KYBERNETIKA — VOLUME 17 (1981), NUMBER 4

ON COMPOSITIONAL AND CONVOLUTIONAL DISCRETE SYSTEMS

BEDŘICH PONDĚLÍČEK

In this paper, the following crucial notions of discrete system theory in the multidimensional time area are considered: linearity, causality, time independence and stability.

During the last several years many publications concerning two and multidimensional digital filters have appeared (see [1], [2] and [3]). Having presented six counterexamples E. I. Jury [4] has shown that in extending the theorems developed for the one dimensional case to two and multidimensional systems many difficulties are encountered. It seems to be useful to investigate the relationship among fundamental concepts of multidimensional digital filters.

In this paper we shall consider the following crucial notions of discrete system theory in the multidimensional time area: linearity, causality, time independence and stability. We shall use standard notation and terminology of linear space theory: linear spaces and subspaces, bases, and linear mappings.

1. COMPOSITIONAL DISCRETE SYSTEMS

By R we denote the set of all real numbers. Let N denote the set of all nonnegative integers. In this paper let n be a fixed positive integer and let N^n denote the cartesian power of N. Elements of N^n will be called *multiindices*. If $u, v \in N^n$, then $u = (u_1, u_2, \ldots, u_n)$ and $v = (v_1, v_2, \ldots, v_n)$, where $u_i, v_i \in N$ for $i = 1, 2, \ldots, n$. We shall put

$$u \leqq v \quad \text{if} \quad u_i \leqq v_i \quad \text{for} \quad i = 1, 2, \ldots, n\,.$$

In such case, by $v - u$ we denote the multiindex $(v_1 - u_1, v_2 - u_2, \ldots, v_n - u_n)$. Finally, for $m \in N$ by $\bar{m}$ the multiindex $(m, m, \ldots, m)$ will be denoted.

Let $\mathscr{S}$ be the set of all *sequences* $a: N^n \to R$. It is well known that $\mathscr{S}$ forms a real linear space with respect to addition and scalar multiple of sequences. In this paper we shall consider linear mappings of $\mathscr{S}$ into itself.

For any multiindex w we define the following sequence $t^w \in \mathscr{S}$:

$$t^w(u) = \begin{cases} 1 & \text{for} \quad u = w\,; \\ 0 & \text{for} \quad u \neq w\,; \end{cases} \tag{1.1}$$

where $u \in N^n$.

A mapping $I: N^n \times N^n \to R$ is said to be a *kernel*, if for any multiindex u

$$\text{the set } \{v : v \in N^n \quad \text{and} \quad I(u, v) \neq 0\} \quad \text{is finite}\,. \tag{1.2}$$

By $\mathscr{K}$ we denote the set of all kernels. Now, we can define the *composition* of a kernel $I \in \mathscr{K}$ and a sequence $a \in \mathscr{S}$ as the following sequence b:

$$b(u) = \sum_{v \in N^n} I(u, v)\, a(v) \tag{1.3}$$

for any multiindex u. It follows from (1.2) that the right-hand side of (1.3) is well defined. It is clear that $b \in \mathscr{S}$. We shall write $b = I \circ a$.

In this paper by a discrete system Φ we mean a mapping Φ of the linear space $\mathscr{S}$ into itself.

Definition 1.1. A discrete system Φ is called *compositional* if there exists a kernel I from $\mathscr{K}$ such that for any sequence a of $\mathscr{S}$ we have $\Phi(a) = I \circ a$.

Note 1.1. We shall show that the kernel I of a compositional discrete system Φ is uniquely determined. Suppose that there exists another kernel J such that $\Phi(a) = J \circ a$ for any sequence $a \in \mathscr{S}$. Let w be an arbitrary multiindex. Then $I \circ t^w = \Phi(t^w) = J \circ t^w$ and therefore, by (1.1) and (1.3), we have $I(u, w) = J(u, w)$ for every multiindex u, hence $I = J$.

Theorem 1.1. Every compositional discrete system is linear.

Proof. It follows from (1.3) that $I \circ (\alpha a + \beta b) = \alpha(I \circ a) + \beta(I \circ b)$ for $I \in \mathscr{K}$, $a,\ b \in \mathscr{S}$ and $\alpha, \beta \in R$. □

Note 1.2. There exist linear discrete systems, which which are not compositional.

Proof. Since the set $\mathscr{T}$ of all sequences t^w (see (1.1)) is linearly independent, there exists a base $\mathscr{P}$ of the linear space $\mathscr{S}$ such that $\mathscr{T} \subset \mathscr{P}$. Evidently $\mathscr{T} \neq \mathscr{P}$. The zero sequence and the unit sequence will be denoted by o and j, respectively; i.e. for any multiindex u $o(u) = 0$ and $j(u) = 1$. There exists a linear mapping Ψ of $\mathscr{S}$ into itself such that

$$\Psi(a) = \begin{cases} j & \text{for} \quad a \in \mathscr{T}\,; \\ o & \text{for} \quad a \in \mathscr{P} \setminus \mathscr{T}\,. \end{cases} \tag{1.4}$$

Now, we shall show that the linear discrete system Ψ is not compositional. Suppose that there exists a kernel I from $\mathscr{K}$ such that $\Psi(a) = I \circ a$ for every sequence a of $\mathscr{S}$. Let v be an arbitrary multiindex. Then $I \circ t^v = \Psi(t^v) = j$ and so, by (1.3), we have $I(u, v) = j(u)$ for all multiindices u, which contradicts (1.2). □

For any multiindex w we denote by $\mathscr{V}_w$ the set of all sequences $a \in \mathscr{S}$ such that

$$(1.5) \qquad a(u) = 0 \quad \text{for every multiindex} \quad u \leqq w\,.$$

It is easy to show that $\mathscr{V}_w$ is a linear subspace of the linear space $\mathscr{S}$ and $\bigcap_{w \in N^n} \mathscr{V}_w = \{o\}$.

Definition 1.2. A discrete system Φ is called *causal* if for any multiindex w the following implication holds:

$$(1.6) \qquad \text{If} \quad a, b \in \mathscr{S} \quad \text{and} \quad b - a \in \mathscr{V}_w\,, \quad \text{then} \quad \Phi(b) - \Phi(a) \in \mathscr{V}_w\,.$$

Note 1.3. A linear discrete system Φ is causal if and only if for any multiindex w we have

$$(1.7) \qquad \Phi(c) \in \mathscr{V}_w \quad \text{for every} \quad c \in \mathscr{V}_w\,.$$

Proof. Assume that a discrete system Φ is linear. Let w be an arbitrary multiindex. If the implication (1.6) holds, then we can put $b = c$ and $a = o$. Thus we have $\Phi(c) \in \mathscr{V}_w$ for $c \in \mathscr{V}_w$ because $\Phi(o) = o$. If (1.7) is true, then for $a, b \in \mathscr{S}$, $b - a \in \mathscr{V}_w$, we have $\Phi(b) - \Phi(a) = \Phi(b - a) \in \mathscr{V}_w$. □

By $\mathscr{M}$ we denote the set of all mappings $I : N^n \times N^n \to R$ satisfying the following implication:

$$(1.8) \qquad \text{If} \quad I(u, v) \neq 0 \quad \text{for some multiindices} \quad u, v, \,, \quad \text{then} \quad \tau \leqq u\,.$$

According to (1.2) and (1.8) we have $\mathscr{M} \subset \mathscr{K}$.

Theorem 1.2. Every compositional discrete system with the kernel belonging to $\mathscr{M}$ is causal.

Proof. Let Φ be a discrete system such that there exists $I \in \mathscr{M}$ and $\Phi(a) = I \circ a$ for any sequence $a \in \mathscr{S}$. Let w be an arbitrary multiindex. Suppose that $b - a \in \mathscr{V}_w$ for some sequences $a, b \in \mathscr{S}$. It follows from (1.5) that $a(u) = b(u)$ for every multiindex $u \leqq w$. Put $x = \Phi(a) = I \circ a$ and $y = \Phi(b) = I \circ b$. Let u be a multiindex such that $u \leqq w$. By (1.3) and (1.8) we have $x(u) = \sum_{v \in N^n} I(u, v)\, a(v) = \sum_{v \in N^n} I(u, v)\,.$ $.\, b(v) = y(u)$. It follows from (1.5) that $\Phi(b) - \Phi(a) = y - x \in \mathscr{V}_w$. Thus Φ is a causal discrete system. □

Theorem 1.3. A discrete system is linear and causal if and only if it is compositional with the kernel belonging to $\mathscr{M}$.

Proof. If a discrete system is compositional with the kernel belonging to $\mathcal{M}$, then according to Theorem 1.1, it is linear and, by Theorem 1.2, it is causal.

Suppose that a discrete system Φ is linear and causal. For any multiindex w we put $z_w = \Phi(t^w) \in \mathcal{S}$. Let us define a mapping $I : N^n \times N^n \to R$ in the following way:

$$(1.9) \qquad I(u, v) = z_v(u) \quad \text{for all multiindices} \quad u \quad \text{and} \quad v\,.$$

We shall show that $I \in \mathcal{M}$. Assume that $I(u, v) \neq 0$ for some multiindices u and v. Then by (1.9) and (1.5) we have $z_v \in \mathcal{S} \setminus \mathcal{V}_u$ and according to (1.7), we obtain that $t^v \in \mathcal{S} \setminus \mathcal{V}_u$. It follows from (1.1) and (1.5) that $v \leqq u$. Hence, by (1.8), $I \in \mathcal{M}$.

Now, we shall prove that $\Phi(a) = I \circ a$ for every sequence $a \in \mathcal{S}$. Let $a \in \mathcal{S}$. Choose a multiindex w and put $p = \sum_{v \leqq w, v \in N^n} a(v)\, t^v$. It follows from (1.1) that $p(v) = a(v)$ for every multiindex $v \leqq w$ and $p(v) = 0$ for every multiindex v non $\leqq w$. Since Φ is a linear discrete system, we have $q = \Phi(p) = \sum_{v \leqq w, v \in N^n} a(v)\, \Phi(t^v) = \sum_{v \leqq w, v \in N^n} a(v)\, z_v$. It follows from (1.9) that $q(u) \sum_{v \leqq w, v \in N^n} a(v)\, z_v(u) = \sum_{v \in N^n} I(u, v)\, p(v)$ and thus, by (1.3), we have $\Phi(p) = q = I \circ p$. If a multiindex $v \leqq w$, then $a(v) - p(v) = 0$ and therefore, by (1.5), $a - p \in \mathcal{V}_w$. According to Theorem 1.2 and (1.6), we have $I \circ a - I \circ p \in \mathcal{V}_w$ and $\Phi(a) - \Phi(p) \in \mathcal{V}_w$. Since $\mathcal{V}_w$ is a linear subspace pf $\mathcal{S}$, we have $\Phi(a) - I \circ a = \Phi(a) - \Phi(p) - (I \circ a - I \circ p) \in \mathcal{V}_w$. Then $\Phi(a) - I \circ a \in \bigcap_{w \in N^n} \mathcal{V}_w = \{o\}$, hence $\Phi(a) = I \circ a$ for all sequences a belonging to $\mathcal{S}$. □

2. CONVOLUTIONAL DISCRETE SYSTEMS

Recall that by *convolution* $a * b$ of sequences a, $b \in \mathcal{S}$ we mean the sequence $c \in \mathcal{S}$ such that for any multiindex u we have

$$(2.1) \qquad c(u) = \sum_{v \leqq u, v \in N^n} a(u - v)\, b(v)\,.$$

It is well known (and it is easy to show) that

$$(2.2) \qquad a * b = b * a\,,$$

$$a * (b * c) = (a * b) * c$$

for all sequences a, b, $c \in \mathcal{S}$. By δ we denote the sequence t^w, where the multiindex $w = \bar{0}$. For any sequence $a \in \mathcal{S}$ there is

$$(2.3) \qquad a * \delta = a = \delta * a\,.$$

Definition 2.1. A discrete system Φ is called *convolutional* if there exists a sequence $g \in \mathcal{S}$ such that for any sequence $a \in \mathcal{S}$ we have $\Phi(a) = g * a$.

Note 2.1. If Φ is a convolutional discrete system, then it follows from (2.3) that $\Phi(a) = \Phi(\delta) * a$ for every sequence $a \in \mathcal{S}$ because $\Phi(\delta) = g * \delta = g$.

Theorem 2.1. Every convolutional discrete system is compositional with the kernel belonging to $\mathscr{M}$.

Proof. If Φ is a convolutional discrete system, then there exists a sequence $g \in \mathscr{S}$ such that $\Phi(a) = g * a$ for all $a \in \mathscr{S}$. For multiindices u and v we put

$$G(u, v) = \begin{cases} g(u - v) & \text{for} \quad v \leqq u ; \\ 0 & \text{for} \quad v \text{ non} \leqq u . \end{cases}$$

It is easy to show that $G \in \mathscr{M}$ and comparing (1.3) and (2.1) we obtain that $g * a = G \circ a$ for every sequence $a \in \mathscr{S}$. □

Let w be a multiindex. By τ_w we denote the convolutional discrete system such that $\tau_w(a) = t^w * a$ for all $a \in \mathscr{S}$. Put $b = t^w * a$. It follows from (2.1) that for any multiindex u we have

$$b(u) = \begin{cases} a(u - w) & \text{for} \quad w \leqq u ; \\ 0 & \text{for} \quad w \text{ non} \leqq u . \end{cases} \tag{2.4}$$

Hence τ_w is a time-translation on the linear space $\mathscr{S}$.

Definition 2.2. A discrete system Φ is called *time-invariant* if for every multiindex w have $\Phi\tau_w = \tau_w\Phi$, i.e. $\Phi(t^w * a) = t^w * \Phi(a)$ for all sequences $a \in \mathscr{S}$.

Theorem 2.2. Every linear time-invariant discrete system is causal.

Proof. Let Φ be a linear and time-invariant discrete system. Let $c \in \mathscr{V}_w$ for some multiindex w. We shall prove that $\Phi(c) \in \mathscr{V}_w$. Define sequences $c_k \in \mathscr{S}$ for $k = 1, 2, \ldots, n$ as follows: For any multiindex u we put

$$c_k(u) = \begin{cases} c(u) & \text{for} \quad u_i = w_i,\ i < k, \quad \text{and} \quad w_k < u_k ; \\ 0 & \text{otherwise} , \end{cases} \tag{2.5}$$

where $u = (u_1, u_2, \ldots, u_n)$ and $w = (w_1, w_2, \ldots, w_n)$. Put $d = \sum_{k=1}^{n} c_k$. We shall show that $c = d$. Let u be a multiindex, $u \leqq w$. Since $c \in \mathscr{V}_w$, we have $c(u) = 0$. It follows from (2.5) that $c_k(u) = 0$ for $k = 1, 2, \ldots, n$ and therefore $c(u) = d(u)$. Suppose that a multiindex u non $\leqq w$. Then there exists a positive integer $k \leqq n$ such that $w_k < u_k$ and $u_i = w_i$ for all positive integers $i < k$ (if $k > 1$). It follows from (2.5) that $c_k(u) = c(u)$ and $c_i(u) = 0$ for $i \neq k$ and again $c(u) = d(u)$.

Put $v_k = (v_{1k}, v_{2k}, \ldots, v_{nk})$, where $v_{ik} = 0$ for $i \neq k$ and $v_{kk} = w_k + 1$. Further, we define sequences $b_k \in \mathscr{S}$ for $k = 1, 2, \ldots, n$ as follows: $b_k(u) = c_k(u + v_k)$ for all multiindices u. Denote $d_k = t^{v_k} * b_k$. We shall show that $d_k = c_k$. Let $u = (u_1, u_2, \ldots, u_n)$ be a multiindex. If $v_k \leqq u$, then according to (2.4) we have $d_k(u) = c_k(u)$. If v_k non $\leqq u$, then $(w_k + 1)$ non $\leqq u_k$ and therefore $u_k \leqq w_k$. Then, by (2.4) and (2.5), we have $d_k(u) = 0 = c_k(u)$. Therefore $c_k = t^{v_k} * b_k$.

Let us denote $z_k = \Phi(c_k)$ and $y_k = \Phi(b_k)$. Since Φ is time-invariant, we have

$z_k = \Phi(t^{v_k} * b_k) = t^{v_k} * y_k$. We shall show that $z_k \in \mathscr{V}_w$. Let $u = (u_1, u_2, \ldots, u_n)$ be a multiindex. If $u \leqq w$, then $u_k \leqq w_k$ and therefore v_k non $\leqq u$. It follows from (2.4) that $z_k(u) = 0$. Since Φ is linear, we have $\Phi(c) = \Phi(\sum_{k=1}^{n} c_k) = \sum_{k=1}^{n} \Phi(c_k) = \sum_{k=1}^{n} z_k \in \mathscr{V}_w$ because $\mathscr{V}_w$ is a linear subspace of $\mathscr{S}$. □

Theorem 2.3. A discrete system is convolutional if and only if it is compositional and time-invariant.

Proof. Let Φ be a convolutional discrete system. Then there exists a sequence $g \in \mathscr{S}$ such that $\Phi(a) = g * a$ for any sequence $a \in \mathscr{S}$. It follows from Theorem 2.1 that Φ is compositional. Now, we shall prove that Φ is time-invariant. Let u be a multiindex. According to (2.2), for any $a \in \mathscr{S}$ we have $\Phi(t^w * a) = g * (t^w * a) = (g * t^w) * a = (t^w * g) * a = t^w * (g * a) = t^w * \Phi(a)$.

Assume that Φ is a compositional time-invariant discrete system. Then there exists a kernel $I \in \mathscr{K}$ such that $\Phi(a) = I \circ a$ for all $a \in \mathscr{S}$. Put $g = \Phi(\delta) = I \circ \delta$. It follows from (1.3) that $g(u) = I(u, \bar{0})$ for all multiindices u. Since Φ is time-invariant, we have, by (2.3), $I \circ t^v = \Phi(t^v) = \Phi(t^v * \delta) = t^v * \Phi(\delta) = g * t^v$ for all multiindices v. From (1.3) and (2.1) now follows that

$$I(u, v) = \begin{cases} g(u - v) & \text{for } v \leqq u\,, \\ 0 & \text{for } v \text{ non } \leqq u \end{cases} \tag{2.6}$$

and therefore $\Phi(a) = I \circ a = g * a$ for all $a \in \mathscr{S}$. Hence Φ is convolutional. □

Theorem 2.4. Let Φ be a discrete system.

If Φ is linear and time-invariant, then it is convolutional.

If Φ is convolutional, then it is linear, causal and time-invariant.

The proof follows from Theorems 2.2, 1.3, 2.3 and 2.1. □

Note 2.2. Our results from Sections 1 and 2 can be generalized for systems in which inputs are sequences from $\mathscr{S}$ and outputs are either functions or distributions. In the second case we can use modified conclusions of [5] and [6].

3. STABLE DISCRETE SYSTEMS

For any sequence a of $\mathscr{S}$ we put $\nu(a) = \sup_{u \in N^n} |a(u)|$. A sequence a is said to be *bounded* if $\nu(a) < +\infty$. The set of all bounded sequences of $\mathscr{S}$ will be denoted by $\mathscr{B}$. It is well known that $\mathscr{B}$ is a linear subspace of the linear space $\mathscr{S}$.

Definition 3.1. A discrete system Φ is called *stable* if for every bounded sequence $a \in \mathscr{B}$ we have $\Phi(a) \in \mathscr{B}$. A discrete system Φ is said to be *uniformly stable* if for any positive real number β there exists a positive real number α such that the following implication holds:

$$\text{(3.1)} \qquad \text{If} \quad a \in \mathscr{S} \quad \text{and} \quad v(a) \leqq \alpha\,, \quad \text{then} \quad v(\Phi(a)) \leqq \beta\,.$$

Note 3.1. A linear discrete system Φ is uniformly stable if and only if there exists a positive real number γ such that

$$\text{(3.2)} \qquad v(\Phi(a)) \leqq \gamma \quad \text{for every sequence} \quad a \quad \text{of} \quad \mathscr{S} \quad \text{with} \quad v(a) \leqq 1\,.$$

Proof. Assume that a linear discrete system Φ satisfies the condition (3.2). Let α be a positive real number. Put $\beta = \alpha\gamma$. Let $a \in \mathscr{S}$ with $v(a) \leqq \alpha$. Then $v(\alpha^{-1}a) \leqq 1$, hence, by (3.2), we have $v(\Phi(\alpha^{-1}a)) \leqq \gamma$. Thus $v(\Phi(a)) \leqq \alpha\gamma = \beta$. From (3.1) it follows that Φ is uniformly stable. □

Note 3.2. It is easy to prove that every uniformly stable discrete system is stable. We shall show that the linear discrete system Ψ as described in Note 1.2 is stable but not uniformly stable.

Let a be a sequence belonging to the base $\mathscr{P}$ of $\mathscr{S}$ (see Note 1.2). It follows from (1.4) that $\Psi(a) \in \{j, o\} \subset \mathscr{B}$. Since $\mathscr{B}$ is a linear subspace of the linear space $\mathscr{S}$, we obtain that Ψ is stable. On the other hand, for every $m \in N$ we put $a_m = t^w$, where the multiindex $w = \overline{m}$, and $b_m = \sum_{k=1}^{m} a_k$. Then $v(b_m) = 1$. According to (1.4), we have $\Psi(b_m) = \sum_{k=1}^{m} \Psi(a_k) = mj$ and so $v(\Psi(b_m)) = m$. Then Ψ cannot be uniformly stable. □

By $\mathscr{C}$ we denote the set of all kernels I from $\mathscr{K}$ satisfying the following conditon:

$$\text{(3.3)} \qquad \sup_{u \in N^n} \sum_{v \in N^n} |I(u, v)| < +\infty\,.$$

Theorem 3.1. Let Φ be a compositional discrete system. Then the following conditions are equivalent:

1. Φ is uniformly stable;
2. Φ is stable;
3. the kernel of Φ belongs to $\mathscr{C}$.

The proof is a multidimensional modification of one part of the proof of Kojima-Schur's theorem from [8].

1 ⇒ 2. It is evident.

2 ⇒ 3. Let Φ be a stable compositional discrete system with a kernel I. Let v be a multiindex. Evidently $t^v \in \mathscr{B}$ and so $\Phi(t^v) \in \mathscr{B}$. It follows from (1.1) and (1.3) that there exists a positive real number $\lambda(v)$ such that

$$\text{(3.4)} \qquad 2|I(u, v)| \leqq \lambda(v) \quad \text{for all multiindices} \quad u\,.$$

Suppose that $I \in \mathscr{K} \setminus \mathscr{C}$. Then, by (3.3), there exists a multiindex u_1 such that

$$\sum_{v \in N^n} |I(u_1, v)| > 1 + \lambda(\overline{0})\,.$$

It follows from (1.2) that there exists a positive integer k_1 such that if $I(u_1, v) \neq 0$, then $v \leqq \bar{k}_1$.

Now, we shall define by induction a bounded sequence a. Put

$$a(\bar{0}) = 0 \tag{3.5}$$

and

$$a(v) = \operatorname{sgn} I(u, v) \quad \text{for any multiindex} \quad v \leqq \bar{k}_1, v \neq \bar{0}\,.$$

Let m be a positive integer. Suppose that there exists a multiindex u_m and an integer $k_m > k_{m-1}$, where $k_0 = 0$, such that

$$\sum_{v \in N^n} |I(u_m, v)| > m + \sum_{v \in N^n, v \leqq \bar{k}_{m-1}} \lambda(v)\,. \tag{3.6}$$

$$\text{If} \quad I(u_m, v) \neq 0\,, \quad \text{then} \quad v \leqq \bar{k}_m\,. \tag{3.7}$$

For any multiindex $v \leqq \bar{k}_m$ and v non $\leqq \bar{k}_{m-1}$ we have

$$a(v) = \operatorname{sgn} I(u_m, v)\,. \tag{3.8}$$

Since $I \in \mathscr{K} \setminus \mathscr{C}$, according to (3.3), there exists a multiindex u_{m+1} such that

$$\sum_{v \in N^n} |I(u_{m+1}, v)| > m + 1 + \sum_{v \in N^n, v \leqq \bar{k}_m} \lambda(v)\,.$$

The condition (1.2) implies that there exists an integer $k_{m+1} > k_m$ such that if $I(u_{m+1}, v) \neq 0$, then $v \leqq \bar{k}_{m+1}$. For any multiindex $v \leqq \bar{k}_{m+1}$ and v non $\leqq \bar{k}_m$ we put $a(v) = \operatorname{sgn} I(u_{m+1}, v)$.

Let $v = (v_1, v_2, \ldots, v_n) \neq \bar{0}$ be an arbitrary multiindex. Then there exists a positive integer m such that

$$k_{m-1} < \max\{v_1, v_2, \ldots, v_n\} \leqq k_m$$

and so $v \leqq \bar{k}_m$ and v non $\leqq \bar{k}_{m-1}$. This implies that the sequence a is defined for all multiindices. It follows from (3.8) that

$$|a(v)| = 1 \quad \text{for all multiindices} \quad v \neq \bar{0}\,. \tag{3.9}$$

Put $b = \Phi(a) = I \circ a$. Now, we shall show that $b(u_m) > m$ for every positive integer m. It follows from (3.5), (3.4) and (3.9) that for all multiindices v we have

$$- I(u_m, v)\, a(v) \leqq |I(u_m, v)| \leqq \lambda(v) - |I(u_m, v)|$$

and therefore

$$\lambda(v) + I(u_m, v)\, a(v) \geqq |I(u_m, v)|\,. \tag{3.10}$$

Denote by L_m the set of all multiindices u such that $u \leqq \bar{k}_m$. According to (1.3), (3.7), (3.8) and (3.6), we obtain that

$$b(u_m) = \sum_{v \in N^n} I(u_m, v)\, a(v) = I_1 + I_2\,,$$

where

$$I_1 = \sum_{v \in L_m \setminus L_{m-1}} I(u_m, v) \operatorname{sgn} I(u_m, v) = \sum_{v \in L_m \setminus L_{m-1}} |I(u_m, v)| > m + \sum_{v \in L_{m-1}} (\lambda(v) - |I(u_m, v)|)$$

and

$$I_2 = \sum_{v \in L_{m-1}} I(u_m, v)\, a(v)\,.$$

It follows from (3.10) that $b(u_m) = I_1 + I_2 > m$ for all positive integers m and therefore $b \in \mathscr{S} \setminus \mathscr{B}$. According to (3.9), we have $a \in \mathscr{B}$, which is a contradiction because Φ is stable. Therefore $I \in \mathscr{C}$.

$3 \Rightarrow 1$. Let Φ be a compositional discrete system with the kernel I belonging to $\mathscr{C}$. Let a be a sequence with $\nu(a) \leqq 1$. Put $b = \Phi(a)$. For any multiindex u, by (1.3), we have $|b(u)| = |\sum_{v \in N^n} I(u, v)\, a(v)| \leqq \sum_{v \in N^n} |I(u, v)|\, |a(v)| = \sum_{v \in N^n} |I(u, v)|$. Then (3.3) implies that

$$\nu(\Phi(a)) = \sup_{u \in N^n} \sum_{v \in N^n} |I(u, v)|\,.$$

It follows from Theorem 1.1 and Note 3.1 that Φ is uniformly stable. □

Theorem 3.2. A linear and causal discrete system is uniformly stable if an only if it is stable.

The proof follows from Theorem 1.3 and Theorem 3.1. □

In what follows, we denote by $\mathscr{A}$ the set of all sequences $a \in \mathscr{S}$ satisfying the following condition:

$$\sup_{u \in N^n} \sum_{v \in N^n, v \leqq u} |a(v)| < +\infty\,.$$

Let g be an arbitrary sequence of $\mathscr{S}$. Using (2.6) we can define a kernel I such that $I \circ a = g * a$ for all sequences $a \in \mathscr{S}$. From (2.6) it follows that for any multiindex u we have

$$\sum_{v \in N^n} |I(u, v)| = \sum_{v \in N^n, v \leqq u} |g(u - v)| = \sum_{w \in N^n, w \leqq u} |g(w)|\,,$$

where we put $w = u - v$. This implies that $I \in \mathscr{C}$ if and only if $g \in \mathscr{A}$. Hence, by Theorem 3.1 and Note 2.1, we have the following:

Theorem 3.3. Let Φ be a convolutional discrete system. Then the following conditions are equivalent:

1. Φ is uniformly stable;
2. Φ is stable;
3. $\Phi(\delta)$ belongs to $\mathscr{A}$.

(See Theorem 1 in [7].)

Theorem 3.4. A linear and time-invariant discrete system is uniformly stable if and only if it is stable.

The proof follows from Theorem 2.4 and Theorem 3.1. □

(Received November 11, 1980.)

REFERENCES

[1] L. Rabiner, B. Gold: Theory and Application of Digital Signal Processing. Prentice Hall, Englewood Cliffs, New Jersey 1975.

[2] T. S. Huang (ed.): Picture Processing and Digital Filtering. Springer - Verlag, Berlin — Heidelberg — New York 1975.

[3] N. K. Bose: Problems and progress in multidimensional systems theory. Proc. IEEE *65* (1977), 6, 824 — 840.

[4] E. I. Jury: Counterexamples in multidimensional system theory. IEEE Circuits and Systems Magazine *2* (1980), 2, 1M4.

[5] B. Ponděliček: A contribution to the foundations of network theory using the distribution theory. Czech. Math. J. *19* (*94*) (1969), 4, 697 — 710.

[6] B. Ponděliček: A contribution to the foundations of network theory using the distribution theory, II. Czech. Math. J. *21* (*96*) (1971), 1, 35 — 45.

[7] A. F. Humes, E. I. Jury: Stability tests for two-dimensional linear multivariable digital filters. Internat. J. Control *26* (1977), 2, 225 — 234.

[8] R. G. Cooke: Infinite Matrices and Sequence Spaces. London 1950.

Doc. RNDr. Bedřich Ponděliček, CSc., elektrotechnická fakulta ČVUT (Faculty of Electrical Engineering — Czech Technical University), Suchbátarova 2, 166 27 Praha. Czechoslovakia.