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Discrete Stochastic Regulation and Tracking

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The problem of following a random trajectory in the presence of disturbances is solved in the minimum variance sense for discrete-time linear single-input-output plants. The plant is described by an input-output relation, namely a controlled autoregressive moving average (ARMA) model. The random inputs are zero-mean covariance-stationary sequences modeled as ARMA processes.

This tracking problem is solved by reformulating it as a regulator problem for an augmented system. The optimal control law is shown to contain both feedback and feedforward terms and it is obtained by applying polynomial techniques. The design procedure consists in spectral factorization and the solution of linear polynomial equations.

1. INTRODUCTION

The problem of signal tracking in the face of disturbances, known also as a stochastic servo problem, has found various industrial applications. In practice, it is often required that the output of a given plant be optimally reset to a new reference level or trajectory. This must be accomplished despite the presence of random disturbances and measurement noise.

This problem has not been satisfactorily solved within the classical control theory. This is perhaps due to the prevalent philosophy that the control law must operate on the tracking error, see Newton, Gould, and Kaiser [5] or Youla, Bongiorno, and Jabr [6]. As noted by Gawthrop [1], the resultant control law is suboptimal unless known set points and unknown disturbances are properly distinguished.

Modern control theory has overcome this tradition by introducing a state-variable feedback. When augmenting the plant and reference dynamics into a single composite system, the stochastic servo problem can be reformulated as a standard LQG problem, see Kwakernaak and Sivan [4]. The resultant feedback control law can then be interpreted as a combination of feedback and feedforward terms relative to the original plant. However, this technique requires the identification of state-variable models and the solution of coupled algebraic Riccati equations.

The aim of this paper is to solve the stochastic servo problem by applying the polynomial methods developed by Kučera [2; 3]. We start with the input-output description of the plant and reference as ARMA processes and proceed to determine the optimal control law by manipulating polynomials only. The design procedure consists in solving linear polynomial equations whose polynomial coefficients are obtained by spectral factorization. The computational feasibility of this approach is discussed elsewhere [3].

2. PROBLEM FORMULATION

Consider a discrete-time stochastic plant modeled by the controlled ARMA process

$$A(d)\, y = B(d)\, u + C(d)\, w \tag{1}$$

where y is the output sequence, u is the input sequence, and w is a zero-mean covariance-stationary white random sequence with variance ψ. The observed output z is assumed in the form

$$z = y + v \tag{2}$$

where v is another zero-mean covariance-stationary white random sequence, with variance φ. The A, B, and C are coprime polynomials in the delay operator d, with arbitrary relative degrees, and such that $A(0) \neq 0$ and $B(0) = 0$. The φ and ψ are nonnegative but not both zero.

Further consider a reference sequence r modeled by the ARMA process

$$\bar{A}(d)\, r = \bar{C}(d)\, \bar{w} \tag{3}$$

where $\bar{w}$ is a zero-mean covariance-stationary white random sequence with variance $\bar{\psi}$. Let s, the available version of the reference, be of the form

$$s = r + \bar{v} \tag{4}$$

where $\bar{v}$ is another zero-mean covariance-stationary white random sequence, with variance $\bar{\varphi}$. The $\bar{A}$ and $\bar{C}$ are coprime polynomials in d, with arbitrary relative degrees, and such that $\bar{A}(0) \neq 0$. The $\bar{\varphi}$ and $\bar{\psi}$ are again nonnegative but not simultaneously zero.

Let us suppose that all four random sources v, w and $\bar{v}$, $\bar{w}$ are pairwise uncorrelated. Moreover, it is natural to assume that the plant is free of unstable hidden modes and that the reference is covariance stationary. In particular, these assumptions entail stability of the greatest common divisor of A and B as well as of the polynomial $\bar{A}$.

For a given plant and reference, the design of the optimal control law

$$(5)\qquad P(d)\,u = -Q(d)\,z + R(d)\,s$$

with polynomial P, Q, and R and $P(0) \neq 0$ evolves from minimization of the cost

$$(6)\qquad J = \lambda J_u + \mu J_e$$

where J_u and J_e are respectively the steady-state variances of the plant input and of the tracking error, defined as $e = y - r$. The λ and μ are constants weighting the relative importance of the two components; they are nonnegative but not simultaneously zero.

Thus the objective of our design is to minimize (6) subject to the constraint that the control system defined by (1) through (5) be (asymptotically, internally) stable. The resulting control system is shown in Fig. 1.

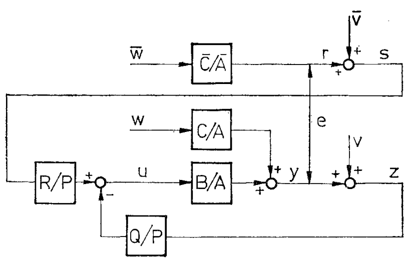

Fig. 1. Stochastic Servo System.

This problem of simultaneous regulation and tracking will be referred to as the stochastic servo problem; if there is no reference ($s = 0$) we speak of the associated regulator problem.

3. THE REGULATOR PROBLEM

It will be seen in the next section that any servo problem can be reformulated as a regulator problem for an appropriately defined system. The purpose of this section, therefore, is to summarize the statement and solution of the regulator problem for multivariate processes as developed by Kučera [2; 3].

Let the process be modeled by

$$(7)\qquad \hat{A}(d)\,\hat{y} = \hat{B}(d)\,\hat{u} + \hat{C}(d)\,\hat{w}$$

where $\hat{y}$ is the l-vector output sequence, $\hat{u}$ is the m-vector input sequence, and $\hat{w}$ is a zero-mean covariance-stationary white n-vector sequence with covariance matrix Ψ. The observed output $\hat{z}$ is assumed in the form

$$\hat{z} = \hat{y} + \hat{v} \tag{8}$$

where $\hat{v}$ is another zero-mean covariance-stationary white random l-vector sequence, uncorrelated with $\hat{w}$ and having covariance matrix Φ. The $\hat{A}$, $\hat{B}$, and $\hat{C}$ are left coprime polynomial matrices, of compatible dimensions, in the delay operator d. We assume that $\hat{A}(0)$ is invertible, $\hat{B}(0) = 0$, and that the process model is devoid of unstable hidden modes.

The optimal control law

$$\hat{P}(d)\,\hat{u} = -\hat{Q}(d)\,\hat{z} \tag{9}$$

with $\hat{P}(0)$ invertible is obtained by minimizing the cost

$$J = \operatorname{tr}\,(\Lambda J_u + M J_y) \tag{10}$$

subject to stability of the closed-loop system (7)–(9). The J_u and J_y are covariance matrices of the plant input and output, respectively, in steady state while Λ and M are nonnegative definite weighting matrices.

For any polynomial matrix H, define $H_*(d) = H^T(d^{-1})$. The design procedure then consists of the following steps:

1° Calculate right coprime polynomial matrices $\hat{A}_1$ and $\hat{B}_1$ satisfying

$$\hat{A}^{-1}\hat{B} = \hat{B}_1\hat{A}_1^{-1}$$

2° Calculate polynomial matrices $\hat{F}$ and $\hat{G}$ with stable determinants, called spectral factors, defined by

$$\hat{A}_{1*}\Lambda\hat{A}_1 + \hat{B}_{1*}M\hat{B}_1 = \hat{F}_*\hat{F}$$

$$\hat{A}\Phi\hat{A}_* + \hat{C}\Psi\hat{C}_* = \hat{G}\hat{G}_*$$

3° Determine polynomial matrices $\hat{G}_1$, $\hat{G}_2$ and $\hat{B}_2$, $\hat{A}_2$ satisfying

$$\hat{G}^{-1}\hat{B} = \hat{B}_2\hat{G}_1^{-1}\,, \quad \hat{G}^{-1}\hat{A} = \hat{A}_2\hat{G}_2^{-1}$$

4° Solve the equations

$$\hat{F}_*\hat{X} - \hat{Z}_*\hat{B}_2 = \hat{A}_{1*}\Lambda\hat{G}_1$$

$$\hat{F}_*\hat{Y} + \hat{Z}_*\hat{A}_2 = \hat{B}_{1*}M\hat{G}_2$$

for polynomial matrices $\hat{X}$, $\hat{Y}$, and $\hat{Z}$ such that

$$\hat{Z}(0) = 0\,.$$

5° Determine polynomial matrices G_0 and X_0, Y_0 such that

$$[\hat{X}\hat{G}_1^{-1} \ \hat{Y}\hat{G}_2^{-1}] = G_0^{-1}[X_0 \ Y_0].$$

6° **The optimal control law is then** given by

$$\hat{P} = X_0, \quad \hat{Q} = Y_0$$

and must be realized without unstable hidden modes.

The above optimal solution exists whenever step 2° can be performed and if it does exist, it is unique.

4. THE SERVO PROBLEM

The stochastic servo problem can be reformulated as a regulator problem by augmenting the plant and reference into a single system (7) as follows:

$$\hat{y} = \begin{bmatrix} y \\ r \end{bmatrix}, \quad \hat{u} = u, \quad \hat{w} = \begin{bmatrix} w \\ \bar{w} \end{bmatrix}$$

hence inducing

$$\hat{z} = \begin{bmatrix} z \\ s \end{bmatrix}, \quad \hat{v} = \begin{bmatrix} v \\ \bar{v} \end{bmatrix}$$

in (8). The system matrices then become

$$\hat{A} = \begin{bmatrix} A & 0 \\ 0 & \bar{A} \end{bmatrix}, \quad \hat{B} = \begin{bmatrix} B \\ 0 \end{bmatrix}, \quad \hat{C} = \begin{bmatrix} C & 0 \\ 0 & \bar{C} \end{bmatrix}$$

and

$$\Phi = \begin{bmatrix} \varphi & 0 \\ 0 & \bar{\varphi} \end{bmatrix}, \quad \Psi = \begin{bmatrix} \psi & 0 \\ 0 & \bar{\psi} \end{bmatrix}.$$

The definition of the tracking error implies the weighting matrices

$$\Lambda = [\lambda], \quad M = \begin{bmatrix} \mu & -\mu \\ -\mu & \mu \end{bmatrix}$$

in the measure of performance (10) and, finally, the matrices in the control law (9) can be identified as follows:

$$\hat{P} = [P], \quad \hat{Q} = [Q \ -R].$$

Now, following the steps of the design procedure given in the preceding section, we first determine the matrices

$$\hat{B}_1 = \begin{bmatrix} B \\ 0 \end{bmatrix}, \quad \hat{A}_1 = [A]$$

The spectral factors are

$$\hat{F} = [F], \quad \hat{G} = \begin{bmatrix} G & 0 \\ 0 & \bar{G} \end{bmatrix}$$

where F, G, and $\bar{G}$ are stable polynomials satisfying

(11)
$$\begin{aligned} A_*\lambda A + B_*\mu B &= F_*F \\ A\varphi A_* + C\psi C_* &= GG_* \\ \bar{A}\bar{\varphi}\bar{A}_* + \bar{C}\bar{\psi}\bar{C}_* &= \bar{G}\bar{G}_* \end{aligned}$$

Further determine the matrices

$$\hat{B}_2 = \begin{bmatrix} B \\ 0 \end{bmatrix}, \quad \hat{G}_1 = [G]$$

and

$$\hat{A}_2 = \begin{bmatrix} A & 0 \\ 0 & \bar{A} \end{bmatrix}, \quad \hat{G}_2 = \begin{bmatrix} G & 0 \\ 0 & \bar{G} \end{bmatrix}$$

and obtain the polynomial equations

(12)
$$\begin{aligned} F_*X - Z_*B &= A_*\lambda G \\ F_*Y + Z_*A &= B_*\mu G \\ F_*\bar{Y} + \bar{Z}_*\bar{A} &= B_*\mu\bar{G} \end{aligned}$$

Among all solutions

$$\hat{X} = [X], \quad \hat{Y} = [Y \; -\bar{Y}] \quad \hat{Z} = \begin{bmatrix} Z \\ -\bar{Z} \end{bmatrix}$$

choose the one satisfying

(13)
$$Z(0) = 0, \quad \bar{Z}(0) = 0$$

and the optimal control law is obtained in the form

(14)
$$P = X\bar{G}, \quad Q = Y\bar{G}, \quad R = \bar{Y}G$$

Let us now summarize:

(15) Design Procedure for the Servo Problem

1° Perform the spectral factorizations (11) to obtain stable polynomials $F, G,$ and $\bar{G}$.

2° Solve the equations (12) for polynomials X, Y, Z and $\bar{Y}$, $\bar{Z}$ such that relations (13) are satisfied.

3° The optimal control law (5) is then given by (14).

This optimal control law exists whenever step 1° can be executed and if it does exist, it is unique. Needless to say, it must be realized without unstable hidden modes. Moreover, it can be shown that the minimized value J of (6) equals

$$J = J_0 + \bar{J}_0 \tag{16}$$

where

$$J_0 = \left\langle \frac{Z_* Z}{F_* F} \right\rangle + \mu \left\langle \lambda \frac{G G_*}{F_* F} - \varphi \right\rangle$$

$$\bar{J}_0 = \left\langle \frac{\bar{Z}_* \bar{Z}}{F_* F} \right\rangle + \mu \left\langle \lambda \frac{\bar{G} \bar{G}_*}{F_* F} \frac{A_* A}{\bar{A} \bar{A}_*} - \bar{\varphi} \right\rangle$$

using the notation $\langle H \rangle = H(0)$ for any H.

For the associated regulator problem, we have $\bar{G} = 0$ by convention. It follows that $\bar{Y} = 0$, $\bar{Z} = 0$ and the resulting optimal control law is a purely feedback one:

$$P = X, \quad Q = Y, \quad R = 0$$

Several remarks are now in order. First of all, the optimal control law for the servo problem consists, in general, of the feedback part Q and the feedforward part R. It does not operate on the tracking error $e = y - r$ but rather it operates on y and r in different ways. Therefore, every control configuration using only error feedback to regulate and track must be suboptimal.

A close examination of the design procedure (15) reveals that the last equation (12) is independent of the preceding two. These two equations, however, yield a solution to the associated regulator problem. This means that the feedback part of the control law is independent of the reference sequence.

5. EXAMPLE

To illustrate the design procedure, consider a stochastic plant modeled by the controlled ARMA process (1) with

$$A = 2 - d, \quad B = 2d - d^2, \quad C = 1$$

and a random reference modeled by (3) with

$$\bar{A} = 3 - d, \quad \bar{C} = 3$$

The random sources have the following variances:

$$\varphi = 2, \quad \psi = 7$$

and

$$\bar{\varphi} = 0\,, \quad \bar{\psi} = 1$$

The purpose of the control is to make the output of the plant follow the given reference in the minimum variance sense. Thus we take the weighting coefficients

$$\lambda = 0\,, \quad \mu = 1$$

in the measure of performance (6).

Following the design procedure (15), we first calculate the spectral factors F, G, and $\bar{G}$ defined in (11):

$$\begin{aligned}(2d^{-1} - d^{-2})(2d - d^2) &= F_*F \\ (2 - d^{-1})\,2(2 - d) + 7 &= GG_* \\ 3\,.\,3 &= \bar{G}\bar{G}_*\end{aligned}$$

We obtain quite easily

$$F = 2 - d\,, \quad G = 4 - d\,, \quad \bar{G} = 3$$

Now we form the equations (12):

$$\begin{aligned}(2 - d^{-1})X - Z_*(2d - d^2) &= 0 \\ (2 - d^{-1})\,Y + Z_*(2 - d) &= (2d^{-1} - d^{-2})(4 - d) \\ (2 - d^{-1})\,\bar{Y} + \bar{Z}_*(3 - d) &= (2d^{-1} - d^{-2})\,3\end{aligned}$$

The general solution of the first two equations reads

$$\begin{aligned}X &= 4 - d^2 - (2d - d^2)\,\tau \\ Y &= -1 + d + (2 - d)\,\tau \\ Z &= 2 + 3d - 2d^2 - (2 - d)\,\tau\end{aligned}$$

and that of the third equation becomes

$$\begin{aligned}\bar{Y} &= 4 - d + (3 - d)\,\bar{\tau} \\ \bar{Z} &= -2 + 3d - d^2 - (2 - d)\,\bar{\tau}\end{aligned}$$

for any reals τ and $\bar{\tau}$. The solution satisfying (13) is obtained on putting $\tau = 1, \bar{\tau} = -1$:

$$X = 4 - 2d\,, \quad Y = 1\,, \quad Z = 4d - 2d^2$$

and

$$\bar{Y} = 1\,, \quad \bar{Z} = 2d - d^2$$

Thus the optimal control law (5) is given by (14):

$$P = 12 - 6d\,, \quad Q = 3\,, \quad R = 4 - d$$

and it must be realized without unstable hidden modes. A possible realization is shown in Fig. 2. The resultant steady-state variance of the tracking error is given by (16):

$$J = 2 + 1 = 3$$

6. CONCLUSION

A simple optimal solution to the stochastic servo problem has been presented for discrete-time linear single-input-output plants and covariance-stationary random inputs, the optimality being taken in the minimum variance sense. The problem was solved by reformulating it as a regulator problem for multivariate processes and then

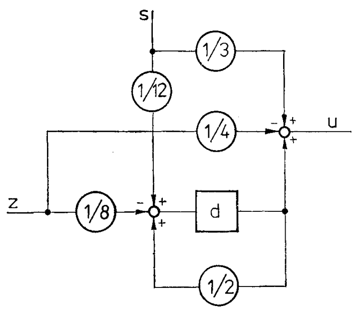

Fig. 2. A Realization of the Control Law.

applying the polynomial equation approach. The design procedure consists in solving linear polynomial equations whose polynomial coefficients are obtained by spectral factorization. The resultant control law was shown to contain both feedback and feedforward terms, the feedback one being independent of the reference to be followed. Thus any control law based upon the error feedback must be suboptimal.

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