \ldots, \langle A_n, \mathcal{S}, t_{o_n}\rangle \in \mathcal{comp}$,

   $\Delta t_{o_1}, \ldots, \Delta t_{o_n} \subseteq \Delta t' \subseteq \Delta t$, $\Delta t_{o_1} \cup \Delta t_{o_2} \cup \ldots \cup \Delta t_{o_n} = \Delta t'$, $\langle \mathcal{S}, \Delta t\rangle \in \mathcal{Syst}$
2. Γ is a set, for which following propositions hold:

   a) $G_{i,j} \in \Gamma$, if a pair of compartments A_i, A_j exists so that $G_{i,j}$ is a coupling between A_i, A_j in the interval $\Delta t'$, $\langle A_i, \mathcal{S}, \Delta t_{o_i}\rangle \in \mathcal{comp}$, $\langle A_j, \mathcal{S}, \Delta t_{o_j}\rangle \in \mathcal{comp}$

   b) ${}^{(s)}G_{i,j} \in \Gamma$ if a pair $\langle D_i, D_j\rangle$ exists so that at least one of its elements D_i, D_j is identical with any ${}^{(s-1)}G_{i_1,i_2} \in \Gamma$, (or ${}^{(s-1)}G_{i_2,i_1} \in \Gamma$), the other element may be any ${}^{(s-o)}G_{i_1,i_2} \in \Gamma$/or ${}^{(s-o)}G_{i_2,i_1} \in \Gamma$/,$(s - o) \geqq 1$, and ${}^{(s)}G_{i,j}$ is a relation between the elements D_i, D_j existing in the interval $\Delta t^{(2)} \subseteq \Delta t'$,

   c) nothing more belongs to the set Γ.

3. Each compartment is at least in one instant $t_i \in \Delta t'$ in the field of at least one coupling of compartments or of another relation from the set Γ, (abbr $\langle \mathcal{A}, \mathcal{S}, \Delta t'\rangle \in \in \mathcal{Comps}$).

Remarks to D 8.

(i) We see that the class Γ of relations in a compartmental system may include not only couplings among compartments, but also couplings among couplings, couplings of couplings among couplings etc without potentional limitation.

(ii) Let x be an element of the universe U of the system $\mathscr{S}$. The element x belongs to the lowest type level of the 0-order. The equivalence class of the form R_{e_j} and the compartment A_j (according to D 1) will then belong to the type level of the 1st order. The classes of the form Q_{j,t_i}, Q_j will belong to the type level of 3rd order. The classes U_{j,t_i}, U_j belong to the level of the 3rd order. The cardinals $\text{card}_{t_i}(R_{e_j})$ are of the 2nd order. The relation ϱ_{A_j} (the operation of aggregation on the class R_{e_j} in a given interval) then belongs to the type level of the 4th order. The relations R_λ, R_ϑ (according to D 4) belong to the type level of the 4th order as well as the relations R_ξ (D 5) and $G_{i,j}$ (D 7). The compartmental system $\mathscr{A} = \langle A, \Gamma \rangle$ (D 8) belongs to the type level of the 5th order under the condition, that the class Γ is exclusively a class of couplings of the form $G_{i,j}$ among compartments (but not among couplings). Generally it is of the order $s \geqq 5$.

(iii) On the base system $\mathscr{S}$ various compartments and compartmental systems may be defined and systems of those systems as well, without potentional limitation. We can so obtain higher order hierarchicly structurea compartmental systems. Such systems are important e.g. in biology.

D 9. The union $\bigcup_{j \in j'} R_{e_j} = B_c$ is called "*the pool B_c of the compartmental system $\mathscr{A}$ in the interval $\Delta t'$*", if:

1. $\langle \mathscr{A}, \mathscr{S}, \Delta t' \rangle \in \mathscr{C}omps$, $\mathscr{S} = \langle U, \mathscr{R} \rangle$, $\langle \mathscr{S}, \Delta t \rangle \in \mathscr{S}yst$, $\Delta t' \subseteq \Delta t$;

2. According to the condition 2 from the definition D 1, there is a relation R_e, partitioning the universe U into equivalence classes R_{e_j}, $j \in J$;

3. $J' \subseteq J$, J' is a set of indices of all these equivalence classes of the form R_{e_j}, whose elements have in addition to their specific properties $P_{j''}$ (condition 2 from D 1) at least one common property C. If $P_{j'}$ is the union of all classes P_{j,t_i} (condition 4 from D 1), then let $P_{j'} = P_{j''} \cap C$.

4. The set $\bigcup_{j' \in J'} R_{e_{j'}}$ is in the field of an operation ϱ_{B_c}, $\varrho_{B_c} \subset C \times \text{card}\left(\bigcup_{j' \in J'} R_{e_{j'}}\right) \times \Delta t' \times Q_{B_c}$ (in the sense of D 1 is the operation of aggregation on the set $\bigcup_{j' \in J'} R_{e_{j'}}$ in the interval $\Delta t'$). The properties of the pool B_c will be represented by the union $\text{card}\left(\bigcup_{j' \in J'} R_{e_{j'}}\right) \cup Q_{B_c}$.

Remark to D 9. The pool may be conceived as a compartment, resulting from the aggregation of such elements of the universe U, which have the property C. So the term "pool" might seem superfluous. In biological sciences however we often find the elements, which have the property C, out of the already compartmentalised universe; then we aggregate them according to the property C, with abstraction from $P_{j''}$. The term "pool" is here introduced to be in accordance with this practice in biology.

D 10. By the term "*aggregation of the properties of the compartments of the system*" will be denoted the operation $\varrho_{\mathscr{A}}$, which is a correspondence at the time t_i of each ordered j-tuple ($1 \leqq j \leqq n$, n is the number of the compartments of the system $\mathscr{A}$) of properties of the form U_{o,t_i} of the individual compartments (according to D 1) to a property U_{A,t_i} of the compartmental system $\mathscr{A}$. The k-th property of the system $\mathscr{A}$ corresponding to a value of a pertinent magnitude at the instant t_i will be designated U^k_{A,t_i}. The class of all properties of the compartmental system $\mathscr{A}$ at the instant t_i will be designated $U_{\mathscr{A},t_i}$, $U^k_{A,t_i} \in U_{\mathscr{A},t_i}$. The class of all properties, which the compartmental system $\mathscr{A}$ has in the interval of its existence (i.e. in any instant $t_i \in \Delta t'$) will be denoted $U_{\mathscr{A}}$, $U_{\mathscr{A}} = \bigcup\limits_{t_i \in \Delta t'} U_{\mathscr{A},t_i}$.

D 11. As "*the k-th input property of the compartmental system $\mathscr{A}$ as the instant t_i*" will be denoted the property V^k_{A,t_i}, iff following conditions are satisfied:

1. $\langle \mathscr{A}, \mathscr{S}, \Delta t' \rangle \in \mathscr{C}omps\,, \quad \langle \mathscr{S}, \Delta t \rangle \in \mathscr{S}yst\,, \quad \Delta t' \subseteq \Delta t\,, \quad t_i \in \Delta t'$

2. There is an interval $\Delta t^{(2)}$, such that $t_i - |\Delta t^{(2)}|$ is the instant, which precedes the instant t_i just by the length of the interval $\Delta t^{(2)}$.

3. There is an object y, distinct from $\mathscr{A}$, being at the instant $t_i - |\Delta t^{(2)}|$ neither an element, nor a relation among the properties of any compartment $A_j \in A$, nor any compartment $A_j \in A$, nor any element of the set Γ, where $\mathscr{A} = \langle A, \Gamma \rangle$.

4. There is a property Y, such that for at least one instant $t_i \in \Delta t'$ the following proposition holds: if y at the instant $t_i - |\Delta t^{(2)}|$ has the property Y, then the system $\mathscr{A}$ lawfully has at the instant t_i the property V^k_{A,t_i}, $V^k_{A,t_i} \in U_{\mathscr{A},t_i}$.

D 12. As "*the 1-th output property of the compartmental system $\mathscr{A}$ at the instant t_i*" will be denoted the property W^l_{A,t_i}, iff following conditions are satisfied:

1. $\langle \mathscr{A}, \mathscr{S}, \Delta t' \rangle \in \mathscr{C}omps\,, \quad \langle \mathscr{S}, \Delta t \rangle \in \mathscr{S}yst\,, \quad \Delta t' \subseteq \Delta t\,, \quad t_i \in \Delta t'$

2. There is an interval $\Delta t^{(2)}$, such that $t_i + |\Delta t^{(2)}|$ is the instant, which follows the instant t_i just after the interval $\Delta t^{(2)}$ elapsed.

3. There is an object y' distinct from $\mathscr{A}$, such that it is at the instant $t_i + |\Delta t^{(2)}|$ neither an element, nor a property, nor a relation among the properties of any compartment $A_j \in A$, nor a compartment $A_j \in A$, nor an element of the set Γ, where $\mathscr{A} = \langle A, \Gamma \rangle$.

4. There is a property Y', such that for at least one instant $t_i \in \Delta t'$ the following

proposition holds: if the system $\mathscr{A}$ at the instant t_i has the property W^l_{A,t_i}, $W^l_{A,t_i} \in U_{\mathscr{A},t_i}$, then y' lawfully has at the instant $t_i + |\Delta t^{(2)}|$ the property Y'.

D 13. The subset $Z_{\mathscr{A},t_i}$, $Z_{\mathscr{A},t_i} \subseteq U_{\mathscr{A},t_i}$, of properties of the compartmental system $\mathscr{A}$ at the time t_i will be called "*the value of the state of the system $\mathscr{A}$ at the instant t_i*", the set $V_{\mathscr{A},t_i}$ of the input properties of the system $\mathscr{A}$ at the time t_i and the set $W_{\mathscr{A},t_l}$ of the output properties of the system $\mathscr{A}$ at the time t_k will be called "*the value of the input and the value of the output of the system at the instants $t_i t_k$ respectively*", if there are the relations $R_{\vartheta\mathscr{A}}$, $R_{\lambda\mathscr{A}}$ such that for every instant $t_i \in \Delta t'$, $t_k \in \Delta t'$, $t_l \in \Delta t'$, $t_i < t_k$, $t_i < t_l$ and $\langle \mathscr{A}, \mathscr{S}, \Delta t' \rangle \in \mathscr{C}omps$, $\langle \mathscr{S}, \Delta t \rangle \in \mathscr{S}yst$, following propositions hold:

$$\langle\langle V_{\mathscr{A},t_i}, Z_{\mathscr{A},t_i}\rangle, W_{\mathscr{A},t_k}\rangle \in R_{\lambda\mathscr{A}} \quad \text{and} \quad \langle\langle V_{\mathscr{A},t_i}, Z_{\mathscr{A},t_i}\rangle, Z_{\mathscr{A},t_l}\rangle \in R_{\vartheta\mathscr{A}} .$$

D 14. As "*the behaviour of the compartmental system $\mathscr{A}$ in the interval $\Delta t'$*" will be denoted the couple $R_{\xi\mathscr{A}}$ of the relations $R_{\lambda\mathscr{A}}$, $R_{\vartheta\mathscr{A}}$, where $R_{\lambda\mathscr{A}}$, $R_{\vartheta\mathscr{A}}$ satisfy the conditions of the definition D 13 in the whole interval $\Delta t'$.

D 15. We say that "*the compartmental system $\mathscr{A} = \langle A, \Gamma \rangle$ exchanges the element x with its enviroment in the interval $\Delta t^{(2)}$*", if following conditions are satisfied:

1. $\langle \mathscr{A}, \mathscr{S}, \Delta t' \rangle \in \mathscr{C}omps$, $\langle \mathscr{S}, \Delta t \rangle \in \mathscr{S}yst$, $\Delta t^{(2)} \subseteq \Delta t' \subseteq \Delta t$
2. There are $t_i \in \Delta t'$, $t_i + |\Delta t^{(2)}| \in \Delta t'$.
3. There is a compartment $A_i \in A$, but there is not any compartment $A_j \in A$, $A_i \neq A_j$ so that:

a) $$\langle x, \mathscr{S}, t_i \rangle \notin El , \quad \langle x, t_i \rangle \in R^{(1)}_{e_j} , \quad \langle x, t_i + |\Delta t^{(2)}| \rangle \in R^{(1)}_{e_i}$$

or

b) $$\langle x, t_i \rangle \in R^{(1)}_{e_i} , \quad \langle x, t_i + |\Delta t^{(2)}| \rangle \in R_{ej} , \quad \langle x, \mathscr{S}, t_i + |\Delta t^{(2)}| \rangle \notin El$$

where R_{e_i}, R_{e_j} are defined by D 1 and $\langle A_i, \mathscr{S}, \Delta t' \rangle \in comp$, $\langle A_j, \mathscr{S}, \Delta t' \rangle \notin comp$.

In the case a) we are speaking on the "*assimilation of the element x by the system $\mathscr{A}$ in the interval $\Delta t^{(2)}$*", in the case b) on "*the dissimilation of the element x by the system $\mathscr{A}$ in the interval $\Delta t^{(2)}$*". Let us designate eq_i, eq_j the conversion factors of the exchange of the elements x_i, x_j. These factors are analogic to the stoichiometric factors in chemistry. ($eq_i \neq eq_j$, if n elements x_i cannot be transformed into n elements x_j). For every compartmental system $\mathscr{A}$ following balance conditions are always satisfied:

$$\sum_{i=1}^{n} eq_i \underset{\Delta t^{(2)}}{\Delta} \text{in } \mathscr{A} \text{ card } (R_{e_i}) + \sum_{i=1}^{n} eq_i \text{ card}_{t_b} (R_{e_i}) =$$

$$= \sum_{i=1}^{n} eq_i \underset{\Delta t^{(2)}}{\Delta} \text{ex } \mathscr{A} \text{ card } (R_{e_i}) + \sum_{i=1}^{n} eq_i \text{ card}_{t_b} (R_{e_i})$$

where n is the number of compartments of the system $\mathscr{A}$ in the interval $\Delta t^{(2)}$, $\underset{\Delta t^{(2)}}{\Delta}$ in $\mathscr{A}$ card (R_{e_i}) is the number of elements assimilated in the interval $\Delta t^{(2)}$ by the system $\mathscr{A}$, $\underset{\Delta t^{(2)}}{\Delta}$ ex $\mathscr{A}$ card (R_{e_i}) is the number of elements dissimilated by the system $\mathscr{A}$ in the interval $\Delta t^{(2)}$, t_a, t_b are the boundary points of the interval $\Delta t^{(2)}$. The exchange of elements may also take place within the compartmental system among its compartments. If compartments $A_1, A_2, \ldots, A_c$ of the system $\mathscr{A}$ in the interval $\Delta t^{(2)}$ have dissimilated elements so that all these elements have been assimilated by other compartments $A_{c+1}, \ldots, A_d$ of the system $\mathscr{A}$, then the balance condition is:

$$\sum_{i=1}^{c} eq_i \underset{\Delta t^{(2)}}{\Delta} \text{ ex card } (R_{e_i}) = \sum_{i=c+1}^{d} eq_i \underset{\Delta t^{(2)}}{\Delta} \text{ in card } (R_{e_i})$$

We might find many trivial and non trivial examples of compartmental systems with compartments on the same type level. Here we give an example of hierarchical structure of biological compartmental systems:

Living cells will be conceived as systems with the universe of particles of various substances, present and interacting in the cell, including the enzymes and their substrates. These particles will be the entities on the type level 0. (The particles of the cell environment, exchangeable with the particles of the cells, will also belong to the type level 0.) The particles are aggregated in compartments on the level 1. On the cell a compartmental system on the level s, $s \gg 1$, will be defined. The competition of various enzymes for one and the same substrate with simultaneous consumption and supply on chemical particles is often studied on such systems.

Yet in multicellular organism the cells may be considered as entities on the type level s, aggregated in compartments of various cell sorts and we may study the situation, where various cell compartments compete for compounds from the nutritional pool of the organism, with simultaneous consumption and supply of it. The cell compartments will be the entities of the level $s + 1$, whereas the compartments of nutritional compounds will belong to the type level 1 (with an aggregation different from the aggregation within the cells). On the multicellular organism a compartmental system on the type level s', $s' \gg s + 1$, will be defined.

We may further study the organisms in an ecological system. Organisms of various species will be aggregated in compartments on the type level $s' + 1$. On the ecological system a compartmental system will be defined on the type level s'', $s'' \gg s' + 1$. The competition among various species for foods may here be studied as well as on other levels of systems. The compartments of foodstufs will here remain on the level 1 (with another aggregation then within the cells or in the pools of the organisms).

The matter of fact, that there is an exchange of entities among compartments or among subsystems and systems on different type levels, often remains a non pronounced assumption in biological sciences.

CONCLUDING REMARK

It is well known, that an exact description of large systems, e.g. biological, is not always possible and that the main sources of inexactness are the stochasticity and ambiguity of relations in the system [20]. Therefore formulations of stochastic and fuzzy systems are attempted. The authors suppose that the foregoing definitions also may be of use to this development of the theory.

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