KYBERNETIKA — VOLUME 17 (1981), NUMBER 2

ASSIGNING THE INVARIANT FACTORS BY FEEDBACK

VLADIMÍR KUČERA

An alternative proof is given of a theorem concerning the limits of state variable feedback in modifying the dynamics and structure of linear constants systems. The proof is based upon polynomial matrix considerations and provides a simple explicit algorithm to construct a feedback which effects the desired change.

INTRODUCTION

We consider a linear, constant, discrete-time system over an arbitrary field K represented by the equation

$$zx = Fx + Gu \tag{1}$$

where x is a sequence of states from K^n, u is a sequence of inputs from K^m, and z is the advance operator. We shall restrict ourselves to reachable systems and study the effect of linear state variable feedback

$$u = -Lx \tag{2}$$

upon the system dynamics. It is well known, see e.g. [6] or [9], that given any monic polynomial $c \in K[z]$ with $\deg c = n$, there exists a matrix L such that $F - GL$ has the characteristic polynomial c.

This result, however, gives only a partial picture as to what can be accomplished by a feedback of the type (2) with regard to altering the dynamics of system (1). Let $r = \operatorname{rank} G$ and let $\nu_1 \geqq \nu_2 \geqq \ldots \geqq \nu_r$ be the ordered set of input dynamical indices (synonyms: minimal indices, controllability indices, control invariants, Kronecker invariants) of system (1). To recall, the input dynamical indices are positive integers, whose sum is n, which constitute a complete set of invariants for system (1) under feedback and change of bases in K^n and K^m, see [1] for details. The following result is due to Rosenbrock [10]; alternative proofs can be found in [5], [2], and [3]. Given any set of monic polynomials $c_1, c_2, \dots, c_q$ of $K[z]$ satisfying the conditions

$$c_{i+1} \mid c_i, \quad i = 1, 2, \dots, q-1 \quad \text{and} \quad q \leqq r$$

and

$$\sum_{i=1}^{q} \deg c_i = n$$

there is a matrix L such that $F - GL$ has the invariant factors $c_1, c_2, \dots, c_q$ if and only if

$$\sum_{i=1}^{k} \deg c_i \geqq \sum_{i=1}^{k} v_i, \quad k = 1, 2, \dots, q .$$

Thus the input dynamical indices provide bounds on the sizes of the cyclic components of $F - GL$ thereby limiting the ability to modify the dynamics and algebraic structure of system (1) by feedback (2).

The purpose of this paper is to present an alternative proof of this remarkable result. The proof is algebraic in nature and makes use of polynomial matrices. Unlike the original proof, it yields a simple explicit algorithm to construct a feedback L. This algorithm consists in performing elementary operations on a polynomial matrix and then solving a linear equation.

PRELIMINARIES

For any $m \times n$ polynomial matrix P with elements in $K[z]$, write $\deg P$ for the degree of P and $\deg_i P$ for the degree of the ith column of P. Thus $\deg P$ is the highest degree occurring among the entries of P whereas $\deg_i P$ is the highest degree occurring in the ith column of P. Denote P_H the highest column-degree-coefficient matrix of P, i.e., the constant matrix whose ith column consists of the coefficients of $z^{\deg_i P}$ in the ith column of P. The P is said to be *column reduced* if P_H has rank n. If the column degrees of P are arranged in order of magnitude, that is $\deg_i P \geqq \deg_j P$ for $i < j$, we say that P is *degree ordered.*

An $m \times m$ polynomial matrix D_1 is a left divisor of P if there is a polynomial matrix P_1 such that $P = D_1 P_1$; an $n \times n$ polynomial matrix D_2 is a right divisor of P if there is a polynomial matrix P_2 such that $P = P_2 D_2$. Two polynomial matrices P and Q having the same number of rows (or columns) are said to be *left* (or *right*) *coprime* if every common left divisor of theirs is unimodular, i.e. has a polynomial inverse.

Elementary row operations on a polynomial matrix with elements in $K[z]$ are defined as (i) the interchange of two rows, (ii) the multiplication of a row by a nonzero element from K, and (iii) the addition of any $K[z]$-multiple of one row to another row. The *elementary column operations* are defined in an entirely analogous fashion.

By *invariant factors* of an $n \times n$ constant matrix C we mean the non-unit invariant polynomials of $zI_n - C$, where I_n stands for the $n \times n$ identity matrix. These polynomials are exhibited in the Smith normal form of $zI_n - C$, see [8, 10].

Note that the matrices F and G in (1), while describing the system, do not exhibit its input dynamical indices. Therefore write

$$(zI_n - F)^{-1} G := BA^{-1} \tag{3}$$

where A and B are right coprime polynomial matrices in z, respectively $m \times m$ and $n \times n$, and A is column reduced. Then, as shown by Wolovich [12]

$$\deg_i B < \deg_i A\,, \quad i = 1, 2, \ldots, m\,. \tag{4}$$

If, in addition, the A is degree ordered, then the input dynamical indices of (1) are given simply by

$$v_i = \deg_i A\,, \quad i = 1, 2, \ldots, r\,,$$

as shown in [10] or [4]; the last $m - r$ columns of A have zero degrees.

The reachability of system (1) implies that $zI_n - F$ and G are left coprime matrices, see [10]. On the other hand, the polynomial matrices A and B are right coprime. The Smith-McMillan form [8, 10] of the rational matrices in (3) then tells us that the non-unit invariant polynomials of $zI_n - F$ are the same as those of A.

When feedback is applied according to (2), the system equation becomes

$$z\dot{x} = (F - GL)\,x\,.$$

As a preliminary step the following lemma is established.

Lemma 1. The matrices $zI_n - F + GL$ and $A + LB$ have the same non-unit invariant polynomials.

Proof. Using (3),

$$(zI_n - F)\,B = GA\,.$$

Add GLB to both sides of the above identity and rearrange to give

$$(zI_n - F + GL)^{-1}\,G = B(A + LB)^{-1}\,.$$

Because $zI_n - F$ and G are left coprime (by reachability) and

$$[zI_n + F + GL \quad G] = [zI_n - F \quad G]\begin{bmatrix} I_n & 0 \\ L & I_m \end{bmatrix}$$

it follows that $zI_n - F + GL$ and G are left coprime. On the other hand, A and B are right coprime (by definition) and

$$\begin{bmatrix} B \\ A + LB \end{bmatrix} = \begin{bmatrix} I_n & 0 \\ L & I_m \end{bmatrix} \begin{bmatrix} B \\ A \end{bmatrix}$$

whence $A + LB$ and B are right coprime. Therefore the matrices $zI_n - F + GL$ and $A + LB$ must share the same non-unit invariant polynomials. □

The next lemma was first used by Rosenbrock [10] and it is the key technical result in our development.

Lemma 2. Let an $m \times m$ matrix C over $K[z]$ be given, with $C_H = I_m$. Suppose $\deg_i C < \deg_j C$ for some i and j. Then C can be transformed by elementary operations to C' for which $C'_H = I_m$ and $\deg_i C' = \deg_i C + 1$, $\deg_j C' = \deg_j C - 1$, and $\deg_k C' = \deg_k C$, $k \neq i, j$.

Proof. Add z times row i to row j. This leaves the degree od each column but i and j unchanged, places a monic polynomial of degree $\deg_i C + 1$ in position (j, i) and does not increase the degree of the element in position (j, j). Let α be the coefficient (possibly zero) of $z^{\deg_j C}$ in this element. Put $d = \deg_j C - \deg_i C - 1$ and subtract αz^d times column i from column j. This reduces the degree of column j by one while keeping the matrix column reduced. Normalizing its highest column-degree-coefficient matrix to identity we obtain C'. □

NEW PROOF AND CONSTRUCTION

We now present a new, simple, and constructive proof of the following result, already mentioned in the Introduction.

Theorem. Let F, G in (1) define a reachable system with $\nu_1 \geqq \nu_2 \geqq \ldots \geqq \nu_r$ the ordered input dynamical indices and $r = \operatorname{rank} G$. Further, let $c_1, c_2, \ldots, c_q$ be any monic polynomials in $K[z]$ satisfying the conditions

$$c_{i+1} \mid c_i\,, \quad i = 1, 2, \ldots, q - 1 \quad \text{and} \quad q \leqq r \tag{5}$$

and

$$\sum_{i=1}^{q} \deg c_i = n\,. \tag{6}$$

Then there is a matrix L such that $F - GL$ has the invariant factors $c_1, c_2, \ldots, c_q$ if and only if

$$\sum_{i=1}^{k} \deg c_i \geqq \sum_{i=1}^{k} \nu_i\,, \quad k = 1, 2, \ldots, q\,. \tag{7}$$

Proof. To establish necessity, let a (constant) matrix L exist such that $F - GL$ has the invariant factors $c_1, c_2, \ldots, c_q$. Then, by Lemma 1, these polynomials constitute the non-unit invariant polynomials of the matrix

$$C := A + LB .$$

The inequalities (4) further imply that A and C have the same column degrees. Therefore, assuming A degree ordered,

$$\begin{aligned} \deg_i C &= v_i , \quad i = 1, 2, \ldots, r \\ &= 0 , \quad \text{otherwise} . \end{aligned}$$

Now it is easy to see that inequalities (7) must be satisfied. Indeed, the product $c_{k+1} \ldots c_q$ is the greatest common divisor of all minors of order $m - k$ in C. It follows that

$$\sum_{i=k+1}^{q} \deg c_i \leqq \sum_{j=k+1}^{r} v_j$$

with equality holding for $k = 0$, and hence these inequalities can be reordered to give (7).

Sufficiency is proved by construction. If $c_1, c_2, \ldots, c_q$ are monic polynomials satisfying (7) as well as the intrinsic properties (5) and (6) of invariant factors, form the matrix

$$\bar{C} := \operatorname{diag}\left[c_1, c_2, \ldots, c_q, I_{m-q}\right] .$$

If $\deg_i \bar{C} = v_i$, $i = 1, 2, \ldots, r$ we put $C := A_H \bar{C}$. If there is a column $i \leqq r$ for which $\deg_i \bar{C} < v_i$ there must be a column $j \leqq r$ for which $\deg_j \bar{C} < v_j$, for

$$\sum_{k=1}^{q} \deg c_k = n = \sum_{k=1}^{r} v_k .$$

Then Lemma 2 can be applied, several times if necessary, in order to change the column degrees of $\bar{C}$ so that

$$\begin{aligned} \deg_i \bar{C} &= v_i , \quad i = 1, 2, \ldots, r \\ &= 0 , \quad \text{otherwise} \end{aligned}$$

without changing its invariant polynomials. Finally we put $C := A_H \bar{C}$.

Now let P, Q be polynomial matrices satisfying the equation

$$PA + QB = C . \tag{8}$$

Their existence is guaranteed by the right coprimeness of A and B, see [7]. Perform the division

$$Q = T(zI_n - F) + \bar{Q}$$

to obtain the quotient T and the remainder $\bar{Q}$ satisfying $\deg \bar{Q} < \deg (zI_n - F)$. Hence $\bar{Q}$ is a constant matrix and the pair $\bar{P} := P + TG$, $\bar{Q}$ also satisfies (8). The inequalities (4) then imply that $\bar{P}$ is constant as well; in fact, $\bar{P}A_H = C_H(=A_H)$ entails that $\bar{P}$ is identity.

The required feedback is then simply $L := \bar{Q}$. To see this, observe that $A + LB$ has the desired invariant polynomials by (8) and the claim follows from Lemma 1. □

The sufficiency part of the proof suggests a simple procedure to construct the feedback L. This procedure consists of three major steps, which are summarized below in algorithmic form.

Step 1. Given F and G, calculate matrices A and B as follows. Form the matrix

$$\begin{bmatrix} zI_n - F & -G \\ I_n & 0 \\ 0 & I_m \end{bmatrix}$$

and using elementary column operations, bring it to the form

$$\begin{bmatrix} D_{11} & 0 \\ D_{21} & D_{22} \\ D_{31} & D_{32} \end{bmatrix}$$

where D_{11} is $n \times n$, D_{22} is $n \times m$, and D_{32} is column reduced and degree ordered. Then $B := D_{22}$ and $A := D_{32}$; the B has r nonzero columns and $v_i = \deg_i A$, $i = 1, 2, \ldots, r$. For details consult Sain [11] and Kučera [7].

Step 2. Given polynomials $c_1, c_2, \ldots, c_q$ satisfying the hypotheses of the Theorem, construct a matrix C as follows. Form the matrix

$$\bar{C} := \operatorname{diag}\left[c_1, c_2, \ldots, c_q, I_{m-q}\right].$$

If $\deg_i \bar{C} = v_i$, $i = 1, 2, \ldots, r$ set $C := A_H\bar{C}$. If not, take any column $i \leqq r$ for which $\deg_i C < v_i$ and any column $j \leqq r$ for which $\deg_j C > v_j$. By elementary operations raise the degree of column i by one, lower the degree of column j by one, and preserve the other column degrees; then normalize $\bar{C}_H$ to identity. Repeat the process until $\deg_i \bar{C} = v_i$, $i = 1, 2, \ldots, r$. Finally set $C := A_H\bar{C}$.

Step 3. Given A, B, and C, write the ith column of A in the form

$$A_i = A_{i,0} + A_{i,1}z + \ldots + A_{i,v_i}z^{v_i}$$

and similarly for B and C. Then solve the set of equations

$$L[B_{i,0} \ldots B_{i,v_i-1}] = [C_{i,0} - A_{i,0} \ldots C_{i,v_i-1} - A_{i,v_i-1}], \quad i = 1, 2, \ldots, m$$

for a desired feedback L.

EXAMPLE

To illustrate the construction of feedback, consider the system over the field of reals described by

$$F = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix}, \quad G = \begin{bmatrix} 0 & 1 \\ 0 & 0 \\ 1 & 1 \\ 0 & 0 \end{bmatrix}.$$

The task is to find a matrix L, if one exists, such that $F - GL$ has the invariant factors

$$c_1 = z^3 - z\,, \quad c_2 = z\,.$$

To execute Step 1, form the matrix

$$\left[\begin{array}{cccc|cc} z-1 & 0 & 0 & -1 & 0 & -1 \\ 0 & z & -1 & 0 & 0 & 0 \\ 0 & 0 & z & -1 & -1 & -1 \\ -1 & 0 & 0 & z & 0 & 0 \\ \hline 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{array}\right]$$

and reduce it to the form

$$\left[\begin{array}{cccc|cc} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & -z & 1 & 0 & 0 & 0 \\ 0 & 0 & -z & 1 & 0 & 0 \\ \hline 0 & 0 & 0 & -1 & 0 & z \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & -1 & 0 & 0 & z & z \\ 0 & 0 & -1 & 0 & 0 & 1 \\ \hline 1 & 0 & -1 & z-1 & z^2 & z \\ -1 & 0 & 1 & -z+1 & 0 & z^2-z-1 \end{array}\right]$$

by elementary column operations. Thus

$$A = \begin{bmatrix} z^2 & z \\ 0 & z^2 - z - 1 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & z \\ 1 & 1 \\ z & z \\ 0 & 1 \end{bmatrix}$$

and observe that the system is cyclic with the characteristic polynomial $z^4 - z^3 - z^2$ and has the input dynamical indices $\nu_1 = 2$, $\nu_2 = 2$.

The inequalities (7) amount to

$$\deg c_1 \geqq 2$$

$$\deg c_1 + \deg c_2 \geqq 4$$

and are clearly satisfied. Hence our problem is solvable.

Step 2 is begun by forming the matrix

$$\bar{C} = \begin{bmatrix} z^3 - z^2 & 0 \\ 0 & z \end{bmatrix}.$$

Adding z times row 2 to row 1 and then subtracting z times column 2 from column 1 we obtain

$$\bar{C} = \begin{bmatrix} -z^2 & z^2 \\ -z^2 & z \end{bmatrix}.$$

Normalizing its highest column-degree-coefficient matrix to equal

$$A_H = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

we finally get

$$C = \begin{bmatrix} z^2 & 0 \\ z & z^2 - z \end{bmatrix}.$$

To carry out Step 3 write

$$B_1 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} z, \quad B_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix} z$$

$$C_1 - A_1 = \begin{bmatrix} 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} z, \quad C_2 - A_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix} + \begin{bmatrix} -1 \\ 0 \end{bmatrix} z$$

and solve the equation

$$L\left[\begin{array}{cc|cc} 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{array}\right] = \left[\begin{array}{cc|cc} 0 & 0 & 0 & -1 \\ 0 & 1 & 1 & 0 \end{array}\right]$$

to obtain a desired feedback

$$L = \begin{bmatrix} -1 & 0 & 0 & 0 \\ -1 & 0 & 1 & 1 \end{bmatrix}.$$

DISCUSSION

A new proof has been given of a fundamental theorem by Rosenbrock on changing the algebraic structure of a linear constant system by linear state variable feedback. The proof is based on the algebra of polynomial matrices and provides a simple algorithm to construct the feedback.

It may be worthwhile to note that the feedback which effects the desired change is by no means unique. The degrees of freedom in choosing an appropriate feedback are embodied in Step 2, that is, in the variety of ways in which elementary operations can be performed.

All the results which have been given for the system

$$zx = Fx + Gu$$

translate at once into dual theorems concerning the system

$$y = Hx\,, \quad zx = Fx\,.$$

The easiest way to obtain these is to consider H^T, F^T. Note that the input dynamical indices, which limit the existence of an L such that $F - GL$ has a desired structure, are replaced by the output dynamical indices in the dual problem of changing the structure of $F - LH$.

(Received May 27, 1980.)

REFERENCES

[1] P. Brunovský: A classification of linear controllable systems. Kybernetika *6* (1970), 3, 173—187.

[2] B. W. Dickinson: On the fundamental theorem of linear state variable feedback. IEEE Trans. Autom. Control *AC-19* (1974), 5, 577—579.

[3] D. S. Flamm: A new proof of Rosenbrock's theorem on pole assignment. IEEE Trans. Autom. Control *AC-25* (1980), 6, 1128—1133.

[4] G. D. Forney: Minimal bases of rational vector spaces, with applications to multivariable linear systems. SIAM J. Control *13* (1975), 3, 493—520.

[5] R. E. Kalman: Kronecker invariants and feedback. In: Ordinary Differential Equations (L. Weiss, Ed.). Academic, New York 1972.
[6] R. E. Kalman, P. L. Falb, M. A. Arbib: Topics in Mathematical System Theory. McGraw-Hill, New York 1969.
[7] V. Kučera: Discrete Linear Control — The Polynomial Equation Approach. Wiley, Chichester 1979.
[8] C. C. MacDuffee: The Theory of Matrices. Chelsea, New York 1946.
[9] S. K. Mitter, R. Foulkes: Controllability and pole assignment for discrete time linear systems defined over arbitrary fields. SIAM J. Control *9* (1971), 1, 1—7.
[10] H. H. Rosenbrock: State-space and Multivariable Theory. Wiley, New York 1970.
[11] M. K. Sain: A free-modular algorithm for minimal design of linear multivariable systems. In: Proc. 6th IFAC Congress, Boston 1975.
[12] W. A. Wolovich: Linear Multivariable Systems. Springer, New York 1974.

Ing. Vladimír Kučera, DrSc., Ústav teorie informace a automatizace ČSAV (Institute of Information Theory and Automation — Czechoslovak Academy of Sciences), Pod vodárenskou věží 4, 182 08 Praha 8, Czechoslovakia.