KYBERNETIKA — VOLUME 17 (1981), NUMBER 2

NORMALIZATION OF GENERAL COALITION-GAMES

MILAN MAREŠ

It is shown in the presented paper that the normalization concept used in the classical theory of games is senseful also for some important classes of general coalition-games. The normalization of such games is connected with some difficulties caused by the general form and wide scale of possible types of the general characteristic function of the game. Even if the common concept of normalization applicable for all general coalition-games does not exist, the concepts of normalization suggested here are suitable for many interesting and important types of coalition-games.

0. INTRODUCTION

The normalization concept was introduced in the classical coalition-games theory, especially in the theory of coalition-games with side-payments, in order to create a proper tool for the comparison of different games. It enables to compare the properties and cooperation possibilities of coalition-games with the same (or equivalent) set of players by means of their normalized forms.

The main problem investigated in the presented paper is whether there exists some analogical concept in case of the general coalition-games introduced in [2]. It is not difficult to see that any normalization of a general coalition-game is a transformation of the imputations space. Such transformations were generally investigated in [5] and some of their further properties were introduced in [6]. The main results obtained in those papers imply also the possibilities and properties of the normalization of general coalition-games.

It is shown here that the normalization of the general coalition-games is not generally as simple as the normalization of the classical coalition-games with side payments. Not every general coalition-game can be normalized in senseful way, and even if the game can be normalized, there usually exist more equivalent normalizations of the same game. It means that the normalization of the general coalition-games cannot be generally as important as the normalization of the side-payments coalition-games.

However, there exists a wide class of general coalition-games which can be normalized and for which the normalization is also useful. That class is investigated here, and some of its main properties are introduced in the following sections.

1. GENERAL COALITION-GAME

The concept of the general coalition-game was suggested in [2]. Let us denote by R the set of all real numbers. Let us consider a finite and non-empty set I, and a mapping V of 2^I into the class of subsets of the more dimensional real space R^I such that for any set $K \in 2^I$ the set $V(K) \subset R^I$ fulfils the following conditions:

(1.1) $$V(K) \text{ is closed;}$$

(1.2) $$\text{if } \mathbf{x} = (x_i)_{i \in I} \in V(K), \quad \mathbf{y} = (y_i)_{i \in I} \in R^I \text{ and } x_i \geqq y_i \text{ for all } i \in K \text{ then } \mathbf{y} \in V(K);$$

(1.3) $$V(K) \neq \emptyset, \quad V(K) = R^I \Leftrightarrow K = \emptyset.$$

Then the pair (I, V) is called a *general coalition-game*, or briefly a *game*, elements of the set I are called *players*, and the mapping V is called a *general characteristic function*.

Any set $K \in 2^I$ is called a *coalition*, and any partition of the set I into disjoint subsets is called a *coalition structure*. Any vector $\mathbf{x} \in R^I$ is called an *imputation*.

If $K \in 2^I$ is a coalition then we introduce

(1.4) $$V^*(K) = \{\mathbf{x} = (x_i)_{i \in I} \in R^I : \text{for all } \mathbf{y} = (y_i)_{i \in I} \in V(K) \text{ is either } x_i > y_i \text{ for some } i \in K \text{ or } x_i = y_i \text{ for all } i \in K\},$$

and the mapping V^* of 2^I into the class of subsets of R^I defined by (1.4) will be called a *superoptimum* function of the game (I, V).

If $\mathcal{M} \subset 2^I$ is a class of coalitions then we denote

$$V(\mathcal{M}) = \bigcap_{M \in \mathcal{M}} V(M), \quad V^*(\mathcal{M}) = \bigcap_{M \in \mathcal{M}} V^*(M).$$

It was shown in [2] that for any coalition $K \in 2^I$ is

(1.5) $$V(K) \cup V^*(K) = R^I.$$

An imputation $\mathbf{x} \in R^I$ is called *strongly stable* in the considered game (I, V) iff

(1.6) $$\mathbf{x} \in V(\mathcal{K}) \text{ for some coalition structure } \mathcal{K} \subset 2^I, \text{ and}$$

(1.7) $$\mathbf{x} \in V^*(K) \text{ for all coalitions } K \in 2^I.$$

A coalition structure $\mathscr{K} \subset 2^I$ is called strongly stable in the game (I, V) iff there exists a strongly stable imputation $\mathbf{x} \in R^I$ such that $\mathbf{x} \in V(\mathscr{K})$.

It is useful to introduce here the following statements and concepts.

Lemma 1. Let (I, V) and (I, W) be general coalition-games with the same set of players, and let $\mathbf{x} \in R^I$ be an imputation such that $\mathbf{x} \in V(\mathscr{K})$ for some coalition structure $\mathscr{K}$. Let us suppose that for every coalition $K \in 2^I$ is $V(K) \subset W(K)$. If $\mathbf{x}$ is strongly stable imputation in the game (I, W) then it is strongly stable even in (I, V).

Proof. It follows from the definitions of the general characteristic function V and superoptimum function V^* that for any coalition $K \in 2^I$ is

$$V(K) \cup V^*(K) = R^I = W(K) \cup W^*(K)\,. \tag{1.8}$$

Consequently, the assumption $V(K) \subset W(K)$ implies that $V^*(K) \supset W^*(K)$ for any $K \in 2^I$. If $\mathbf{x} \in R^I$ is a strongly stable imputation in (I, W) then $\mathbf{x} \in W^*(K)$ for all $K \in 2^I$, and by (1.8) is also $\mathbf{x} \in V^*(K)$ for all $K \in 2^I$. If $\mathbf{x} \in V(\mathscr{K})$ for some coalition structure $\mathscr{K}$ then $\mathbf{x}$ is strongly stable in the game (I, V), and the statement is proved. □

Let (I, V) be a general coalition-game with a general characteristic function V. The game (I, V) is called *superadditive* iff for any pair of disjoint coalitions $K, L \in 2^I$, $K \cap L = \emptyset$, is

$$V(K \cup L) \subset V(K) \cap V(L)\,. \tag{1.9}$$

The game (I, V) is called *subadditive* iff for any pair of disjoint coalitions $K, L \in 2^I$, $K \cap L = \emptyset$, is

$$V^*(K \cup L) \supset V^*(K) \cap V^*(L)\,. \tag{1.10}$$

The game (I, V) is called *additive* iff it is superadditive and subadditive.

The general characteristic function V is called *convex* iff all sets $V(K)$, $K \in 2^I$, are convex*). It is called *concave* iff the sets $V^*(K)$ are convex for all $K \in 2^I$, and it is called *linear* iff it is convex and concave.

2. TRANSFORMATIONS OF GAMES

The normalization concept is based on the idea of a proper transformation of the considered game into another one with equivalent properties. Such transformations of games were investigated in [5] and [6].

The concepts and results presented in [5] which are frequently referred in the

*) This convexity of the general characteristic functions is not equivalent to the convexity of the side-payments coalition-games known from the literature, e.g. from [8] or [7].

following sections will be briefly mentioned here. However, also some other, less important, concepts from [5] will be referred in other sections of this paper, if necessary.

Let us consider a set of players I and a *one-to-one transformation* T *of* R^I *onto* R^I. If $M \subset R^I$ is a set of imputations then we denote

$$(2.1) \qquad TM = \{\mathbf{x} \in R^I : \exists \mathbf{y} \in M \text{ such that } \mathbf{x} = T\mathbf{y}\}\,.$$

If $\mathbf{x} = (x_i)_{i\in I}$ then we denote by $(T\mathbf{x})_i$ the elements of the transformed vector $T\mathbf{x}$. The transformation T is called a *game-preserving transformation* iff for any general coalition-game (I, V) the pair (I, TV), where for all $K \in 2^I$ the set $T\,V(K)$ is defined from $V(K)$ by (2.1), is also a general coalition-game. The transformation T is called a *coordinatewise strictly increasing* transformation iff for any pair of imputations $\mathbf{x}, \mathbf{y} \in R^I$ and any $i \in I$ the inequality $x_i > y_i$ implies $(T\mathbf{x})_i > (T\mathbf{y})_i$.

It was proved in [5], Theorem 1, that a one-to-one transformation of R^I onto R^I is a game-preserving one if and only if it is coordinatewise strictly increasing.

Two general coalition-games (I, V) and (I, W) with the same set of players are called *equivalent* iff there exists a game-preserving one-to-one transformation T of R^I onto R^I such that

$$(2.2) \qquad W(K) = T\,V(K) \quad \text{for all} \quad K \in 2^I.$$

The relation between the strong solutions of equivalent general coalition-games was investigated in [5], where the following results were derived. Let (I, V) be a general coalition-game and let T be a game-preserving one-to-one transformation of R^I onto R^I. Then $\mathbf{x} \in R^I$ is strongly stable in the game (I, V) if and only if $T\mathbf{x}$ is strongly stable in (I, TV) ([5], Theorem 3). A coalition structure $\mathscr{K} \subset 2^I$ is strongly stable in (I, V) if and only if it is strongly stable in (I, TV), ([5], Theorem 4).

Let $\boldsymbol{a} = (a_0, (a_i)_{i\in I}) \in R \times R^I$ be a real-valued vector, and let $a_i > 0$ for $i \in I$. Then we denote the hyperplane

$$(2.3) \qquad H_{\boldsymbol{a}} = \{\mathbf{x} \in R^I : \sum_{i\in I} x_i a_i = a_0\}\,,$$

and the halfspaces

$$(2.4) \qquad H_{\boldsymbol{a}}^+ = \{\mathbf{x} \in R^I : \sum_{i\in I} x_i a_i \geqq a_0\}\,,$$

$$H_{\boldsymbol{a}}^- = \{\mathbf{x} \in R^I : \sum_{i\in I} x_i a_i \leqq a_0\}\,.$$

A general coalition-game (I, V) is called *constrained* iff for every coalition $K \in 2^I$, every imputation $\mathbf{x} \in V(K)$ and every player $i \in K$ there exists an imputation $\mathbf{y} \in R^I - V(K)$ such that $y_j = x_j$ for all $j \in I$, $j \neq i$, and $y_i > x_i$. It was shown in [7], (hat if (I, V) is a constrained game and T is a game-preserving transformation then tI, $TV)$ is also constrained.

A one-to-one transformation T of R^I onto R^I is called *coordinatewise decomposable* iff there exist transformations T_i, $i \in I$, of R onto R such that for any $\mathbf{x} \in R^I$ and any $i \in I$ is $T_i x_i = (T\mathbf{x})_i$. A coordinatewise decomposable transformation T of R^I onto is called ***convex*** iff all partial transformations T_i, $i \in I$, are convex. Analogously, T is called *linear* iff all transformations T_i are linear, and T is called *concave* iff all transformations T_i are concave.

It was proved in [5], Theorem 2, that any game-preserving transformation is coordinatewise decomposable. It was shown in [6] that if a general characteristic function V is convex and a game-preserving transformation T is concave then the general characteristic function TV is convex; if V is concave and T is convex then the general characteristic function TV is concave; if both, V and T are linear then TV is also linear (see [6]).

3. 0-NORMALIZATION

It is possible to consider more kinds of normalization of general coalition-games. One of the possibilities is to normalize the pay-off (the values of the general characteristic function) of the minimal, i.e. one-element, coalitions.

Definition 1. Let (I, V) and (I, W) be general coalition-games, and let for all $i \in I$ be

$$W(\{i\}) = \{\mathbf{x} \in R^I : x_i \leqq 0\} .$$

Then the game (I, W) is called a 0-*normalization* of the game (I, V) iff there exists a game-preserving one-to-one transformation T of R^I onto R^I such that $W(K) = T\,V(K)$ for all $K \in 2^I$. The game (I, W) is called a 0-*normalized* game.

The existence of the 0-normalizations of general coalition-games is proved by the following theorem.

Theorem 1. If (I, V) is a general coalition-game then there always exists its 0-normalization.

Proof. Condition (1.2) implies that for any general coalition-game (I, V) and any player $i \in I$ there exists a real number a_i such that

$$V(\{i\}) = \{\mathbf{x} \in R^I : x_i \leqq a_i\} .$$

Let us define a one-to-one transformation T of R^I onto R^I such that for all $\mathbf{x} = (x_i)_{i \in I}$ is $(T\mathbf{x})_i = x_i - a_i$ for any $i \in I$. Then for all $i \in I$ is

$$\begin{aligned} TV(\{i\}) &= \{\mathbf{y} \in R^I : \exists \mathbf{x} \in V(\{i\}), \mathbf{y} = T\mathbf{x}\} = \\ &= \{\mathbf{y} \in R^I : \exists \mathbf{x} \in V(\{i\}), x_i \leqq a_i, y_i = x_i - a_i\} = \\ &= \{\mathbf{y} \in R^I : y_i + a_i \leqq a_i\} = \{\mathbf{y} \in R^I : y_i \leqq 0\} . \end{aligned}$$

The transformation T is a game-preserving transformation, as it is coordinatewise strictly increasing. It means that the game (I, TV) is a 0-normalization of the game (I, V). □

4. 1-NORMALIZATION

Another of the generally possible attitudes to the normalization concept is to normalize the pay-offs (the values of the general characteristic function) of the largest possible coalition, i.e. of the coalition of all players.

Definition 2. Let (I, V) and (I, W) be general coalition-games, let for any coalition structure $\mathscr{K}$ be $W(\mathscr{K}) \subset \{\mathbf{x} \in R^I : \sum_{i \in I} x_i \leqq 1\}$, and let there exists al least one coalition structure $\mathscr{L} \subset 2^I$ such that

$$W(\mathscr{L}) \cap \{\mathbf{x} \in R^I : \sum_{i \in I} x_i = 1\} \neq \emptyset .$$

Then the game (I, W) is called a 1-*normalization* of the game (I, V) iff there exists a one-to-one game-preserving transformation T of R^I onto R^I such that $W(K) = T V(K)$ for all $K \in 2^I$. The game (I, W) is called a 1-*normalized* game.

The existence of the 1-normalization for a wide class of general coalition-games is proved by the following statements.

Theorem 2. Let (I, V) be a general coalition-game, and let there exists a coalition structure $\mathscr{K} \subset 2^I$ such that $V(\mathscr{K})$ is a convex subset of R^I, and for any coalition structure $\mathscr{L} \subset 2^I$ is $V(\mathscr{K}) \subset V(\mathscr{L})$. Then there exists a 1-normalization of the game (I, V).

Proof. The convexity of the set $V(\mathscr{K})$ for some coalition structure $\mathscr{K}$ means that there exists a real-valued vector $(b_0, (b_i)_{i \in I}) \in R \times R^I$, and a hyperplane

$$H_{\mathbf{b}} = \{\mathbf{x} \in R^I : \sum_{i \in I} x_i b_i = b_0\} ,$$

such that $b_i > 0$ for all $i \in I$,

$$V(\mathscr{K}) \subset \{\mathbf{x} \in R^I : \sum_{i \in I} x_i b_i \leqq b_0\} ,$$

and $V(\mathscr{K}) \cap H_{\mathbf{b}} \neq \emptyset$ (c.f. [6], Lemma 2). If $b_0 > 0$ then we may consider a transformation T of R^I onto R^I such that for every $\mathbf{x} \in R^I$ and every $i \in I$ is $(T\mathbf{x})_i = x_i/b_0$. It is obvious that T is coordinatewise strictly increasing one-to-one transformation of R^I onto R^I, and that

$$T V(\mathscr{K}) \subset \{\mathbf{y} \in R^I : \sum_{i \in I} y_i \leqq 1\} \quad \text{and} \quad T V(\mathscr{K}) \cap TH_{\mathbf{b}} \neq \emptyset ,$$

where

$$TH_{\boldsymbol{b}} = \{\mathbf{y} \in R^I : \exists \mathbf{x} \in H_{\boldsymbol{b}}, \mathbf{y} = T\mathbf{x}\} = \{\mathbf{y} \in R^I : \sum_{i \in I} y_i = 1\} .$$

If $b_0 \leqq 0$ then it is possible to construct a vector $\boldsymbol{a} \in R^I$ and a transformation U of R^I onto R^I such that for every $\mathbf{x} \in R^I$ and every $i \in I$ is $a_i < b_0/n$, where n is the number of elements of the set I, and $(U\mathbf{x})_i = x_i - a_i$. Then U is a coordinatewise strictly increasing one-to-one transformation of R^I onto R^I, and

$$U\ V(\mathscr{K}) \subset \{\mathbf{y} \in R^I : \sum_{i \in I} y_i \leqq c_0\} , \quad U\ V(\mathscr{K}) \cap UH_{\boldsymbol{b}} \neq \emptyset ,$$

$$UH_{\boldsymbol{b}} = \{\mathbf{y} \in R^I : \exists \mathbf{x} \in H_{\boldsymbol{b}}, \mathbf{y} = U\mathbf{x}\} = \{\mathbf{y} \in R^I : \sum_{i \in I} y_i = c_0\} ,$$

where

$$c_0 = b_0 - \sum_{i \in I} a_i > b_0 - (nb_0)/n = 0 .$$

Then the transformation T described above may be applied to the game (I, UV). It was shown in [6] that a composition of two game-preserving transformations is also a game-preserving transformation. Consequently, the pair (I, TUV) where for any $K \in 2^I$ is

$$TU\ V(K) = T(U\ V(K)) = \{\mathbf{x} \in \mathrm{R}^I : \exists \mathbf{y} \in V(K),\ \mathbf{x} = T\mathbf{z}, \mathbf{z} = U\mathbf{y}\}$$

is also a general coalition-game. It means that in both cases there was constructed a one-to-one game-preserving transformation of R^I onto R^I such that the transformed game fulfils the first condition of the 1-normalization. It was proved in [6] that for any coalition structure $\mathscr{L}$ such that $V(\mathscr{L}) \subset V(\mathscr{K})$ is

$$T\,V(\mathscr{L}) \subset T\,V(\mathscr{K}) \subset TH_{\boldsymbol{b}} \quad \text{or} \quad TU\ V(\mathscr{L}) \subset TU\ V(\mathscr{K}) \subset TUH_{\boldsymbol{b}}$$

in the first or second considered case, respectively. Consequently, the transformed game (I, TV) or (I, TUV) is a 1-normalization of the game (I, V). □

Corollary. If (I, V) is a constrained convex and superadditive general coalition-game then there exists its 1-normalization.

Theorem 3. If a general coalition-game (I, V) is subadditive then there always exists its 1-normalization.

Proof. If (I, V) is a subadditive game then the assumptions of Theorem 2 are fulfilled. The coalition structure $\mathscr{K}$ fulfilling the assumptions of that theorem is the coalition structure of exactly all one-element coalitions. □

It is not difficult to see that the definitions of the 0-normalization and 1-normalization may be easily generalized. Let $\boldsymbol{a} = (a_i)_{i \in I} \in R^I$ be a real-valued vector, and let (I, V) and (I, W) be equivalent general coalition-games. Let us denote by T the game-preserving one-to-one transformation of R^I onto R^I for which $W(K) = T\,V(K)$ for all $K \in 2^I$. Then the game (I, W) could be called an $\boldsymbol{a}$-normalization from below of (I, V) iff for all $i \in I$ is

$$W(\{i\}) = \{\mathbf{x} \in R^I : x_i \leqq a_i\} .$$

Let us consider a hyperplane

$$H_{\boldsymbol{b}} = \{\mathbf{x} \in R^I : \sum_{i \in I} x_i b_i = b_0\}$$

for some real-valued vector $\boldsymbol{b} = (b_0, (b_i)_{i \in I}) \in R \times R^I$ such that $b_i > 0$ for all $i \in I$. Then the game (I, W) could be called a $\boldsymbol{b}$-normalization from above of the game (I, V) iff for all coalition structures $\mathscr{K} \subset 2^I$ is $W(\mathscr{K}) \subset H_{\boldsymbol{b}}^-$, and for at least one coalition structure $\mathscr{L} \subset 2^I$ is $W(\mathscr{L}) \cap H_{\boldsymbol{b}} \neq \emptyset$.

Such generalization of Definitions 1 and 2 would be a formal one, only. It can be easily shown that if a general coalition-game is an $\boldsymbol{a}$-normalization form below of a game (I, V) for some $\boldsymbol{a} \in R^I$ then there exists another coalition-game which is a 0-normalization of (I, V). Analogously, for any $\boldsymbol{b}$-normalization from above of a game (I, V) there exists another equivalent game which is a 1-normalization of (I, V).

5. (0,1)-NORMALIZATION

The most desirable kind of normalization of the general coalition-game is the one normalizing in certain sense all pay-offs in both senses – "from above" and "from below", i.e. normalizing the payments of the smallest one-player coalitions and the largest all-players coalition. This normalization is senseful especially for the games which are superadditive, subadditive or additive in which the pay-offs of all coalitions are really limited by the pay-offs of the smallest and largest ones.

The (0,1)-normalization introduced in this section is a natural combination of the previous two kinds of normalization introduced in Sections 3 and 4.

Definition 3. Let (I, V) and (I, W) be general coalition-games. Then the game (I, W) is called a (0,1)-*normalization* of the game (I, V) iff it is both, the 0-normalization and the 1-normalization of (I, V). The game (I, W) is then called also a (0,1)-*normalized* game.

It can be easily seen that for most of the general coalition-games which can be 0-normalized, 1-normalized or (0,1)-normalized there exist more game-preserving one-to-one transformations of R^I onto R^I fulfilling the conditions of the respective type of normalization. It means that there exist more games which are the 0-normalization, 1-normalization or the (0,1)-normalization of a given game. Results proved in [5] and mentioned also in Section 2 of this paper imply that all such normalizations are equivalent in the sense of [5] and that there exists a strong correspondence between their solutions.

Remark 1. If (I, V) is a general coalition-game, and if $(I, \boldsymbol{W}_0)$, $(I, \boldsymbol{W}_1)$ and $(I, \boldsymbol{W}_{(0,1)})$ are its 0-normalization, 1-normalization and (0,1)-normalization, respectively, then all the games are equivalent, and $(I, \boldsymbol{W}_0)$ is a 0-normalization of $(I, \boldsymbol{W}_1)$ and $(I, \boldsymbol{W}_{(0,1)})$, $(I, \boldsymbol{W}_1)$ is a 1-normalization of $(I, \boldsymbol{W}_0)$ and $(I, \boldsymbol{W}_{(0,1)})$, and $(I, \boldsymbol{W}_{(0,1)})$ is a (0,1)-normalization of $(I, \boldsymbol{W}_0)$, and $(I, \boldsymbol{W}_1)$, too.

The existence of the (0,1)-normalization of a game (I, V) depends especially on the existence of its 1-normalization, as follows immediately from Theorem 1. The results formulated in Theorems 2 and 3 can be completed by the following statements interesting especially for the case of the (0,1)-normalization.

Theorem 4. Let (I, V) be a general coalition-game, let there exists its 1-normalization $(I, \boldsymbol{W})$, and let for $\mathbf{x} \in R^I$ be

$$\mathbf{x} \in \bigcap_{i \in I} \boldsymbol{W}(\{i\}) \Rightarrow \sum_{i \in I} x_i < 1 .$$

Then there exists a (0,1)-normalization of (I, V).

Proof. If $(I, \boldsymbol{W})$ is a 1-normalization of (I, V) then there exists a hyperplane

$$H = \{\mathbf{x} \in R^I : \sum_{i \in I} x_i = 1\}$$

such that, according to the assumptions of this statement,

$$(5.1) \qquad \bigcap_{i \in I} \boldsymbol{W}(\{i\}) \cap H = \emptyset \quad \text{and} \quad \bigcap_{i \in I} \boldsymbol{W}(\{i\}) \subset \{\mathbf{x} \in R^I : \sum_{i \in I} x_i < 1\} .$$

Let us denote by $\boldsymbol{a} = (a_i)_{i \in I}$ the real-valued vector for which

$$\boldsymbol{W}(\{i\}) = \{\mathbf{x} \in R^I : x_i \leqq a_i\} \quad \text{for all} \quad i \in I .$$

Then (5.1) and the assumptions of this theorem imply that

$$(5.2) \qquad \sum_{i \in I} a_i < 1 , \quad \text{i.e.} \quad \boldsymbol{a} \notin H .$$

Let us define a mapping S of R^I onto R^I such that for all $\mathbf{x} \in R^I$ and all $i \in I$ is $(S\mathbf{x})_i = x_i - a_i$. The mapping S is a game-preserving mapping, as it is coordinatewise strictly incteasing, and consequently, the pair $(I, S\boldsymbol{W})$ is a general coalition-game. Moreover, $(I, S\boldsymbol{W})$ is a 0-normalization of $(I, \boldsymbol{W})$ and of (I, V). Let us denote by SH the hyperplane

$$SH = \{\mathbf{y} \in R^I : \exists \mathbf{x} \in H,\ \mathbf{y} = S\mathbf{x}\} = \{\mathbf{y} \in R^I : \sum_{i \in I} y_i < 1 - \sum_{i \in I} a_i\} ,$$

where

$$1 - \sum_{i \in I} a_i \neq 0 \,,$$

as follows from (5.2). Then there exists a mapping U of R^I onto R^I such that for all $\mathbf{y} \in R^I$ and all $i \in I$ is

$$(T\mathbf{y})_i = y_i/(1 - \sum_{i \in I} a_i)\,.$$

It is not difficult to verify that U is a one-to-one and game-preserving mapping of R^I onto R^I such that the game (I, USW) is a 0-normalization and 1-normalization of (I, W). If we denote by T the game-preserving transformation for which $W(K) = T\,V(K)$ for all $K \in 2^I$ then Theorem 5 from [5] implies that the composed mapping UST of R^I onto R^I is also a game-preserving one-to-one mapping, and the game $(I, USW) = (I, USTV)$ is a (0,1)-normalization of the game (I, V), and also of the games (I, W) and (I, SW). □

Corollary. If the game (I, V) is superadditive but not additive, and if there exists its 1-normalization then there exists its (0,1)-normalization as follows from the previous theorem.

Remark 2. It can be easily seen that no subadditive general coalition-game can be (0,1)-normalized.

The (0,1)-normalization of any general coalition-game is equivalent to the original game. It means that the general properties of the game-preserving transformations introduced in [5] and [6] can be applied, and that the normalization can be chosen in such a way that it preserves some important properties of the transformed games.

Theorem 5. Let (I, V) be a general coalition-game, and let there exists a (0.1)-normalization of (I, V), Then the general characteristic function V is convex, concave or linear if and only if there exists a (0,1)-normalization (I, W) of (I, V) such that the general characteristic function W is convex, concave or linear, respectively. The game (I, V) is constrained if and only if there exists its (0,1)-normalization which is constrained.

Proof. If there exists a (0,1)-normalization of the game (I, V) then the game-preserving transformation transforming (I, V) into the (0,1)-normalized game is is always coordinatewise decomposable, as follows from [5], Theorem 2. Then there exists a transformation T of R^I onto R^I transforming (I, V) into a (0,1)-normalized game (I, W) such that T is a combination of a few linear transformations, namely the shift transformation S such that for any $\mathbf{x} \in R^I$ and any $i \in I$ is

$$(S\mathbf{x})_i = x_i + a_i\,, \quad \text{where} \quad a_i \in R\,,$$

and the scale transformation U such that for any $\mathbf{x} \in R^I$ and any $i \in I$ is

$$(U\mathbf{x})_i = b_i x_i\,, \quad \text{where} \quad b_i \in R\,, \quad b_i > 0\,.$$

Both of these transformations are linear. It means that their combination is also a linear, coordinatewise increasing transformation, and consequently, the pair (I, W) is a general coalition-game and, moreover, theorems from [6] imply the first part if the statement. As the game-preserving transformation of (I, V) onto its (0,1)-normalization is coordinatewise strictly increasing, the constrainedness of the transformed game is equivalent to the same property of the original game (I, V), as follows from [6]. □

Remark 3. An analogous theorem can be easily proved even for the 0-normalization and 1-normalization of a game (I, V).

If a game with linear characteristic function is considered then the following statement describing the properties of the maximal coalition in superadditive games holds.

Theorem 6. If (I, V) is a superadditive general coalition-game with linear general characteristic function V and 1-normalized, then

$$V(I) = \{\mathbf{x} \in R^I : \sum_{i \in I} x_i \leqq 1\}\,. \tag{5.3}$$

Proof. Since (I, V) is 1-normalized, then for all coalition structures $\mathscr{K}$ and all imputations $\mathbf{x} \in V(\mathscr{K})$ is

$$\sum_{i \in I} x_i \leqq 1\,,$$

and there exists a coalition-structure $\mathscr{L}$ and an imputation $\mathbf{y} \in V(\mathscr{L})$ such that

$$\sum_{i \in I} y_i = 1\,.$$

The superadditivity of (I, V) implies that the coalition structure $\mathscr{L}$ fulfilling the preceding condition is the coalition structure $\{I\}$ formed exactly by the all-players coalition I. The linearity of the considered coalition-game (I, V) means that both sets $V(I)$ and $V^*(I)$ are convex subsets of R^I. Then (1.5) implies the validity of (5.3). □

It was already mentioned above that any general coalition-game and its 0-normalization, 1-normalization and (0,1)-normalization are equivalent in the sense defined in [5] and introduced also in Section 2. It means that there also exists a strong correspondence between their solutions. This correspondence was proved in [5], Theorems 3 and 4, and it was briefly mentioned also in Section 2. The following results concern some more special properties of the strong solutions of the normalized games which can be used, according to the equivalence property, even for the investigation of the properties of some further general coalition-games.

Lemma 2. Let (I, V) be a 1-normalized superadditive general coalition-game such that

$$V(I) = \{\mathbf{x} \in R^I : \sum_{i \in I} x_i \leqq 1\}.$$

If $\mathbf{x} \in R^I$ is a strongly stable imputation in the game (I, V) then $\sum_{i \in I} x_i = 1$.

Proof. If $\mathbf{x}$ is strongly stable in (I, V) then, by definition, there exists a coalition-structure $\mathscr{K}$ such that $\mathbf{x} \in V(\mathscr{K})$, and $\mathbf{x} \in V^*(K)$ for all $K \in 2^I$. Hence, $\mathbf{x} \in V^*(I)$, too. and also $\mathbf{x} \in V(I) \supset V(\mathscr{K})$. As

$$V(I) = \{\mathbf{x} \in R^I : \sum_{i \in I} x_i \leqq 1\}, \quad \text{and}$$

$$V^*(I) = \{\mathbf{x} \in R^I : \sum_{i \in I} x_i \geqq 1\},$$

the inequality $\sum_{i \in I} x_i = 1$ is necessarily true. □

Theorem 7. Let (I, V) and (I, W) be $(0,1)$-normalized superadditive general coalition-games such that

$$V(I) = W(I) = \{\mathbf{x} \in R^I : \sum_{i \in I} x_i \leqq 1\},$$

and such that $V(K) \subset W(K)$ for all coalitions $K \in 2^I$. Let $\mathbf{x} \in R^I$ be an imputation. If $\mathbf{x}$ is strongly stable in the game (I, W) then it is strongly stable also in (I, V).

Proof. This theorem follows immediately from Lemma 1, where the coalition structure $\mathscr{K}$ for which $\mathbf{x} \in V(\mathscr{K})$ is $\mathscr{K} = \{I\}$, as follows from Lemma 2. □

6. CONCLUSIONS

The general coalition games include a relatively rich class of more special coalition-games, and their normalization is not as easy as the normalization of the classical coalition-games with side-payments. For some special cases the normalization does not exist or is not natural for their structure. However, it exists and it is senseful for some interesting and important subclasses of the considered class of games.

Some properties of the normalizations of general coalition-games were introduced in the previous sections. Some other properties may be derived in case of necessity for some more limited and more specialized games. It concerns, especially, the propeties of the strong solutions of the normalized games which especially depend on the actual form of the general characteristic function of the considered game.

(Received May 12, 1980.)

REFERENCES

[1] R. D. Luce, H. Raiffa: Games and Decisions. Introduction and Critical Survey. J. Wiley and Sons, New York 1957.

[2] M. Mareš: General coalition-games. Kybernetika *14* (1978), 4, 245—260.

[3] M. Mareš: Dynamic solution of general coalition-games. Kybernetika *14* (1978), 4, 261—284.

[4] M. Mareš: Additivity in general coalition-games. Kybernetika *14* (1978), 5, 350—368.

[5] M. Mareš: Transformations of general coalition-games. Problems of Control and Information Theory *9* (1980), 2, 103—110.

[6] M. Mareš: Combinations and transformations of some general coalition-games. Kybernetika *17* (1981), 1, 45—61.

[7] J. Rosenmüller: Kooperative Spiele und Märkte. Springer-Verlag, Heidelberg—Berlin—New York 1971.

[8] J. Rosenmüller: Extreme Games and Their Solutions. Springer-Verlag, Heidelberg—Berlin—New York 1977.

RNDr. Milan Mareš, CSc., Ústav teorie informace a automatizace ČSAV (Institute of Information Theory and Automation — Czechoslovak Academy of Sciences), Pod vodárenskou věží 4, 182 08 Praha 8. Czechoslovakia.