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ON COVARIANCE COEFFICIENTS ESTIMATES OF FINITE ORDER MOVING AVERAGE PROCESSES

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In the present paper the necessary and sufficient conditions for the estimates of covariance coefficients of moving average processes are presented. Further the modification for estimates of Wilson's method covariance coefficients is introduced.

1. INTRODUCTION

Covariance coefficients provide important information about the structure of the given time series in most cases. We shall treat the covariance coefficients of finite order MA processes.

Let $\{Y_n, n = 0, \pm 1, \pm 2, \ldots\}$ be uncorrelated random variables with

$$\mathsf{E}[Y_n] = 0\,, \quad \mathsf{E}[Y_n^2] = \sigma^2 > 0 \quad \text{for all } n\,. \tag{1}$$

Let $\theta_0, \theta_1, \ldots, \theta_k$ be real numbers ($\theta_0 \neq 0, \theta_k \neq 0$) and consider the process defined by

$$X_n = \theta_0 Y_n + \theta_1 Y_{n-1} + \theta_2 Y_{n-2} + \ldots + \theta_k Y_{n-k}\,. \tag{2}$$

This process is called Moving Average Process (MA process). For the covariance, we have (for simplicity we put $\sigma^2 = 1$)

$$\mathsf{E}[X_n] = \mathsf{E}[\theta_0 Y_n + \theta_1 Y_{n-1} + \ldots + \theta_k Y_{n-k}] = 0 \tag{3}$$

$$\mathsf{E}[X_n^2] = (\theta_0^2 + \theta_1^2 + \ldots + \theta_k^2) \tag{4}$$

$$R(v) = \mathsf{E}[X_n X_{n-|v|}] = \mathsf{E}[(\theta_0 Y_n + \theta_1 Y_{n-1} + \ldots + \theta_k Y_{n-k}) \times \times (\theta_{|v|} Y_n + \theta_{1+|v|} Y_{n-1} + \ldots + \theta_{k+|v|} Y_{n-k})] = \begin{cases} \sum_{j=0}^{k-|v|} \theta_j \theta_{j+|v|} & \text{for } |v| \leqq k \\ 0 & \text{for } |v| \geqq k+1\,. \end{cases} \tag{5}$$

Let us define two generating function

$$\theta(z) = \sum_{j=0}^{k} \theta_j z^j \tag{6}$$

and

$$C(z) = \sum_{j=-k}^{k} R(j)\, z^j \,. \tag{7}$$

By substitution (5) into (7) we obtain

$$C(z) = \theta(z) \,.\, \theta(z^{-1}) \quad \text{for} \quad z \neq 0 \,. \tag{8}$$

The necessary and sufficient condition for the existence of MA process (2) with covariance coefficients $R(v)$ for $v = 0, 1, \ldots, k$ is that the auxiliary equation $P(x) = 0$ derived from (7), where $x = (z + z^{-1})/2$, has no real roots of odd multiplicity in the interval $-1 < x < 1$ (Wold's theorem [1]). For $|z| = 1$ e.g. $z = e^{i\lambda}$ is $(1/2\pi)\, C(z)$ spectral density and then it is real and nonnegative, i.e.

$$C(z) \geqq 0 \,. \tag{9}$$

Now, we shall outline the principals of Wilson's method [2]. We denote the generating function for coefficients of MA process in t-iterative step by $\theta^{(t)}(z)$ and $\bar{y}$ is a complex conjugative y. For $t \geqq 0$ we have

$$\begin{aligned} \theta^{(t)}(z)\, \theta^{(t+1)}(z^{-1}) + \theta^{(t+1)}(z)\, \theta^{(t)}(z^{-1}) &= \\ = \theta^{(t)}(z)\, \theta^{(t)}(z^{-1}) &+ \theta(z)\, \theta(z^{-1}) \,. \end{aligned} \tag{10}$$

From (10) we get

$$\begin{aligned} &\theta^{(t+1)}(z)\, \theta^{(t+1)}(z^{-1}) = \\ &= (\theta^{(t+1)}(z) - \theta^{(t)}(z))\, (\theta^{(t+1)}(z^{-1}) - \theta^{(t)}(z^{-1})) + \theta(z)\, \theta(z^{-1}) \,. \end{aligned} \tag{11}$$

For $|z| = 1$ and $z^{-1} = \bar{z}$ and $\theta(z^{-1}) = \overline{\theta(z)}$ we get from (11)

$$|\theta^{(t+1)}(z)|^2 = |\theta^{(t+1)}(z) - \theta^{(t)}(z)|^2 + |\theta(z)|^2 \tag{12}$$

and then we get

$$|\theta^{(t+1)}(z)|^2 \geqq |\theta^{(t+1)}(z) - \theta^{(t)}(z)|^2 \,. \tag{13}$$

We shall assume that our MA processes are invertible. Since we start from nonzero covariance coefficients,

$$|\theta^{(t+1)}(z)|^2 > |\theta^{(t+1)}(z) - \theta^{(t)}(z)|^2 \tag{14}$$

holds, and consequently (as it is proved in [2])

$$\theta^{(t)}(z)\, \theta^{(t)}(z^{-1}) \to \theta(z)\, \theta(z^{-1}) = C(z) \quad \text{for} \quad t \to \infty \tag{15}$$

and $z \in \langle -1, 1 \rangle$ holds.

As we have seen before the condition

$$C(z) > 0 \quad \text{for} \quad |z| = 1 \tag{16}$$

is the necessary and sufficient for the existence MA invertible process (2) with non-zero covariance coefficients. In processing time series we have only estimates of co-variance coefficients $\hat{R}(v)$, $v = 0, 1, \ldots, k$, which were obtained by

$$\hat{R}(v) = \frac{1}{N} \sum_{i=1}^{N} x_j x_{j+v} \tag{17}$$

where N is the length of realization of the given time series. For these estimates (16) need not be valid (see examples). A practical verification of (16) is very difficult and therefore we shall derive (16) is another form.

Let $P(x)$ be a polynomial derived from (7) where $x = (z + z^{-1})/2$ and let $|z| = 1$. Then from (16) we have

$$P(x) > 0 \quad \text{for} \quad -1 \leqq x \leqq 1 \,. \tag{18}$$

Now we construct sequence of polynomials

$$P_1(x), P_2(x), \ldots, P_m(x) \tag{19}$$

such that

$$P_1(x) = P(x)\,, \quad P_2(x) = P'(x)$$

and

$$\begin{aligned} P_{i-1}(x) &= P_i(x) \,.\, Q_{i-1}(x) - P_{i+1}(x)\,, \quad i = 2, 3, \ldots, m-1 \\ P_{m-1}(x) &= Q_{m-1}(x)\, P_m(x)\,, \quad P_m(x) \neq 0 \quad \text{for} \quad -1 \leqq x \leqq 1 \end{aligned} \tag{20}$$

and $P'(x)$ denotes derivative of $P(x)$. Let $V(1)$ be the number of sign inversions in sequence $P_1(1), P_2(1), \ldots, P_m(1)$ and let $V(-1)$ be the number of sign inversions in sequence $P_1(-1), P_2(-1), \ldots, P_m(-1)$. (If $P_i(1) = 0$ (or $P_i(-1) = 0$) we take the sign $+$, in case that there exist left (or, right) neighborhood of a point 1 (or, -1), so that $P_i(x) > 0$ for all x from the given neighborhood, and we take the sign $-$, if $P_i(x) < 0$.) Since sequence (19) forms the Sturm's sequence of polynomials, the number of real roots of $P(x)$ in the interval $-1 < x < 1$ is equal $V(1) - V(-1)$ (see [3]). From (18) we have

$$V(1) = V(-1)\,.$$

Thus we have obtained a new necessary and sufficient condition equivalent to (16), which can be written in the form

$$\begin{aligned} P(1) &> 0 \\ P(-1) &> 0 \\ V(1) &= V(-1)\,. \end{aligned} \tag{21}$$

If for some $\hat{R}(i)$, $i = 1, 2, \ldots, k$ (21) is not valid, we shall modify these coefficients. It is obvious that the modification can be done by different ways. We choose one way giving sufficient results in many practical examples, when Wilson's method gives no results.

2. MODIFICATION ALGORITHM OF COVARIANCE COEFFICIENT

Let $\hat{R}(0), \hat{R}(1), \ldots, \hat{R}(k)$ be estimates obtained from (17) and let (21) be not valid. Let δ be a given sufficiently small number and let $\tilde{R}(i)$, $i = 1. 2, \ldots, k$ be modificated covariance coefficients. We construct sequences $R_j(i), j = 0, 1, \ldots, n$, $i = 0, 1, \ldots, k$, in the following way:

1. $R_0(i) = \hat{R}(i) \quad i = 0, 1, \ldots, k$.
2. $R_1(0) = \hat{R}(0) \quad R_1(i) = \frac{1}{2}\hat{R}(i) \quad i = 1, 2, \ldots, k$.
3. Let be given $R_0(i), R_1(i), \ldots, R_{j-1}(i), \quad i = 0, 1, \ldots, k$.

   If for $R_{j-1}(i)$, $i = 0, 1, \ldots, k$, (21) is valid, let us put

$$R_j(i) = R_{j-1}(i) + \gamma_i U_j(i) ,$$

   otherwise

$$R_j(i) = R_{j-1}(i) - U_j(i)$$

   where

$$U_j(i) = \tfrac{1}{2}|R_{j-2}(i) - R_{j-1}(i)|$$

   $i = 0, 1, \ldots, k$; $\gamma_i = \operatorname{sign}(R_0(i))$.

4. n is chosen in such a way that it is the minimum of numbers 1, so that

$$U_1(i) < \delta \quad \text{for all} \quad i = 0, 1, \ldots, k$$

   and so that (21) is valid for $R_n(i)$, $i = 0, 1, \ldots, k$.

5. Let us put $\tilde{R}(i) = R_n(i)$, $i = 0, 1, \ldots, k$.

3. APPLICATION OF THE FOREGOING ALGORITHM ON INVERTIBLE MA PROCESSES OF THE ORDER 1 AND 2

Example 1. Let us have

$$X_n = Y_n + \theta_1 Y_{n-1}$$

where $\{Y_n, n = 0, \pm 1, \pm 2, \ldots\}$ are uncorrelated Gaussian random variables with $\mathsf{E}[Y_n] = 0$, $\mathsf{E}[Y_n^2] = 1$, θ_1 is an real number. From (7) we have

$$C(z) = R(1) z^{-1} + R(0) + R(1) z .$$

Therefore

$$P(x) = R(0) + 2R(1)\,x\,.$$

From (19) we have

(22) $$P_1(x) = R(0) + 2R(1)\,x$$

$$P_2(x) = 2R(1)$$

($m = 2$). From (21) we have

(23) $$R(0) + 2R(1) > 0$$

$$R(0) - 2R(1) > 0$$

The number of sign inversions in (22) is the same for both points 1 and -1. Therefore (23) is equivalent to (21) for $k = 1$. For this case (21) has a very simple form. In Table 1 the values $\hat{R}(i)$ and $\tilde{R}(i)$ $i = 0, 1$ are presented for MA process $X_n = Y_n - 0.9\,Y_{n-1}$, $N = 500$, $\delta = 10^{-5}$. $R(i)$ was obtained from (17), $\tilde{R}(i)$ was obtained by application of the foregoing algorithm and $\tilde{\theta}_1$ is computed by Wilson's method for values $\tilde{R}(i)$ $i = 0,1$. Note that the second condition in (23) is not valid for estimates $\hat{R}(i)$ $i = 0, 1$, but is valid for $\tilde{R}(i)$ $i = 0, 1$.

Table 1.

$\hat{R}(0)$	$\hat{R}(1)$	$\tilde{R}(0)$	$\tilde{R}(1)$	$\tilde{\theta}_1$
2·179832	−1·158923	2·179832	−1·089915	−0·99943

Example 2. Let us have

$$X_n = Y_n + \theta_1 Y_{n-1} + \theta_2 Y_{n-2} \quad n = 0, \pm 1, \pm 2, \ldots$$

where Y_n is the same as in Example 1. θ_1, θ_2 are real numbers. Then

$$C(z) = R(2)\,z^{-2} + R(1)\,z^{-1} + R(0) + R(1)\,z + R(2)\,z^2\,.$$

Therefore (we assume that $P_3(x) \neq 0$)

$$P_1(x) = R(0) + 2R(1)\,x + 4R(2)\,x^2$$

$$P_2(x) = 2R(1) + 8R(2)\,x$$

$$P_3(x) = R(0) - 2R(2) - \frac{R^2(1)}{4R(2)} \quad (m = 3).$$

From (21) we shall obtain that

$$R(0) + 2R(1) + 2R(2) > 0 \tag{24}$$
$$R(0) - 2R(1) + 2R(2) > 0$$

and

$$\text{if} \quad 4R(2) + R(1) < 0 \quad \text{or} \quad 4R(2) - R(1) < 0 \quad \text{then}$$
$$R^2(1) + 8R^2(2) - 4R(0)\,R(2) < 0$$

must hold. (24) represents the equivalent condition to (21) for case $k = 2$. In Table 2 the values $\hat{R}(i)$ and $\tilde{R}(i)$ $i = 0, 1, 2$ are presented for MA process $X_n = Y_n + 0{\cdot}2Y_{n-1} - 0{\cdot}48Y_{n-2}$, $N = 500$, $\delta = 10^{-5}$. $\tilde{\theta}_1$ and $\tilde{\theta}_2$ are computed by Wilson's method for values $\tilde{R}(i)$ $i = 0, 1, 2$. Note that the second condition in (24) is not valid for $\hat{R}(i)$ $i = 0, 1, 2$ but is valid for $\tilde{R}(i)$ $i = 0, 1, 2$.

Table 2.

$\hat{R}(0)$	$\hat{R}(1)$	$\hat{R}(2)$	$\tilde{R}(0)$
1·138916	0·219642	−0·394317	1·138926
$\tilde{R}(1)$	$\tilde{R}(2)$	$\tilde{\theta}_1$	$\tilde{\theta}_2$
0·203722	−0·365737	0·51574	−0·48076

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