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COMBINATIONS AND TRANSFORMATIONS OF SOME GENERAL COALITION-GAMES

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The general coalition-game model suggested in [2] is considered, and a few types of combinations and transformations of such games are investigated. It is shown that a conjunction, disjunction, and combination of two general coalition-games is, under natural conditions, also a general coalition-game. Also the main relations between the solutions of the original games and the resulting one are derived. The general conditions under which a transformation of the imputation space preserves the existence of general coalition-games were presented in [5]. It is shown in this paper that the same conditions are sufficient for the preservation of some important special properties of the games like their superadditivity, subadditivity and additivity. Also the preservation of the convexity, concavity or linearity of the sets of available imputations is connected with the same conditions.

0. INTRODUCTION

The concept of the general coalition-game was introduced in [2]. There are two kinds of problems connected with those games investigated in this paper. The first one concerns the properties of combinations of general coalition-games. Let us suppose that the same set of players plays two different coalition-games, and that the sets of achievable imputations can be somehow combined (intersected, unified, convexly combined). Such situation may appear in some applications, for example in situations in which players optimize their decisions according to two different criteria or when they should respect different kinds of constrains and interests (e.g. the theoretical budget set and the really existing quantities of goods in a market), The achieved result is valued from the points-of-view of the partial games, and the final evaluation of the result is a combination of the partial ones. It is shown in this paper that the combination of general coalition-games is also a general coalition-game, and the main results describe the main relation between the strongly stable solutions of the original games and their union, intersection and also combination.

The second group of problems investigated in this paper concerns the transformations of a general coalition-game into another one. The transformations, generally investigated in [5], are realized by means of one-to-one transformations of the imputations space onto itself. It is shown here that the game-preserving transformations of the imputations space preserve also some important special properties of the games like their superadditivity, subadditivity and additivity, and under further natural conditions also the convexity, concavity and linearity of the sets of achievable imputations.

1. GENERAL COALITION-GAME

The concept of the general coalition-game was defined in [2]. Let us denote by R the set of all real numbers. Let us consider a finite and non-empty set I, and a mapping V of 2^I into the class of subsets of the space R^I, such that for any set $K \in 2^I$ the set $V(K) \subset R^I$ fulfils the following conditions.

(1.1) $V(K)$ *is closed*;

(1.2) *if* $\mathbf{x} = (x_i)_{i \in I} \in V(K)$, $\mathbf{y} = (y_i)_{i \in I} \in R^I$ *and* $x_i \geqq y_i$ *for all* $i \in K$ *then* $\mathbf{y} \in V(K)$;

(1.3) $V(K) \neq 0$; $V(K) = R^I \Leftrightarrow K = \emptyset$.

Then the pair (I, V) is called a *general coalition-game*, or briefly a *game*, elements of the set I are called *players*, and the mapping V is called a *general characteristic function*.

Any set $K \in 2^I$ is called a *coalition*, and any partition of the set I into disjoint coalitions is called a ***coalition structure***. If $K \in 2^I$ is a coalition then we denote the set

(1.4) $V^*(K) = \{\mathbf{x} = (x_i)_{i \in I} \in R^I$: *for all* $\mathbf{y} = (y_i)_{i \in I} \in V(K)$ *is either* $x_i > y_i$ *for some* $i \in K$ *or* $x_i = y_i$ *for all* $i \in K\}$,

and the mapping V^* of 2^I into the class of subsets of R^I defined by (1.4) will be called a *superoptimum* function of the game (I, V).

If $\mathcal{M} \subset 2^I$ is a class of coalitions then we denote

$$V(\mathcal{M}) = \bigcap_{M \in \mathcal{M}} V(M), \quad V^*(\mathcal{M}) = \bigcap_{M \in \mathcal{M}} V^*(M).$$

It was shown in [2] that for any coalition $K \in 2^I$ is

(1.5) $$V(K) \cup V^*(K) = R^I.$$

The vectors from R^I are called *imputations*. An imputation $\mathbf{x} \in R^I$ is called *strongly stable* in the considered game (I, V) iff

(1.6) $$\mathbf{x} \in V(\mathcal{K}) \text{ for some coalition structure } \mathcal{K} \subset 2^I, \text{ and}$$

(1.7) $$\mathbf{x} \in V^*(K) \text{ for all coalitions } K \in 2^I.$$

A coalition structure $\mathcal{K} \subset 2^I$ is called strongly stable iff there exists a strongly stable imputation $\mathbf{x} \in R^I$ such that $\mathbf{x} \in V(\mathcal{K})$.

2. AUXILIARY CONCEPTS

If $X \subset R^I$ is a set of imputations then we denote by ∂X the *boundary* of X, i.e. the set of exactly those imputations $\mathbf{x} \in R^I$ such that any open neighbourhood of $\mathbf{x}$ contains elements of both X and $R - X$. The imputations from the set $X - \partial X$ are the *inner points* of X.

If $K \in 2^I$ is a coalition, $V(K) \subset R^I$ is the corresponding value of the general characteristic function, and if $K \neq \emptyset$ then always $\partial V(K) \subset V(K)$ as follows from (1.1), (1.2) and (1.3). It is not difficult to verify the following statements.

Remark 1. If V is a general characteristic function and $K \in 2^I$ is a coalition then for any $\mathbf{x} \in V(K) - V^*(K)$ there exists $\mathbf{y} \in \partial V(K)$ such that $y_i \geqq x_i$ for all $i \in K$.

Lemma 1. If $K \in 2^I$ is a coalition and $\mathbf{x} \in R^I$ is an inner point of $V(K)$ then there exists $\mathbf{y} \in \partial V(K)$ such that $y_i > x_i$ for all $i \in K$.

Proof. If $\mathbf{x} \in V(K) - \partial V(K)$ then there exists a neighbourhood $u(\mathbf{x})$ of $\mathbf{x}$ such that $u(\mathbf{x}) \subset V(K)$. As $u(\mathbf{x})$ is an open set, there exists $\mathbf{z} \in u(\mathbf{x})$ such that $z_i > x_i$ for all $i \in K$, and Remark 1 implies that there exists $\mathbf{y} \in \partial V(K)$ such that $y_i \geqq z_i$ for all $i \in K$. □

We say that the general characteristic function V of a game (I, V) is *untruncated* iff for all coalitions $K \in 2^I$ is

$$V(K) \cap V^*(K) = \partial V(K) . \tag{2.1}$$

In the opposite case, the general characteristic function V is said to be *truncated*.

Lemma 2. If V is an untruncated general characteristic function and $\mathbf{x} \in V(K) - V^*(K)$ then there exists $\mathbf{y} \in V(K) \cap V^*(K)$ such that $y_i > x_i$ for all $i \in K$.

Proof. If V is untruncated and $\mathbf{x} \in V(K) - V^*(K)$ then $\mathbf{x} \in V(K) - \partial V(K)$, and $\mathbf{x}$ is an inner poit of the set $V(K)$. Lemma 1 implies this statement immediately. □

Let us consider a real-valued vector $\mathbf{a} = (a_0, (a_i)_{i \in I})$ where at least one a_i, $i \in I$, is different from zero. Then we denote by $H_{\mathbf{a}}$ the hyperplane in R^I defined by

$$H_{\mathbf{a}} = \{\mathbf{x} \in R^I : \sum_{i \in I} a_i x_i = a_0\} . \tag{2.2}$$

The half-spaces defined by the hyperplane $H_{\mathbf{a}}$ will be denoted by

$$H_{\mathbf{a}}^+ = \{\mathbf{x} \in R^I : \sum_{i \in I} a_i x_i \geqq a_0\} , \tag{2.3}$$

$$H_{\mathbf{a}}^- = \{\mathbf{x} \in R^I : \sum_{i \in I} a_i x_i \leqq a_0\} . \tag{2.4}$$

Let (I, V) and (I, W) be general coalition-games with the same set of players.

They are called *co-oriented* iff for every non-empty coalition $K \in 2^I$, $K \neq \emptyset$, there exists a real-valued vector $\mathbf{a}(K) \in R \times R^I$ and a hyperplane $H_{\mathbf{a}(K)}$ in R^I such that the sets $\boldsymbol{V}(K)$ and $\boldsymbol{W}(K)$ are in the same half-space defined by the hyperplane $H_{\mathbf{a}(K)}$.

Remark 2. It can be easily verified that the relation "to be co-oriented" is reflexive and symmetric.

The relation "to be co-oriented" is not transitive, as follows from the next example.

Example 1. Let us consider three two-person games $(I, \boldsymbol{V})$. $(I, \boldsymbol{W})$ and $(I, \boldsymbol{U})$ where $I = \{1, 2\}$, and

$$\boldsymbol{U}(I) = \{\mathbf{x} \in R^2 : x_1 \leqq 0\}, \quad \boldsymbol{V}(I) = \{\mathbf{x} \in R^2 : x_2 \leqq 0\},$$

$$\boldsymbol{W}(I) = \{\mathbf{x} \in R^2 : x_1 \leqq 0,\ x_2 \leqq 0\},$$

$$\boldsymbol{U}(\{i\}) = \boldsymbol{V}(\{i\}) = \boldsymbol{W}(\{i\}) = \{\mathbf{x} \in R^2 : x_i \leqq 0\}, \quad i = 1, 2.$$

Then $(I, \boldsymbol{U})$ and $(I, \boldsymbol{W})$ are co-oriented, and the games $(I, \boldsymbol{V})$ and $(I, \boldsymbol{W})$ are also co-oriented, but $(I, \boldsymbol{U})$ and $(I, \boldsymbol{V})$ are not co-oriented.

3. UNION AND INTERSECTION OF GAMES

The main results of this paper concern the intersection and union of general coalition-games. They represent certain kind of operation on the class of games with a common set of players defined by the conjunction or disjuntion of the possible pay-offs of coalitions. This conjunction or disjunction of pay-offs may be expressed by an intersection or union of the respective set values of the general characteristic functions.

Such operations with the general characteristic function influence the values of the superoptimum mapping, and the validity of the fundamental conditions (1.1), (1.2) and (1.3). That influence is investigated in this section. Further problem investigated here concerns the mutual relation between the strong solutions (strongly stable coalition structures and imputations) of the original games and the strong solution of the final general coalition-game.

Let us consider two general coalition-games $(I, \boldsymbol{V})$ and $(I, \boldsymbol{W})$ with the same set of players. Then we denote for any coalition $K \in 2^I$ the sets

$$(3.1) \qquad [\boldsymbol{V} \cup \boldsymbol{W}](K) = \boldsymbol{V}(K) \cup \boldsymbol{W}(K), \quad [\boldsymbol{V} \cap \boldsymbol{W}](K) = \boldsymbol{V}(K) \cap \boldsymbol{W}(K).$$

The first question which hould be answered here is whether the mappings $[\boldsymbol{V} \cup \boldsymbol{W}]$ and $[\boldsymbol{V} \cap \boldsymbol{W}]$ follow the properties of the general characteristic function.

Theorem 1. If $\boldsymbol{V}$ and $\boldsymbol{W}$ are general characteristic functions on the same class of coalitions 2^I then $[\boldsymbol{V} \cup \boldsymbol{W}]$ and $[\boldsymbol{V} \cap \boldsymbol{W}]$ are also general characteristic functions.

Proof. If $K \in 2^I$ is a coalition then the sets $V(K) \cup W(K)$ and $V(K) \cap W(K)$ are closed as both $V(K)$ and $W(K)$ are closed. If $\mathbf{x} \in V(K) \cap W(K)$ and $\mathbf{y} \in R^I$ is such that $y_i \leqq x_i$ for all $i \in K$ then $\mathbf{y} \in V(K)$ and $\mathbf{y} \in W(K)$. Hence, $\mathbf{y} \in V(K) \cap \cap W(K)$. Analogously, if $\mathbf{x} \in V(K) \cup W(K)$, $\mathbf{y} \in R^I$, $y_i \leqq x_i$ for all $i \in K$ then either $\mathbf{y} \in V(K)$ or $\mathbf{y} \in W(K)$, and consequently $\mathbf{y} \in V(K) \cup W(K)$. Let us choose $\mathbf{x} \in V(K)$ and $\mathbf{y} \in W(K)$ and let us construct $\mathbf{z} \in R^I$ such that

$$z_i = \min(x_i, y_i) \quad \text{for all} \quad i \in I .$$

Then $\mathbf{z} \in V(K) \cap W(K) \neq \emptyset$. Moreover, also $V(K) \cup W(K) \neq \emptyset$ as both sets $V(K)$ and $W(K)$ are non-empty. On the other hand, if $K \neq \emptyset$ then there exists $\mathbf{x} \in \in R^I - V(K)$ and $\mathbf{y} \in R^I - W(K)$. Then also $\mathbf{x} \in R^I - (V(K) \cap W(K))$, and $V(K) \cap \cap W(K) \neq R^I$. Let us construct $\mathbf{z} \in R^I$ such that

$$z_i = \max(x_i, y_i) \quad \text{for all} \quad i \in I .$$

Then

$$\mathbf{z} \in (R^I - V(K)) \cap (R^I - W(K)) = R^I - (V(K) \cup W(K)) ,$$

and consequently $V(K) \cup W(K) \neq R^I$. If $K = \emptyset$ then both sets $V(K)$ and $W(K)$ are equal to R^I. Consequently, also their union and intersection is equal to R^I. It means that both mappings $[V \cup W]$ and $[V \cap W]$ follow conditions (1.1), (1.2) and (1.3), and they are general characteristic functions over the class of coalitions 2^I. □

As the mappings $[V \cup W]$ and $[V \cap W]$ are general characteristic functions, it is possible to use formula (1.4) to define the superoptimum functions $[V \cup W]^*$ and $[V \cap W]^*$ corresponding to those general characteristic functions. The superoptimum functions V^*, W^*, $[V \cup W]^*$ and $[V \cap W]^*$ are mutually connected as follows from the next lemmas.

Lemma 3. If V and W are general characteristic functions on the same set of coalitions 2^I then for any $K \in 2^I$ is

$$[V \cap W]^* (K) \supset V^*(K) \cup W^*(K) \quad \text{and} \quad [V \cup W]^* (K) = V^*(K) \cap W^*(K) .$$

Proof. If $K \in 2^I$ is a coalition then, according to (1.4), is

$$[V \cap W]^* (K) = \{\mathbf{x} = (x_i)_{i \in I} \in R^I : \forall \mathbf{y} \in V(K) \cap W(K) \textit{ is either}$$
$$y_i = x_i \textit{ for all } i \in K \textit{ or } x_i > y_i \textit{ for some } i \in K\} .$$

Let us consider an $\mathbf{x} \in V^*(K)$. Then for all $\mathbf{y} \in V(K)$ is either $y_i = x_i$ for all $i \in K$ or $x_i > y_i$ for some $i \in K$. It means that the same relations are fulfilled also for all $\mathbf{y} \in V(K) \cap W(K) \subset V(K)$, and then $\mathbf{x} \in [V \cap W]^* (K)$. Hence, $V^*(K) \subset \subset [V \cap W]^* (K)$, and, analogously, $W^*(K) \subset [V \cap W]^* (K)$. Consequently,

$$V^*(K) \cup W^*(K) \subset [V \cap W]^* (K) .$$

Let us consider an $\mathbf{x} \in [V \cup W]^*(K)$ for $K \in 2^I$. Then either $x_i = y_i$ for all $i \in K$ or $x_i > y_i$ for some $i \in K$ and for all

$$\mathbf{y} \in [V \cup W](K) = V(K) \cup W(K).$$

It means that $\mathbf{x} \in V^*(K)$ and $\mathbf{x} \in W^*(K)$, and

$$(3.2) \qquad [V \cup W]^*(K) \subset V^*(K) \cap W^*(K) \quad \text{for all} \quad K \in 2^I.$$

Let us suppose, now, that $\mathbf{x} \in V^*(K) \cap W^*(K)$. Then for all $\mathbf{y} \in V(K)$ is either $x_i = y_i$ for all $i \in K$ or $x_i > y_i$ for some $i \in K$, and for all $\mathbf{y}' \in W(K)$ is either $x_i = y'_i$ for all $i \in K$, or $x_i > y'_i$ for some $i \in K$. If $\mathbf{z} \in V(K) \cup W(K)$, then necessarily either $x_i = z_i$ for all $i \in K$ or $x_i > z_i$ for some $i \in K$, as $\mathbf{z}$ is an element of at least one of the sets $V(K)$ and $W(K)$. It means that $\mathbf{x} \in [V \cup W]^*(K)$, and

$$(3.3) \qquad [V \cup W]^*(K) \supset V^*(K) \cap W^*(K) \quad \text{for all} \quad K \in 2^I.$$

Inclusions (3.2) and (3.3) imply the equality which should be proved. □

The relation between the superoptimum functions $[V \cap W]^*$, V^* and W^* is much stronger if the considered general characteristic functions V and W are untruncated.

Lemma 4. If V and W are general characteristic functions over the same set of coalitions 2^I, and if at least one of them is untruncated then for all coalitions $K \in 2^I$ is

$$[V \cap W]^*(K) = V^*(K) \cup W^*(K).$$

Proof. It was proved in Lemma 4 that $[V \cap W]^*(K) \supset V^*(K) \cup W^*(K)$ for all $K \in 2^I$. Let us choose a coalition $K \in 2^I$ and an imputation $\mathbf{x} \in R^I$ such that $\mathbf{x} \notin V^*(K) \cup W^*(K)$, and let us suppose that the general characteristic function V is untruncated. Then

$$\mathbf{x} \in V(K) - V^*(K) \quad \text{and} \quad \mathbf{x} \in W(K) - W^*(K),$$

and $\mathbf{x}$ is an inner point of the set $V(K)$. According to Lemma 1, there exists $\mathbf{y} \in V(K)$ such that $y_i > x_i$ for all $i \in K$, and by Remark 1 there exists $\mathbf{y}' \in W(K)$ such that $y'_i \geqq x_i$ all $i \in K$, and $y'_j > x_j$ for some $j \in K$. Let us construct an imputation $\mathbf{z} = (z_i)_{i \in I} \in R^I$ such that

$$z_i = \min(y_i, y'_i) \quad \text{for all} \quad i \in I.$$

Then $\mathbf{z} \in V(K) \cup W(K)$, as follows from (1.2), and $z_i \geqq x_i$ for all $i \in K$, $z_j > x_j$ for some $j \in K$. Consequently, $\mathbf{x} \notin [V \cap W]^*(K)$, and

$$[V \cap W]^*(K) \subset V^*(K) \cup W^*(K) \quad \text{for all} \quad K \in 2^I. \qquad \square$$

If both general characteristic functions V and W are truncated then the equality from Lemma 4 is not guaranted, as follows from the next example.

Example 2. Let us consider two general coalition-games (I, V) and (I, W) with the same set of players, and let $K = \{i, j\} \in 2^I$ be a coalition. Let

$$V(K) = \{\mathbf{x} = (x_k)_{k \in I} : x_i \leqq 1,\ x_j \leqq 2\},$$

$$W(K) = \{\mathbf{x} = (x_k)_{k \in I} : x_i \leqq 2,\ x_j \leqq 1\}.$$

Then

$$V(K) \cap V^*(K) = \{\mathbf{x} = (x_k)_{k \in I} : x_i = 1,\ x_j = 2\},$$

$$\partial V(K) = \{\mathbf{x} = (x_k)_{k \in I} : x_i = 1,\ x_j \leqq 1\} \cup \{\mathbf{x} = (x_k)_{k \in I} : x_j = 1,\ x_i \leqq 1\},$$

and, consequently, V is truncated. Analogously, also W is truncated. It can be easily seen that

$$[V \cap W](K) = V(K) \cap W(K) = \{\mathbf{x} = (x_k)_{k \in I} : x_i \leqq 1,\ x_j \leqq 1\},$$

$$[V \cap W]^*(K) = \{\mathbf{x} = (x_k)_{k \in I} : x_i > 1\} \cup \{\mathbf{x} = (x_k)_{k \in I} : x_j > 1\} \cup$$

$$\cup \{\mathbf{x} = (x_k)_{k \in I} : x_i = 1,\ x_j = 1\},$$

but

$$V^*(K) \cup W^*(K) = \{\mathbf{x} = (x_k)_{k \in I} : x_i > 1\} \cup \{\mathbf{x} = (x_k)_{k \in I} : x_j > 1\},$$

and then

$$[V \cap W]^*(K) - (V^*(K) \cup W^*(K)) = \{\mathbf{x} = (x_k)_{k \in I} : x_i = 1,\ x_j = 1\}.$$

Lemma 5. If V is a general characteristic function and $K \in 2^I$ is a coalition then

$$V(K) \cap V^*(K) \subset \partial V(K) = \partial V^*(K).$$

Proof. If $\mathbf{x} \in \partial V(K)$ then for any open neighbourhood $u(\mathbf{x})$ of $\mathbf{x}$ there exist $\mathbf{y} \in u(\mathbf{x})$ and $\mathbf{z} \in u(\mathbf{x})$ such that $y_i < x_i < z_i$ for all $i \in K$, and $\mathbf{y} \in V(K) - V^*(K)$, $\mathbf{z} \in V^*(K) - V(K)$. It means that $\mathbf{x} \in \partial V^*(K)$, too, and $\partial V(K) \subset \partial V^*(K)$. It is possible to prove in an analogous way that $\partial V(K) \supset \partial V^*(K)$. Let $\mathbf{x} \in V(K) - \partial V(K)$. then there exists an open neighbourhood $u(\mathbf{x}) \subset V(K) - \partial V(K)$ and an imputation $\mathbf{y} \in u(\mathbf{x})$ such that $y_i > x_i$ for all $i \in K$. It means that $\mathbf{x} \notin V(K) \cap V^*(K)$, and the remaining inclusion is proved. □

As follows from the last lemma, equality (2.1) valid for the untruncated general characteristic functions turns into an inclusion in the general case. The class of untruncated general characteristic functions is closed with respect to the operations of union and intersection as follows from the next statement.

Lemma 6. If V and W are untruncated general characteristic functions over the set of coalitions 2^I, then the general characteristic functions $[V \cup W]$ and $[V \cap W]$ are also untruncated.

Proof. If V and W are untruncated then for all $K \in 2^I$ is

$$V(K) \cap V^*(K) = \partial V(K) \quad \text{and} \quad W(K) \cap W^*(K) = \partial W(K).$$

For any $K \in 2^I$ is

$$\partial[V \cap W](K) = (\partial V(K) \cap W(K)) \cup (\partial W(K) \cap V(K)) =$$
$$= (V(K) \cap V^*(K) \cap W(K)) \cup (W(K) \cap W^*(K) \cap V(K)) =$$
$$= (V(K) \cap W(K)) \cap (V^*(K) \cup W^*(K)) = [V \cap W](K) \cap [V \cap W]^*(K),$$

as follows from Lemma 4. Hence, $[V \cap W]$ is untruncated. On the other hand, as follows from Lemma 5 and Lemma 3,

$$\partial[V \cup W](K) = \partial[V \cup W]^*(K) = \partial(V^*(K) \cap W^*(K)) =$$
$$= (\partial V^*(K) \cap W^*(K)) \cup (\partial W^*(K) \cap V^*(K)) =$$
$$= (V(K) \cap V^*(K) \cap W^*(K)) \cup (W(K) \cap W^*(K) \cap V^*(K) =$$
$$= (V^*(K) \cap W^*(K)) \cap (V(K) \cup W(K)) = [V \cup W]^*(K) \cap [V \cup W](K), \quad \square$$

The relations between the strong solutions of the general coalition-games (I, V), (I, W), $(I, [V \cup W])$ and $(I, [V \cap W])$ are described by the following theorems.

Theorem 2. Let (I, V) and (I, W) be general coalition-games, let $\mathscr{K}$ be a coalition structure, and let $\mathbf{x} \in V(\mathscr{K}) \cap W(\mathscr{K})$. If $\mathbf{x}$ is strongly stable in (I, V) then it is strongly stable in $(I, [V \cap W])$, too.

Proof. It is supposed that

$$\mathbf{x} \in V(K) \cap W(K) = [V \cap W](K) \quad \text{for all} \quad K \in \mathscr{K},$$

and, moreover, $\mathbf{x} \in V^*(L)$ for all $L \in 2^I$. It means that

$$\mathbf{x} \in V^*(L) \cup W^*(L) \subset [V \cap W]^*(L) \quad \text{for all} \quad L \in 2^I,$$

and $\mathbf{x}$ is strongly stable in $(I, [V \cap W])$. □

Theorem 3. Let (I, V) and (I, W) be general coalition-games. If an imputation $\mathbf{x} \in R^I$ is strongly stable in (I, V) and in (I, W) then it is strongly stable in $(I, [V \cup W])$.

Proof. If $\mathbf{x}$ is strongly stable in (I, V) then there exists a coalition structure $\mathscr{K} \subset 2^I$ such that $\mathbf{x} \in V(\mathscr{K})$. It means that $\mathbf{x} \in V(K)$ and also $\mathbf{x} \in V(K) \cup W(K)$ for all $K \in \mathscr{K}$, hence, $\mathbf{x} \in [V \cup W](\mathscr{K})$. Moreover, for all coalitions $L \in 2^I$ is $\mathbf{x} \in V^*(L)$ and $\mathbf{x} \in W^*(L)$. Then

$$\mathbf{x} \in V^*(L) \cap W^*(L) = [V \cup W]^*(L) \quad \text{for all} \quad L \in 2^I,$$

and, consequently, $\mathbf{x}$ is strongly stable in $(I, [V \cup W])$. □

Theorem 4. Let (I, V) and (I, W) be general coalition-games. If $\mathbf{x} \in R^I$ is an imputation such that $\mathbf{x} \in V(\mathscr{K})$ for some coalition structure $\mathscr{K} \subset 2^I$ and if $\mathbf{x}$ is strongly stable in $(I, [V \cup W])$ then $\mathbf{x}$ is strongly stable in (I, V).

Proof. It is assumed that $\mathbf{x} \in V(\mathscr{K})$ for some coalition structure $\mathscr{K}$, and that

$$\mathbf{x} \in [V \cup W]^* (L) = V^*(L) \cap W^*(L) \quad \text{for all} \quad L \in 2^I .$$

It means that $\mathbf{x} \in V^*(L)$ for all $L \in 2^I$, and $\mathbf{x}$ is strongly stable in (I, V). □

4. CONVEX COMBINATION OF GAMES

If there exist two or more games with the same set of players then the final profit of the players may be expressed in more ways according to the actual interpretation of the composed game. The conjunction and disjunction of the partial sets of imputations were investigated in the foregoing section. Here, we shall consider the coalition-game which is defined as a convex combination of two other games of the same players. There is no general rule describing the relation between the solutions of such games, but it is possible to show that any convex combination of two co-oriented games is also a general coalition-game.

Let us consider two general coalition-games (I, V) and (I, W) with the same set of players, and a real number $\lambda \in [0, 1]$. Let U be a mapping of the set of coalitions 2^I into the class of subsets of R^I such that for any coalition $K \in 2^I$ is

$$(4.1) \qquad U(K) = \{\mathbf{x} \in R^I : \exists \mathbf{y} \in V(K),\ \mathbf{z} \in W(K) \ \textit{such that}\ \mathbf{x} = \lambda \mathbf{y} + (1 - \lambda)\, \mathbf{z}\} ,$$

and let us denote

$$(4.2) \qquad U = \lambda V + (1 - \lambda)\, W .$$

Then we say that U is a *convex combination* of the general characteristic functions V and W. It is shown in the following theorem that if the games (I, V) and (I, W) are co-oriented (c.f. section 2) then any convex combination of their general characteristic unctions is also a general characteristic function.

Theorem 5. Let (I, V) and (I, W) be co-oriented general coalition-games, and let $\lambda \in [0, 1]$ be a real number. Then the pair (I, U), where $U = \lambda V + (1 - \lambda)\, W$ is a convex combination of V and W, is also a general coalition-game.

Proof. The closedness of the sets $U(K)$ for all $K \in 2^I$ follows from the closedness of $V(K)$ and $W(K)$ and from (4.1). Let us consider an imputation $\mathbf{x} \in U(K)$ for some $K \in 2^I$, and $\mathbf{x}' \in R^I$ such that $x'_i \leqq x_i$ for all $i \in K$. Let us suppose that $\mathbf{y} \in V(K)$ and $\mathbf{z} \in W(K)$ are such that $\mathbf{x} = \lambda \mathbf{y} + (1 - \lambda)\, \mathbf{z}$, and let us denote for all $i \in I$

$$y'_i = y_i + x'_i - x_i \leqq y_i , \quad z'_i = z_i + x'_i - x_i \leqq z_i .$$

Then $\mathbf{y}' = (y'_i)_{i \in I} \in \boldsymbol{V}(K)$, $\mathbf{z}' = (z'_i)_{i \in I} \in \boldsymbol{W}(K)$, and

$$\lambda y'_i + (1 - \lambda) z'_i = \lambda y_i + (1 - \lambda) z_i + x'_i - x_i = x'_i \quad \text{for all} \quad i \in I .$$

It means that $\mathbf{x}' \in \boldsymbol{U}(K)$ if $0 \leqq \lambda \leqq 1$. (For $\lambda = 0$ or $\lambda = 1$ is (1.2) fulfilled immediately.) As both sets $\boldsymbol{V}(K)$ and $\boldsymbol{W}(K)$ are non-empty for all $K \in 2^I$, the sets $\boldsymbol{U}(K)$ are also non-empty for all $K \in 2^I$. If $K \neq \emptyset$ is a non-empty coalition then there exists a hyperplane H_K such that both sets $\boldsymbol{V}(K)$ and $\boldsymbol{W}(K)$ belong to the same half-space according to H_K. It means that also $\boldsymbol{U}(L)$ belongs to that half-space and, consequently, $\boldsymbol{U}(K) \neq R^I$. If, on the other hand, $K = \emptyset$ then both, $\boldsymbol{V}(K)$ and $\boldsymbol{W}(K)$ are equal to R^I, and then also $\boldsymbol{U}(K) = R^I$. Hence, the mapping $\boldsymbol{U}$ fulfils conditions (1.1), (1.2) and (1.3), and the pair $(I, \boldsymbol{U})$ is a general coalition-game. □

Remark 3. It follows from the previous proof immediately that if $(I, \boldsymbol{V})$ and $(I, \boldsymbol{W})$ are co-oriented general coalition-games, $\lambda \in R$, $0 \leqq \lambda \leqq 1$, and if $(I, \boldsymbol{U})$ is a general-coalition-game such that $\boldsymbol{U} = \lambda \boldsymbol{V} + (1 - \lambda) \boldsymbol{W}$ then the general coalition-game is co-oriented with both games $(I, \boldsymbol{V})$ and $(I, \boldsymbol{W})$.

If the general coalition-games forming the convex combination $(I, \boldsymbol{U})$ are not co-oriented, then the pair $(I, \boldsymbol{U})$ need not be a general coalition-game, i.e. the mapping $\boldsymbol{U}$ does not generally fulfil the conditions (1.1), (1.2) and (1.3). It is illustrated by the following example.

Example 3. Let us consider two general coalition-games $(I, \boldsymbol{V})$ and $(I, \boldsymbol{W})$ with the set of players $I = \{1, 2\}$ containing exactly two elements. Let

$$\boldsymbol{V}(I) = \{\mathbf{y} = (y_1, y_2) \in R^2 : -2y_1 + y_2 \leqq 0\} ,$$

$$\boldsymbol{W}(I) = \{\mathbf{z} = (z_1, z_2) \in R^2 : -z_1 + 2z_2 \leqq 0\} .$$

Let us consider $\lambda = \frac{1}{2}$. Then

$$\boldsymbol{U}(I) = [\lambda \boldsymbol{V} + (1 - \lambda) \boldsymbol{W}](I) =$$

$$= \{\mathbf{x} = (x_1, x_2) \in R^2 : \exists \mathbf{y} \in \boldsymbol{V}(I),\ \mathbf{z} \in \boldsymbol{W}(I) \textit{ such that}$$

$$x_1 = y_1/2 + z_1/2,\ x_2 = y_2/2 + z_2/2\} = R^2 = R^I .$$

It means that the set $\boldsymbol{U}(I)$ does not fulfil condition (1.3) and the pair $(I, \boldsymbol{U})$ is not general coalition-game.

5. GAME-PRESERVING TRANSFORMATIONS

It was already mentioned in the introduction that this work is a free continuation of the author's paper [5]. It is useful to repeat here a few notions and results introduced there. They are presented here without comments and explanations which were already done in the referred paper [5].

Let us consider a *one-to-one transformaion* T of R^I *onto* R^I. If $\mathbf{x} = (x_i)_{i\in I} \in R^I$ then $(T\mathbf{x})_i$ are components of the transformed vector $T\mathbf{x} \in R^I$. If $M \subset R^I$ is a set of imputations then

$$(5.1) \qquad TM = \{\mathbf{x} \in R^I : \exists \mathbf{y} \in M \text{ for which } \mathbf{x} = T\mathbf{y}\}\,.$$

The transformation T is called a *game-preserving* transformation iff for any general coalition-game (I, V) the pair (I, TV), where $TV(K)$ are sets of imputations defined from the sets $V(K)$ by (5.1), is also a general coalition-game. The transformation T is called *coordinatewise strictly increasing* iff for all $i \in I$ and all $\mathbf{x}, \mathbf{y} \in R^I$ the inequality $x_i > y_i$ implies $(T\mathbf{x})_i > (T\mathbf{y})_i$. The transformation T is called *coordinatewise decomposable* iff there exist transformations T_i, $i \in I$, of R onto R such that for every $\mathbf{x} = (x_i)_{i\in I} \in R^I$ is $T\mathbf{x} = (T_i x_i)_{i\in I}$.

It was proved in [5] that a one-to-one transformation of R^I onto R^I is a game-preserving one if and only if it is coordinatewise strictly increasing ([5], Theorem 1), and that any game-preserving one-to-one transformation of R^I onto R^I is coordinatewise decomposable ([5], Theorem 2).

If (I, V) is a game and if T is a game-preserving one-to-one transformation of R^I onto R^I then we denote for every $K \in 2^I$, analogously to (5.1), the set

$$(5.2) \qquad TV^*(K) = \{\mathbf{x} \in R^I : \exists \mathbf{y} \in V^*(K) \text{ such that } \mathbf{x} = T\mathbf{y}\}\,.$$

It is also possible to define for every coalition $K \in 2^I$ and every set $TV(K)$ the set $[TV]^*(K)$ derived from $TV(K)$ by means of (1.4). It was shown in [5] (Lemma 5) that for every $K \in 2^I$ is

$$(5.3) \qquad TV^*(K) = [TV]^*(K)\,.$$

The relation between the strong solution of a general coalition-game and its transformation was derived in [5] and formulated in the following way. Let us consider a game (I, V) and a game-preserving one-to-one transformation T of R^I onto R^I. Then an imputation $\mathbf{x} \in R^I$ is strongly stable in the game (I, V) if and only if the imputation $T\mathbf{x}$ is strongly stable in the transformed game (I, TV) ([5], Theorem 3). Moreover, a coalition structure $K \subset 2^I$ is strongly stable in (I, TV) if and only if it is strongly stable in (I, V) ([5], Theorem 4).

The following auxiliary statement concerns the monotonicity of the game-preserving transformations.

Lemma 7. Let (I, V) be a general coalition-game, and let T be a game-preserving one-to-one transformation of R^I onto R^I. If $\mathscr{K}$ and $\mathscr{L}$ are coalition structures such that $V(\mathscr{K}) \supset V(\mathscr{L})$ then also

$$TV(\mathscr{K}) = \bigcap_{K\in\mathscr{K}} TV(K) \supset TV(\mathscr{L}) = \bigcap_{L\in\mathscr{L}} TV(L)\,.$$

Proof. Let us consider $\mathbf{y} \in TV(\mathscr{L})$, i.e. $\mathbf{y} \in TV(L)$ for all $L \in \mathscr{L}$. It means that there exists $\mathbf{x} \in R^I$ such that $\mathbf{y} = T\mathbf{x}$, and $\mathbf{x} \in V(L)$ for all $L \in \mathscr{L}$, because T is a one-to-one mapping and $T\,V(L)$ fulfil (2.1) for all $L \in \mathscr{L}$. Then $\mathbf{x} \in V(K)$ for all $K \in \mathscr{K}$, as $V(\mathscr{L}) \subset V(\mathscr{K})$, and, consequently, $\mathbf{y} = T\mathbf{x} \in T\,V(K)$ for all $K \in \mathscr{K}$.

6. ADDITIVITY AND TRANSFORMATIONS

The general model of games represented by the pair (I, V) includes also a wide scale of more special games. One of the most important specialized subclasses of the class of general coalition-games is formed by the games with general characteristic functions fulfilling certain kind of the additivity assumptions. Such games and their properties were investigated in [4], and their main property concerning the game-preserving transformations, namely the fact that such transformations preserve also the additivity, is proved in this brief section.

Before introducing the main result of this section, it is useful to remember here the additivity concepts for the general coalition-games.

Let us suppose that (I, V) is a general coalition-game. It is called *superadditive* iff for all coalitions $K, L \in 2^I$ such that $K \cap L = \emptyset$ is

$$V(K \cup L) \supset V(K) \cap V(L)\,. \tag{6.1}$$

It is called *subadditive* iff for all coalitions $K, L \in 2^I$ such that $K \cap L = \emptyset$ is

$$V^*(K \cup L) \supset V^*(K) \cap V^*(L)\,. \tag{6.2}$$

The game (I, V) is called *additive* iff it is superadditive and subadditive.

Now, the result concerning the transformations of those types of general coalition-games may be formulated.

Theorem 6. Let (I, V) be a general coalition-game, and let $(I, T\,V)$ be its transformation by means of a game-preserving one-to-one transformation T of R^I onto R^I. Then (I, V) is superadditive, subadditive or additive if and only if $(I, T\,V)$ is superadditive, subadditive or additive, respectively.

Proof. If T is a game-preserving one-to-one transformation of R^I onto R^I then (5.1) and (5.2) imply that for any pair of coalitions $K, L \in 2^I$, such that $K \cap L = \emptyset$, and any $\mathbf{x} \in R^I$ is

$$\mathbf{x} \in V(K) \Leftrightarrow T\mathbf{x} \in T\,V(K)\,, \quad \mathbf{x} \in V(L) \Leftrightarrow T\mathbf{x} \in T\,V(L)\,, \quad \text{and} \tag{6.3}$$

$$\mathbf{x} \in V(K \cup L) \Leftrightarrow T\mathbf{x} \in T\,V(K \cup L)\,. \tag{6.4}$$

Analogously,

$$\mathbf{x} \in V^*(K) \Leftrightarrow T\mathbf{x} \in [T\,V]^*\,(K)\,, \quad \mathbf{x} \in V^*(L) \Leftrightarrow T\mathbf{x} \in [T\,V]^*\,(L)\,, \tag{6.5}$$

$$\mathbf{x} \in V^*(K \cup L) \Leftrightarrow T\mathbf{x} \in [T\,V]^*\,(K \cup L)\,, \tag{6.6}$$

as follows from (5.2) and (5.3). It means that

$$V(K \cup L) \supset V(K) \cap V(L) \Leftrightarrow T\,V(K \cup L) \supset T\,V(K) \cap T\,V(L)\,,$$

and

$$V^*(K \cup L) \supset V^*(K) \cap V^*(L) \Leftrightarrow [T\,V]^*(K \cup L) \supset [T\,V]^*(K) \cap [T\,V]^*(L)\,. \quad \square$$

7. CONVEXITY OF TRANSFORMATIONS

A special class of general coalition-games including the games with convex, concave or linear sets $V(K)$, $K \in 2^I$, will be considered here, and the transformations of such games will be investigated.

Let us suppose that (I, V) is a general coalition-game with the general characteristic function V. Then V is called *convex* iff for all coalitions $K \in 2^I$ the set $V(K)$ is convex, i.e.

$$(7.1) \qquad \mathbf{x} \in V(K)\,, \quad \mathbf{y} \in V(K)\,, \quad \lambda \in [0, 1] \Rightarrow \lambda\mathbf{x} + (1 - \lambda)\,\mathbf{y} \in V(K)\,.$$

Analogously, V is called *concave* iff for all $K \in 2^I$ the set $V^*(K)$ is convex, i.e.

$$(7.2) \qquad \mathbf{x} \in V^*(K)\,, \quad \mathbf{y} \in V^*(K)\,, \quad \lambda \in [0, 1] \Rightarrow \lambda\mathbf{x} + (1 - \lambda)\,\mathbf{y} \in V^*(K)\,.$$

The general characteristic function V is called *linear* iff it is convex and concave.

Remark 4. It is obvious that the convexity of general characteristic functions introduced here is of completely different nature than the convexity of the coalition-games with side payments known from the literature, e.g. from [7] and [6].

It will be shown here that there exists a strong connection between the convexity, concavity or linearity of a general characteristic function, and the analogous properties of the considered game-preserving transformation.

Let us consider a set of players I and a game-preserving one-to-one transformation T of R^I onto R^I. Then T is coordinatewise decomposable as follows from [5], Theorem 2 (see also Section 5), i.e. there exists a set of mappings T_i, $i \in I$, of R onto R such that for any $\mathbf{x} = (x_i)_{i \in I} \in R^I$ and any $i \in I$ is $(T\mathbf{x})_i = T_i x_i$. The transformation T is *convex* iff all transformations T_i, $i \in I$, are convex, i.e. for arbitrary x_i, $y_i \in R$ and $\lambda \in [0, 1]$ is

$$(7.3) \qquad T_i(\lambda x_i + (1 - \lambda)\,y_i) \leqq \lambda T_i x_i + (1 - \lambda)\,T_i y_i\,.$$

Analogously, T is *concave* iff all T_i, $i \in I$, are concave, i.e. for arbitrary x_i, $y_i \in R$ and $\lambda \in [0, 1]$ is

$$(7.4) \qquad T_i(\lambda x_i + (1 - \lambda)\,y_i) \geqq \lambda T_i x_i + (1 - \lambda)\,T_i y_i\,,$$

and T is *linear* iff all T_i, $i \in I$, are linear, i.e. for arbitrary $x_i, y_i \in R$ and $\lambda \in [0, 1]$ is

$$(7.5) \qquad T_i(\lambda x_i + (1 - \lambda) y_i) = \lambda T_i x_i + (1 - \lambda) T_i y_i .$$

The transformation T is linear if and only if it is convex and concave.

Theorem 7. If (I, V) is a general coalition-game with convex general characteristic function V, and if T is a concave game-preserving one-to-one transformation of R^I onto R^I, then the transformed game $(I, T V)$ is also a game with convex general characteristic function $T(V)$.

Proof. Let us consider a coalition $K \in 2^I$ and arbitrary $\mathbf{x}, \mathbf{y} \in V(K)$. Then the convexity of $V(K)$ implies that for any λ, $0 \leqq \lambda \leqq 1$, is

$$\lambda \mathbf{x} + (1 - \lambda) \mathbf{y} \in V(K) .$$

It follows from (5.1) that

$$(7.6) \qquad T\mathbf{x} \in T V(K), \quad T\mathbf{y} \in T V(K), \quad T(\lambda \mathbf{x} + (1 - \lambda) \mathbf{y}) \in T V(K) .$$

As T is a concave game-preserving transformation, the inequality

$$\lambda T_i x_i + (1 - \lambda) T_i y_i \leqq T_i(\lambda x_i + (1 - \lambda) y_i)$$

holds for all $i \in I$. It means, according to (1.2) and (7.6), that

$$\lambda T\mathbf{x} + (1 - \lambda) T\mathbf{y} \in T V(K)$$

and, consequently, the general characteristic function $T V$ is convex. □

Theorem 8. If (I, V) is a general coalition-game with a concave general characteristic function V, and T is a convex game-preserving one-to-one transformation of R^I onto R^I, then the transformed game $(I, T V)$ is a game with concave general characteristic function $T V$.

Proof. Let us consider a coalition $K \in 2^I$ and arbitrary $\mathbf{x}, \mathbf{y} \in V^*(K)$. Then the concavity of V implies for any λ, $0 \leqq \lambda \leqq 1$, the relation

$$(7.7) \quad T\mathbf{x} \in [T V]^* (K), \quad T\mathbf{y} \in [T V]^* (K), \quad T(\lambda \mathbf{x} + (1 - \lambda) \mathbf{y}) \in [T V]^* (K) .$$

As T is a game-preserving convex transformation, the inequality

$$\lambda T_i x_i + (1 - \lambda) T_i y_i \geqq T_i(\lambda x_i + (1 - \lambda) y_i)$$

holds for all $i \in I$. Then, according to (1.4) and (7.7), is

$$\lambda T\mathbf{x} + (1 - \lambda) T\mathbf{y} \in [T V]^* (K)$$

and, consequently, the game $(I, T V)$ is a game with concave general characteristic function $T V$. □

Theorem 9. If (I, V) is a general coalition-game with linear general characteristic function, and if T is a linear game-preserving one-to-one transformation of R^I onto R^I then the transformed game $(I, T V)$ is a game with linear general characteristic function $T V$.

Proof. The statement follows immediately from Theorems 7 and 8, and from the definitions of the linear general characteristic function and linear game-preserving transformation. □

8. CONSTRAINED GAMES

The class of general coalition-games described by the model suggested in [2] includes also such special types of games which can be hardly expected in the classical interpretations of the coalition-games models. It concerns, for example, also games in which the total profit of coalitions is not limited, i.e. in which the profit of one or more players can increase up to infinity without any proportional change of the profit of other members of the same coalition. Such games represent a rarely appearing sort of situations. More interesting are games of the opposite type in which the profits of players are constrained by some mutual connections. Such constrained games are briefly mentioned in this section, and it is shown here that their constrainedness is also preserved by any game-preserving transformation of the imputations space.

A general coalition-game (I, V) is said to be *constrained* iff for any coalition $K \in 2^I$, any imputation $\mathbf{x} \in V(K)$ and any player $i \in K$ there exists an imputation $\mathbf{y} \in R^I - V(K)$ such that

$$(8.1) \qquad y_j = x_j \quad \text{for all} \quad j \in I\,, \quad j \neq i\,, \quad \text{and} \quad y_i > x_i\,.$$

It is not difficult to see that any classical coalition-game with side-payments is a constrained general coalition-game such that for any coalition $K \in 2^I$ is

$$V(K) = \{\mathbf{x} = (x_i)_{i\in I} \in R^I : \sum_{i\in K} x_i \leqq v(K)\}$$

where v is the classical von Neumann-Morgenstern characteristic function mapping 2^I into R (cf. [1], [6]).

Lemma 8. If a general coalition-game (I, V) is constrained and convex then there for any coalition structure $\mathscr{K} \subset 2^I$ exists a hyperplane

$$(8.2) \qquad H_{\mathbf{b}}(\mathscr{K}) = \{\mathbf{x} \in R^I : \sum_{i\in I} b_i x_i = b_0\}\,,$$

$$\mathbf{b} = (b_0, (b_i)_{i\in I}) \in R \times R^I\,, \quad b_i > 0\,, \quad i \in I\,,$$

such that

$$(8.3) \qquad V(\mathscr{K}) \subset \{\mathbf{x} \in R^I : \sum_{i\in I} b_i x_i \leqq b_0\} \quad \text{and} \quad V(\mathscr{K}) \cap H_{\mathbf{b}}(\mathscr{K}) \neq \emptyset\,.$$

Proof. Since the game (I, V) is convex, the sets $V(\mathscr{K})$ are convex for all coalition structures $\mathscr{K}$, and (1.1) implies that the sets $V(\mathscr{K})$ are closed. It means that there always exists a real valued vector $(b_0, (b_i)_{i \in I}) \in R \times R^I$ and a hyperplane $H_{\mathbf{b}}(\mathscr{K})$ defined by (8.2) such that (8.3) is fulfilled. Assumption (1.2) implies that the hyperplane should be constructed so that $b_i \geqq 0$ for all $i \in I$, and the constrainedness of the considered game (I, V) means that the vector $(b_i)_{i \in I}$ may be chosen in such a way that $b_i > 0$ for all $i \in I$. □

Theorem 10. If (I, V) is a constrained general coalition-game and T is a game-preserving one-to-one transformation of R^I onto R^I then the transformed game $(I, T V)$ is also constrained.

Proof. It follows from [5], Theorem 2 (see also Section 5) that any game-preserving transformation T of R^I onto R^I is necessarily coordinatewise strictly increasing. If $K \in 2^I$, $\mathbf{x} \in V(K)$, $i \in K$ and $\mathbf{y} \in R^I - V(K)$, where $y_j = x_j$ for all $j \in I$, $j \neq i$, and $y_i > x_i$ then $T\mathbf{x} \in T V(K)$, $T\mathbf{y} \in R^I - T V(K)$, and $Ty_j = Tx_j$ for all $j \in I$, $j \neq i$, $Ty_i > Tx_i$. These relations are true for arbitrary $K \in 2^I$, $\mathbf{x} \in V(K)$ and $i \in K$. It means that the transformed game $(I, T V)$ is also constrained. □

9. CONCLUSIONS

It was shown here that different kinds of combinations of the general coalition-games, as well as their game-preserving transformations do not violate their fundamental properties and that they do not change the structure of their natural subclasses. The fact that the considered class of games is closed under the operations of union, intersection, and for co-oriented games also under convex combination means that it is sufficiently "rich". On the other hand, it is also sufficiently "homogeneous" in the sense that it includes naturally related objects. Moreover, the main properties of the general coalition-games can not be broken by the game-preserving transformations.

It is obvious that the concept of the convex combination investigated in Section 4 can be easily substituted by more general linear combination where for some games (I, V) and (I, W) and any coalition $K \in 2^I$ is

$$U(K) = \lambda\, V(K) + \mu\, W(K) =$$

$$= \{\mathbf{x} \in R^I : \exists \mathbf{y} \in V(K),\ \mathbf{z} \in W(K) \text{ such that } \mathbf{x} = \lambda \mathbf{y} + \mathbf{b}\mathbf{z}\}$$

for positive real numbers λ and μ. It is not difficult to verify that the results of Section 4 would keep unchanged.

It was also mentioned in Section 4 already that there exists no general and sufficiently strong relation between solutions of the original games and their convex combination. However, there probably exist some relations of such kind which are valid for special classes of games. These results can be derived if the application of such special games is desirable.

It is probably possible to derive that the game-preserving transformations preserve also some other special properties of the games in addition to the ones which were considered here. However, the properties investigated here belong to the most important ones. Some results analogous to the presented ones can be obtained for particular properties and classes of games if it were useful for actual applications of the general model.

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