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Locally Best Unbiased Estimates of Functionals of Covariance Functions of a Gaussian Stochastic Process

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Using the RKHS (Reproducing Kernel Hilbert Space) methods the characterization of the locally best unbiased estimable functionals of unknown covariance function of a Gaussian stochastic is given.

1. INTRODUCTION

The theory of locally best unbiased estimates was founded by Barankin [2]. Parzen [9] investigated the connection between this theory and the theory of RKHS. Parzen [9], Kailath-Duttweiler [3] and the autor [12] utilised the theory to the problem of unbiased estimation of functionals of unknown mean value function of a Gaussian random process. The aim of this paper is to characterize the locally best unbiased estimable functionals of an unknown covariance function of a Gaussian stochastic process $X = \{X(t);\ t \in [0, T]\}$ having its mean value function identically equal to zero. The unknown covariance function of the process X is assumed to be of the type $R(s, t) = \sum_{k=1}^{\infty} \lambda_k\, e_k(s)\, e_k(t)$, where the $\{\lambda_k\}_{k=1}^{\infty}$ are unknown real numbers such that $\lambda_k > 0$; $k = 1, 2, \ldots$ $\sum_{k=1}^{\infty} \lambda_k < \infty$, and $\{e_k\}_{k=1}^{\infty}$ is a known complete orthonormal system in $L^2([0, T])$.

2. GENERAL THEORY OF LOCALLY BEST UNBIASED ESTIMATES

Now we shall give a brief review of the general theory of locally best unbiased estimates following [9]. Let $\{P_\theta,\ \theta \in \Theta\}$ be a parametric set of probability measures and let $\theta_0 \in \Theta$ be fixed. It is assumed that, for every $\theta \in \Theta$, the measure P_θ is absolutely continuous with respect to P_{θ_0} and the function $\mathrm{d}P_\theta/\mathrm{d}P_{\theta_0}$ belongs to the $L^2(P_{\theta_0})$ space for every $\theta \in \Theta$. Denote by $L^2_{\theta_0}$ the subspace of $L^2(P_{\theta_0})$ generated by the set of functions $\{\mathrm{d}P_\theta/\mathrm{d}P_{\theta_0};\ \theta \in \Theta\}$. Then for the function $f : \Theta \to E^1$ there exists an unbiased estimate having finite variance at θ_0 if and only if $f \in H(K_{\theta_0})$, where

$$K_{\theta_0}(\theta, \theta') = E_{\theta_0}\left[\frac{\mathrm{d}P_\theta}{\mathrm{d}P_{\theta_0}} \cdot \frac{\mathrm{d}P_{\theta'}}{\mathrm{d}P_{\theta_0}}\right];$$

$\theta, \theta' \in \Theta$ is a reproducing kernel of the RKHS $H(K_{\theta_0})$. The spaces $H(K_{\theta_0})$ and $L^2_{\theta_0}$ are isomorphic. For every function $f \in H(K_{\theta_0})$ there exists a random variable $V \in L^2_{\theta_0}$ – the isomorphic image of f, such that

$$E_\theta[V] = E_{\theta_0}\left[V \cdot \frac{\mathrm{d}P_\theta}{\mathrm{d}P_{\theta_0}}\right] = f(\theta)$$

for every $\theta \in \Theta$ and

$$\|f\|^2_{H(K_{\theta_0})} = E_{\theta_0}[V^2] \leqq E_{\theta_0}[U^2]$$

for every $U \in L^2[P_{\theta_0}]$ having the property $E_\theta[U] = f(\theta)$ for all $\theta \in \Theta$.

3. LOCALLY BEST UNBIASED ESTIMATES OF DISPERSION

Now we shall study the simplest case of estimation of functions of dispersion. Let X be a $\mathrm{N}(0, \sigma^2)$ distributed random variable. Then for any $\sigma_0^2, \sigma^2 > 0$ the measure P_σ given on the Borel sets of real line by

$$P_\sigma(A) = \frac{1}{\sqrt{(2\pi)}\,\sigma} \int_A \mathrm{e}^{(-x^2/2\sigma^2)}\,\mathrm{d}x$$

is absolutely continuous with respect to P_{σ_0} and $\mathrm{d}P_\sigma/\mathrm{d}P_{\sigma_0}$ belongs to $L^2[P_{\sigma_0}]$ if and only if $0 < \sigma^2 < 2\sigma_0^2$. Accordingly, we have

$$K_{\sigma_0}(\sigma, \sigma') = E_{\sigma_0}\left[\frac{\mathrm{d}P_\sigma}{\mathrm{d}P_{\sigma_0}} \cdot \frac{\mathrm{d}P_{\sigma'}}{\mathrm{d}P_{\sigma_0}}\right] = \frac{\sigma_0^2}{\sigma\sigma'\sigma_0} \frac{1}{\sqrt{(2\pi)}} \int_{-\infty}^{\infty} \mathrm{e}^{[-x^2/2\,(1/\sigma^2 + 1/\sigma'^2 - 1/\sigma_0^2)]}\,\mathrm{d}x =$$

$$= \left[\frac{\sigma_0^4}{(\sigma'\sigma_0)^2 + (\sigma\sigma_0)^2 - (\sigma\sigma')^2}\right]^{1/2} = \left[1 - \frac{\sigma^2 - \sigma_0^2}{\sigma_0^2} \cdot \frac{\sigma'^2 - \sigma_0^2}{\sigma_0^2}\right]^{-1/2},$$

where $0 < \sigma^2, \sigma'^2 < 2\sigma_0^2$. We shall now characterize the space $H(K_{\sigma_0})$. To do this we need the following lemma.

Lemma 1. Let H be a Hilbert space of functions which are analytic in the unit circle $E = \{z = x + \mathrm{i}y : |z| < 1\}$ and such that $\iint_E |f(z)|^2\,\mathrm{d}x\,\mathrm{d}y < \infty$. Then the system

$$\{\varphi_n(z) = (n/\pi)^{1/2} z^{n-1}\}_{n=1}^{\infty}$$

is a complete orthonormal system in H endowed by the inner product $(f, g)_H = \iint_E f \cdot \bar{g} \, dx \, dy$. Moreover, $H = H(K_0)$ with

$$K_0(z, \bar{u}) = \sum_{n=1}^{\infty} \varphi_n(z) \, \varphi_n(\bar{u}) = \sum_{n=1}^{\infty} \frac{n}{\pi} z^{n-1} \bar{u}^{n-1} = \frac{1}{\pi(1 - z\bar{u})^2} \, ; \quad z, u \in E \, .$$

Proof. Meschkowski [7].

Now let $\mathscr{E}_{\sigma_0} = \{w: |w - \sigma_0^2| < \sigma_0^2\}$ be the circle centred at σ_0^2 and of the radius σ_0^2 and let $h : \mathscr{E}_{\sigma_0} \to E$ be a transformation given by $h(w) = (w - \sigma_0^2)/\sigma_0^2$. Then the following lemma is true.

Lemma 2. Let H_{σ_0} be the Hilbert space of functions that are analytic in the circle $\mathscr{E}_{\sigma_0}$ and such that $\iint_{\mathscr{E}_{\sigma_0}} |f(w)|^2 \, dx \, dy < \infty$. Then the system

$$\left\{\psi_n(w) = \left(\frac{n}{\pi}\right)^{1/2} h(w)^{n-1} \cdot \frac{dh}{dw}\right\}_{n=1}^{\infty} = \left\{\left(\frac{n}{\pi}\right)^{1/2} \cdot \frac{1}{\sigma_0^2} \cdot \left(\frac{w - \sigma_0^2}{\sigma_0^2}\right)^{n-1}\right\}_{n=1}^{\infty}$$

is a complete orthonormal system in H_{σ_0}, $H_{\sigma_0} = H(K^*_{\sigma_0})$, where

$$K^*_{\sigma_0}(w, v) = \frac{h'(w) \, \overline{h'(v)}}{\pi(1 - h(w) \, \overline{h(v)})^2} = \frac{1}{\sigma_0^4 \pi} \frac{1}{\left[1 - \dfrac{(w - \sigma_0^2)}{\sigma_0^2} \cdot \overline{\dfrac{v - \sigma_0^2}{\sigma_0^2}}\right]^2} = \sum_{n=1}^{\infty} \psi_n(w) \, \psi_n(v) \, .$$

Proof. Meschkowski [7].

Now let

$$K(w, v) = \left[1 - \frac{w - \sigma_0^2}{\sigma_0^2} \cdot \overline{\frac{v - \sigma_0^2}{\sigma_0^2}}\right]^{-1/2} ; \quad w, v \in \mathscr{E}_{\sigma_0} \, .$$

Then we have:

$$K(w, v) = \frac{d}{dz} \arc\sin z \, \Bigg| \, \frac{w - \sigma_0^2}{{}^2_0} \cdot \overline{\frac{v - \sigma_0^2}{\sigma_0^2}} = 1 + \frac{1}{2} \frac{w - \sigma_0^2}{\sigma_0^2} \cdot \overline{\frac{v - \sigma_0^2}{\sigma_0^2}} +$$

$$+ \frac{1.3}{2.4} \left(\frac{w - \sigma_0^2}{\sigma_0^2} \cdot \overline{\frac{v - \sigma_0^2}{\sigma_0^2}}\right)^2 + \ldots = \sum_{n=1}^{\infty} c_n \left(\frac{w - \sigma_0^2}{\sigma_0^2} \overline{\frac{v - \sigma_0^2}{\sigma_0^2}}\right)^{n-1} = \sum_{n=1}^{\infty} d_n \, \psi_n(w) \, \overline{\psi_n(v)} \, ,$$

where

$$c_1 = 1 \, ; \quad c_n = \frac{(2n - 3)!!}{(2n - 2)!!} = \frac{(2n - 3) \ldots 3.1}{(2n - 2) \ldots 4.2} \quad \text{for} \quad n \geqq 2$$

and

$$d_n = \frac{\pi c_n \sigma_0^4}{n} \, ; \quad n = 1, 2, \ldots \, .$$

From the expression $K(w, v) = \sum_{n=1}^{\infty} d_n \, \psi_n(w) \, \overline{\psi_n(v)}$ we have the following characterization (see Aronszajn [1]) of $H(K)$:

$$H(K) = \{f \in H_{\sigma_0} : \sum_{n=1}^{\infty} \frac{|(f, \psi_n)_{H\sigma_0}|^2}{d_n} < \infty\}.$$

Using the fact that $f \in H_{\sigma_0}$ is analytic in $\mathscr{E}_{\sigma_0}$ we get

$$f(w) = \sum_{k=0}^{\infty} \frac{f^{(k)}(\sigma_0^2)}{k!} (w - \sigma_0^2)^k = \sum_{n=1}^{\infty} \frac{(\sigma_0^2)^{n-1} f^{(n-1)}(\sigma_0^2)}{(n-1)!} \cdot$$

$$\cdot \left(\frac{\pi}{n}\right)^{1/2} \sigma_0^2 \cdot \psi_n(w) = \sum_{n=1}^{\infty} (f, \psi_n)_{H\sigma_0} \cdot \psi_n(w)$$

we get for

$$H(K) : H(K) = \{f \in H_{\sigma_0} : \sum_{n=1}^{\infty} \frac{(\sigma_0^2)^{2n-2} [f^{(n-1)}(\sigma_0^2)]^2}{[(n-1)!]^2 c_n} = \|f\|^2_{H(K)} < \infty\}.$$

According to the uniqueness extension theorem (Saks [10]) and the restriction theorem (see [1]) in RKHS, the following theorem giving the characterization of $H(K_{\sigma_0})$ can be proved.

Theorem 1. A function $f : (0, 2\sigma_0^2) \to E^1$ has an unbiased estimate with finite dispersion at σ_0^2 if and only if it can be extended to an analytic function in $\mathscr{E}_{\sigma_0}$ such that

$$\|f\|^2_{H(K_{\sigma 0})} = \sum_{n=1}^{\infty} \frac{(\sigma_0^2)^{2(n-1)} [f^{(n-1)}(\sigma_0^2)]^2}{[(n-1)!]^2 c_n} < \infty,$$

where

$$c_1 = 1; \quad c_n = \frac{(2n-3)!!}{(2n-2)!!} \quad \text{for} \quad n \geqq 2.$$

It is easy to see that every polynomial f_n given by

$$f_n(\sigma^2) = \sum_{k=0}^{n} a_k \sigma^{2k}; \quad 0 < \sigma^2 < 2\sigma_0^2$$

belongs to the space $H(K_{\sigma_0})$. Especially, let $h_k(\sigma^2) = (\sigma^2)^k$. Then we have

$$\langle h_k, h_l \rangle_{H(K_{\sigma 0})} = \sum_{n=1}^{\infty} \frac{(\sigma_0^2)^{2(n-1)} h_k^{(n-1)}(\sigma_0^2) h_k^{(n-1)}(\sigma_0^2)}{[(n-1)!]^2 c_n} =$$

$$= \sum_{n=1}^{\min\{k+1, l+1\}} \frac{(\sigma_0^2)^{k+l} \binom{k}{n-1} \binom{l}{n-1}}{c_n}; \quad k, l = 0, 1, 2, \ldots.$$

For $k = 1$ we have:

$$\|h_1\|^2_{H(K_{\sigma 0})} = \sigma_0^4 \sum_{n=1}^{2} \binom{1}{n-1}^2 \cdot \frac{1}{c_n} = 3\sigma_0^4 = E_{\sigma_0}[X^4]$$

so that the random variable X^2 is the locally best unbiased estimate of h_1 at σ_0^2 for h_1 defined by $h_1(\sigma^2) = \sigma^2$; $0 < \sigma^2 < 2\sigma_0^2$. Because this estimate does not depend on σ_0^2 and it is unbiased estimate for the function $f(\sigma^2) = \sigma^2$, $\sigma^2 > 0$ we get the known result that X^4 is the uniformly best unbiased estimate for f.

Now we shall prove that the random variable $U_k = (X^2)^k/(2k-1)!!$ is the uniformly best unbiased estimate for $h_k(\sigma^2) = (\sigma^2)^k$; $\sigma^2 > 0$. Because

$$E_\sigma[U_k] = E_{\sigma_0}[U_k \,.\, \mathrm{d}P_\sigma/\mathrm{d}P_{\sigma_0}] = h_k(\sigma^2) \quad \text{for} \quad 0 < \sigma^2 < 2\sigma_0^2\,;$$

$\sigma_0^2 > 0$, it is enough to prove that U_k belongs to the space $L^2_{\sigma_0}$ for every $\sigma_0^2 > 0$.

Lemma 3. Let $U_k = (X^2)^k$; $k = 0, 1, 2, \ldots$. Then $U_k \in L^2_{\sigma_0}$ for every $\sigma_0 > 0$.

Proof. For $k = 0, 1, 2$ it is easy to prove that $\|U_k\|^2_{L^2(P_{\sigma 0})} = \|h_k\|^2_{H(K_{\sigma 0})}$ and the lemma is proved. For $k \geqq 3$ we can proceed by induction. Because $L^2_{\sigma_0}$ is a subspace of $L^2[P_{\sigma_0}]$, $U_k \in L^2_{\sigma_0}$ iff $U_k \in (L^{2\perp}_{\sigma_0})^\perp$, that is iff $(U_k, V)_{L^2(P_{\sigma 0})} = 0$ for all $V \in L^2(P_{\sigma_0})$ such that

$$\left(V, \frac{\mathrm{d}P_\sigma}{\mathrm{d}P_{\sigma_0}}\right)_{L^2(P_{\sigma 0})} = 0 \quad \text{for all} \quad 0 < \sigma^2 < 2\sigma_0^2\,.$$

Because

$$(U_k, V)_{L^2(P_{\sigma 0})} = (U_{k-1}, U_1 V)_{L^2(P_{\sigma 0})}$$

it is enough to prove according to the induction assumption that $U_1 V \in (L^2_{\sigma_0})^\perp$ if $V \in (L^2_{\sigma_0})^\perp$, so that

$$\int U_1 V \frac{\mathrm{d}P_\sigma}{\mathrm{d}P_{\sigma_0}} \,.\, \mathrm{d}P_{\sigma_0} = 0 \quad \text{for all} \quad 0 < \sigma^2 < 2\sigma_0^2$$

if $V \in (L^2_{\sigma_0})^\perp$. This will be done for $\sigma_0^2 = 1$. In this case the system of Hermite polynomials $\{H_n(x)\}_{n=0}^\infty$ is known to be a complete orthonormal system in $L^2(P_{\sigma_0})$ and we can write:

$$\left(V, \frac{\mathrm{d}P_\sigma}{\mathrm{d}P_{\sigma_0}}\right)_{L^2(P_{\sigma 0})} = \sum_{n=0}^\infty (V, H_n) \,.\, \left(\frac{\mathrm{d}P_\sigma}{\mathrm{d}P_{\sigma_0}}, H_n\right) = \sum_{n=0}^\infty (V, H_n) \,.\, E_{P_\sigma}[H_n]\,.$$

For $n = 2k + 1$; $k = 0, 1, \ldots$ we have

$$\int_{-\infty}^{\infty} H_{2k+1}(x) \frac{1}{\sqrt{(2\pi)}\,\sigma} \mathrm{e}^{(-x^2/2\sigma^2)}\,\mathrm{d}x = 0\,,$$

because $H_{2k+1}(x)$ is a polynomial in x containing only odd powers of x. We shall prove now by induction

$$E_{P_\sigma}[H_{2k}] = \int_{-\infty}^{\infty} H_{2k}(x) \frac{1}{\sqrt{(2\pi)}\,\sigma} \mathrm{e}^{(-x^2/2\sigma^2)}\,\mathrm{d}x = \left(\frac{(2k-1)!!}{2k!!}\right)^{1/2} (\sigma^2 - 1)^k\,.$$

The relation is true for $k = 1$, because

$$\frac{1}{\sqrt{(2\pi)}\,\sigma}\int_{-\infty}^{\infty}\frac{1}{\sqrt{2!}}\,(x^2-1)\,e^{(-x^2/2\sigma^2)}\,dx = \frac{\sigma^2-1}{\sqrt{2!!}}\,.$$

Let the relation be true for $k = n - 1$. Then, because

$$H_{n+2}(x) = \frac{1}{\sqrt{(n+2)}}\,(xH_{n+1}(x) - H'_{n+1}(x))$$

and $H'_n(x) = \sqrt{(n)}\,H_{n-1}(x)$ (see Jarník [5]), we get

$$\int_{-\infty}^{\infty} H_{2n}(x)\,\frac{1}{\sqrt{(2\pi)}\,\sigma}\,e^{(-x^2/2\sigma^2)}\,dx = \int_{-\infty}^{\infty}\frac{1}{\sqrt{(2n)}}\,[x\,H_{2n-1}(x) - H'_{2n-1}(x)]\,.$$

$$.\,\frac{1}{\sqrt{(2\pi)}\,\sigma}\,e^{(-x^2/2\sigma^2)}\,dx = -\sigma^2\int_{-\infty}^{\infty}\frac{1}{\sqrt{(2n)}}\,H_{2n-1}(x)\left(\frac{1}{\sqrt{(2\pi)}\,\sigma}\,e^{(-x^2/2\sigma^2)}\right)dx -$$

$$-\frac{1}{\sqrt{(2n)}}\int_{-\infty}^{\infty} H'_{2n-1}(x)\,\frac{1}{\sqrt{(2\pi)}\,\sigma}\,e^{(-x^2/2\sigma^2)}\,dx = \frac{\sigma^2-1}{\sqrt{(2n)}}\,.$$

$$.\int_{-\infty}^{\infty} H'_{2n-1}(x)\,\frac{1}{\sqrt{(2\pi)}\,\sigma}\,e^{(-x^2/2\sigma^2)}\,dx = \left[\frac{(2n-1)!!}{(2n)!!}\right]^{1/2}(\sigma^2-1)^n\,.$$

According to this we have $V \in (L^2_{\sigma_0})^{\perp}$ iff

$$0 = \left(V, \frac{dP_\sigma}{dP_{\sigma_0}}\right)_{L^2(P_{\sigma_0})} = \sum_{n=0}^{\infty}(V, H_{2n})\left[\frac{(2n-1)!!}{(2n)!!}\right]^{1/2}(\sigma^2-1)^n$$

for all $0 < \sigma^2 < 2$. Hence $(V, H_{2n}) = 0$ for $n = 0, 1, \ldots$ and $V(x) = \sum_{n=0}^{\infty}(V, H_{2n+1})\,.$ $.\,H_{2n+1}(x)$. For $U_1(dP_\sigma/dP_{\sigma_0})$ we get:

$$\left(U_1\,\frac{dP_\sigma}{dP_{\sigma_0}},\, H_{2n+1}\right) = \int_{-\infty}^{\infty} x^2\,H_{2n+1}(x)\,dP_\sigma(x) = 0 \quad \text{for} \quad n = 1, 2, \ldots$$

and using this we obtain:

$$\left(VU_1, \frac{dP_\sigma}{dP_{\sigma_0}}\right)_{L^2(P_{\sigma_0})} = \left(V, U_1\,\frac{dP_\sigma}{dP_{\sigma_0}}\right)_{L^2(P_{\sigma_0})} = \sum_{n=0}^{\infty}(V, H_n)\left(U_1\,\frac{dP_\sigma}{dP_{\sigma_0}},\, H_n\right) = 0$$

for all $0 < \sigma^2 < 2$, and the proof of the lemma for $\sigma_0^2 = 1$ is finished. If $\sigma_0^2 \neq 1$, then we use fact that $\{G_n(x) = 1/\sigma_0\, H_n(x/\sigma_0)\}_{n=0}^{\infty}$ is a complete orthonormal system in $L^2(P_{\sigma_0})$ and the proof is analogous to that given.

Corollary 1. For any nonnegative integers k, l the following combinatorial identity is true:

$$\sum_{n=1}^{\min\{k+1,l+1\}} \binom{k}{n-1}\binom{l}{n-1} c_n^{-1} = \frac{(2k+2l-1)!!}{(2k-1)!!\,(2l-1)!!},$$

where

$$c_1 = 1\,; \quad c_n = \frac{(2n-3)!!}{(2n-2)!!} \quad \text{for} \quad n \geqq 2\,.$$

Proof. The left-hand side of the identity is equal to $1/(\sigma_0^2)^{k+l} \langle h_k, h_l\rangle_{H(K_{\sigma 0})}$; the right-hand side equals to:

$$\frac{1}{(\sigma_0^2)^{k+l}} E_{\sigma_0}\left[\frac{U_k}{(2k-1)!!}\cdot\frac{U_l}{(2l-1)!!}\right] = \frac{1}{(\sigma_0^2)^{k+l}} E_{\sigma_0}\left[\frac{U_{k+l}}{(2k-1)!!\,(2l-1)!!}\right]$$

and these expressions equal each other, according to the Lemma 3.

Corollary 2. If $f \in H(K_{\sigma_0})$, then its locally best unbiased estimate U_{σ_0} at σ_0 is given by

$$U_{\sigma_0} = \sum_{n=0}^{\infty} \frac{f^{(n)}(\sigma_0^2)}{n!} \sum_{j=0}^{n} \binom{n}{j} (-1)^{n-j} \frac{X^{2j}}{(2j-1)!!} (\sigma_0^2)^{n-j}\,,$$

where the series converges in $L^2(P_{\sigma_0})$.

Proof. If $f \in H(K_{\sigma_0})$, then

$$f(\sigma^2) = \sum_{n=0}^{\infty} \frac{f^{(n)}(\sigma_0^2)}{n!} (\sigma^2 - \sigma_0^2)^n = \sum_{n=0}^{\infty} \frac{f^{(n)}(\sigma_0^2)}{n!} \cdot \frac{(\sigma_0^2)^n}{\sqrt{c_{n+1}}} \varkappa_{n+1}(\sigma^2)$$

where

$$\left\{\varkappa_{n+1}(\sigma^2) = \sqrt{(d_{n+1})}\, \psi_{n+1}(\sigma^2) = \sqrt{(c_{n+1})} \left(\frac{\sigma^2 - \sigma_0^2}{\sigma_0^2}\right)^n\right\}_{n=0}^{\infty}$$

is a complete orthonormal system in $H(K_{\sigma_0})$. According to the isomorphism between $H(K_{\sigma_0})$ and $L^2_{\sigma_0}$ we get the desired result, because the system of random variables

$$\left\{\frac{\sqrt{c_{n+1}}}{(\sigma_0^2)^n} \sum_{j=0}^{n} \binom{n}{j} \frac{X^{2j}}{(2j-1)!!} (-1)^{n-2j} (\sigma_0^2)^{2j}\right\}_{n=0}^{\infty}$$

is a complete orthonormal system in $L^2_{\sigma_0}$ for every $\sigma_0^2 > 0$.

Example 1. Let $f(\sigma^2) = (\sigma^2)^{1/2}$. The function $f(z) = z^{1/2}$ is analytic in every circle $\mathscr{E}_{\sigma_0}$, $\sigma_0 > 0$ and

$$f(\sigma^2) = \sum_{n=0}^{\infty} \frac{f^{(n)}(\sigma_0^2)}{n!} (\sigma^2 - \sigma_0^2)^n = \sum_{n=0}^{\infty} \frac{a_n}{n!} (\sigma^2 - \sigma_0^2)^n\,,$$

where

$$a_0 = \sigma_0\,, \quad a_1 = \frac{1}{2}(\sigma_0)^{-1/2}\,, \quad a_n = (-1)^{n+1}\frac{(2n-3)!!}{2^n}(\sigma_0^2)^{-(2n-1)/2}$$

for $n = 2, 3, \ldots$. The series

$$\sum_{n=0}^{\infty} \frac{a_n^2(\sigma_0^2)^{2n}}{(n!)^2\, c_{n+1}}$$

converges, because for $n \geqq 2$ we have

$$\frac{a_n^2(\sigma_0^2)^{2n}}{(n!)^2\, c_{n+1}} = \sigma_0^2\,\frac{((2n-3)!!)^2(2n)!!}{2^{2n}(n!)^2\,(2n-1)!!} = \sigma_0^2\,\frac{(2n-1)!!\,(2n)!!}{2^{2n}(n!)^2\,(2n-1)^2} =$$

$$= \sigma_0^2\,\frac{(2n-1)!!\,2^n n!}{2^{2n}(n!)^2\,(2n-1)^2} = \sigma_0^2\,\frac{(2n-3)!!}{2^n n!\,(2n-1)} \leqq \sigma_0^2\,\frac{(2n-2)!!}{2^n n!\,(2n-1)} =$$

$$= \frac{\sigma_0^2}{2}\,\frac{1}{2n^2-n} \leqq \frac{\sigma_0^2}{2n^2}$$

and the function $f(.)$ has the locally best unbiases estimate at σ_0^2.

Remark. The theory just derived can be used in the case of a random sample, too. If $X_1, \ldots, X_n$ are independent, identically $\mathrm{N}(0, \sigma^2)$ distributed random variables, then

$$K_n(\sigma, \sigma') = \left(1 - \frac{\sigma^2 - \sigma_0^2}{\sigma_0^2}\,.\,\frac{\sigma'^2 - \sigma_0^2}{\sigma_0^2}\right)^{-n/2}.$$

The Hilbert space $H(K_n)$ can be characterized in this case utilizing the fact that

$$K_{2k+1}(z) = (1-z)^{-(2k+1)/2} = \frac{2^k}{(2k-1)!!}\,K_1^{(k)}(z)\,,$$

from which we have

$$K_{2k+1}(z, \bar{u}) = \sum_{m=1}^{\infty} a_m^{(k)}(z\,.\,\bar{u})^{m-1}$$

with

$$a_m^{(k)} = c_{m+k}\,\frac{2^k}{(2k-1)!!}\,(m+k-1)\ldots m\,.$$

Because

$$K_{2k+2}(z) = \frac{1}{(1-z)^{k+1}} = K_2^{(k)}(z)\,.\,1/k!$$

we can proceed by analogy and get:

$$K_{2k+2}(z, \bar{u}) = \frac{1}{(1 - z\bar{u})^{k+1}} = \sum_{m=1}^{\infty} b_m^{(k)}(z \, . \, \bar{u})^{m-1} ,$$

where $b_m^{(1)} = 1$,

$$b_m^{(k)} = \frac{(m + k - 1) \dots m}{k!} \quad \text{for} \quad k \geqq 1$$

and

$$K_2(z, \bar{u}) = \frac{1}{(1 - z\bar{u})} = \sum_{m=1}^{\infty} (z \, . \, \bar{u})^{m-1} .$$

Using Lemma 1 and Lemma 2 we get:

$$H(K_n) = \{f \in \mathscr{E}_{\sigma_0} : \sum_{m=1}^{\infty} \frac{(\sigma_0^2)^{2(m-1)} [f^{(m-1)}(\sigma_0^2)]^2}{[(m - 1)!]^2 \, e_m^{(n)}} ,$$

where

$$e_m^{(2k+1)} = \frac{2^k}{(2k - 1)!!} c_{m+k}(m + k - 1) \dots m \quad \text{if} \quad k = 1, 2, \dots$$

and

$$e_m^{(2k)} = \frac{(m + k - 1) \dots m}{k!} \quad \text{if} \quad k = 1, 2, \dots$$

and it is possible to prove again that the random variables $(\sum X_i^2)^k \in L_{\sigma_0}^2$ for $k = 0, 1, 2, \dots$, where $L_{\sigma_0}^2$ is now generated by the system

$$\left\{ \frac{\mathrm{d}P_\sigma}{\mathrm{d}P_{\sigma_0}} (X_1, \dots, X_n) = \prod_{i=1}^{n} \frac{\mathrm{d}P_\sigma}{\mathrm{d}P_{\sigma_0}} (X_i) \, ; \quad 0 < \sigma^2 < 2\sigma_0^2 \right\}$$

of random variables.

4. LOCALLY BEST UNBIASED ESTIMATES OF FUNCTIONALS OF COVARIANCE FUNCTIONS OF A GAUSSIAN STOCHASTIC PROCESS

Let us assume that we observe a Gaussian measurable stochastic process $X = \{X(t); t \in [0, T]\}$ with zero mean value function and with an unknown covariance function of the type

$$R(s, t) = \sum_{k=1}^{\infty} \lambda_k \, e_k(s) \, e_k(t) \, ; \quad s, t \in [0, T] ,$$

where $\{e_k\}_{k=1}^{\infty}$ is a known given complete orthonormal system in $L^2([0, T])$ and

$\{\lambda_k\}_{k=1}^{\infty}$ are unknown positive real numbers such that $\sum_{k=1}^{\infty} \lambda_k < \infty$. The last condition is sufficient to ensure the existence of a Gaussian probability measure P_R in $L^2([0, T])$ which is completely determined by the integral operator in $L^2([0, T])$ with the kernel $R(s, t)$; $s, t \in [0, T]$.

In order to be able to utilize the general theory of estimation as given in part 2 to the problem of estimation of functionals of a covariance function we need to know the conditions under the measures P_R and P_{R_0} are equivalent and $\mathrm{d}P_R/\mathrm{d}P_{R_0}$ belongs to the space $L^2(P_{R_0})$. These problems were solved by many authors; the approach of Skorochod [11] is convenient for us.

Lemma 4. Let R and R_0 be positive definite covariance operators in $L^2([0, T])$. The Gaussian measures P_R and P_{R_0} are either orthogonal or equivalent. The necessary and sufficient condition for equivalence of P_R and P_{R_0} is the following one: there exist a symmetric, Hilbert-Schmidt operator U such that $I + U$ is invertible and $R = R_0^{1/2}(I + U) R_0^{1/2}$. If P_R is equivalent with P_{R_0} then

$$\frac{\mathrm{d}P_R}{\mathrm{d}P_{R_0}}(x) = \exp\left\{\tfrac{1}{2}\sum_{i,j}(U(I+U)^{-1} e_i, e_j)_{L^2}\left[\frac{(x, e_i)(x, e_j)}{\lambda_i^0 \lambda_j^0} - \delta_{ij}\right] + \eta\right\}$$

where

$$\eta = \tfrac{1}{2}\sum_{k=1}^{\infty}\left[\frac{\gamma_k}{1+\gamma_k} - \log(1+\gamma_k)\right]; \quad \{\lambda_k^0\}_{k=1}^{\infty}, \{e_k\}_{k=1}^{\infty},$$

are proper values and proper vectors of the operator R_0, and $\{\gamma_k\}_{k=1}^{\infty}$ are proper values of the operator U. $\mathrm{d}P_R/\mathrm{d}P_{R_0}$ belongs to the $L^2[P_{R_0}]$ iff $|\gamma_k| < 1$; $k = 1, 2, \ldots$. Because U is Hilbert-Schmidt, $\sum_{k=1}^{\infty}\gamma_k^2 < \infty$.

Proof. Skorochod [11]. Let $R' = R_0^{1/2}(I + U') R_0^{1/2}$ and let $\{\gamma'_k\}_{k=1}^{\infty}$ be the proper values of the operator U', $|\gamma'_k| < 1$, $\sum_{k=1}^{\infty}\gamma_k'^2 < \infty$ and let $R'(s, t) = \sum_{k=1}^{\infty}\lambda'_k e_k(s) e_k(t)$, $\lambda'_k > 0$, $\sum_{k=1}^{\infty}\lambda'_k < \infty$.

Then

$$K_{R_0}(R, R') = E_{P_{R0}}\left[\frac{\mathrm{d}P_R}{\mathrm{d}P_{R_0}} \cdot \frac{\mathrm{d}P_{R'}}{\mathrm{d}P_{R_0}}\right] = \mathrm{e}^{\eta} \cdot \mathrm{e}^{\eta'} \cdot$$

$$\cdot\, E_{P_{R0}}\left[\exp\left\{\frac{1}{2}\sum_k\left[\frac{\gamma_k}{1+\gamma_k} + \frac{\gamma'_k}{1+\gamma'_k}\right]\left[\frac{(x, e_k)^2}{\lambda_k^0} - 1\right]\right\}\right] =$$

$$= \prod_{k=1}^{\infty}\frac{\exp\left\{\frac{1}{2}\left(\frac{\gamma_k}{1+\gamma_k} + \frac{\gamma'_k}{1+\gamma'_k}\right)\right\}\left(\frac{1}{1+\gamma_k}\right)^{1/2}\left(\frac{1}{1+\gamma'_k}\right)^{1/2}}{\exp\left\{\frac{1}{2}\frac{\gamma_k}{1+\gamma_k} + \frac{\gamma'_k}{1+\gamma'_k}\right\}\cdot\left(1 - \frac{\gamma_k}{1+\gamma_k} - \frac{\gamma'_k}{1+\gamma'_k}\right)^{1/2}} =$$

$$= \prod_{k=1}^{\infty} (1 - \gamma_k \gamma'_k)^{-1/2} = \prod_{k=1}^{\infty} \left(1 - \frac{\lambda_k - \lambda_k^0}{\lambda_k^0} \cdot \frac{\lambda'_k - \lambda_k^0}{\lambda_k^0}\right)^{-1/2}$$

where $0 < \lambda_k, \lambda'_k < 2\lambda_k^0$ for $k = 1, 2, \ldots$. This follows from the facts, that $\gamma_k = \lambda_k/\lambda_k^0 - 1$ and $|\gamma_k| < 1$, $k = 1, 2, \ldots$ and

$$\sum_{k=1}^{\infty} \gamma_k^2 = \sum_{k=1}^{\infty} \left(\frac{\lambda_k - \lambda_k^0}{\lambda_k^0}\right)^2 < \infty .$$

As we see, the kernel $K_{R_0}(\cdot, -)$ is defined on the set $\mathscr{R}_0 \times \mathscr{R}_0$, where

$$\mathscr{R}_0 = \left\{R : R(s, t) = \sum_{k=1}^{\infty} \lambda_k \, e_k(s) \, e_k(t) ; \right.$$

$$\left. 0 < \lambda_k < 2\lambda_k^0 , \quad \sum_{k=1}^{\infty} \lambda_k < \infty , \quad \sum_{k=1}^{\infty} \left(\frac{\lambda_k - \lambda_k^0}{\lambda_k^0}\right)^2 < \infty \right\},$$

and the RKHS $H(K_{R_0})$ consisting of the functions defined on $\mathscr{R}_0$ have to be considered in connection with unbiased estimation of functionals of covariance function. The following theorem describes the structure of $H(K_{R_0})$.

Theorem 2. The space of estimable functionals of covariance functions $H(K_{R_0})$ is isomorphic to the infinite tensor product $\bigotimes_{i=1}^{\infty} H(K_i)$ of RKHS $H(K_i)$, where

$$K_i(\lambda_i, \lambda'_i) = \left(1 - \frac{\lambda_i - \lambda_i^0}{\lambda_i^0} \cdot \frac{\lambda'_i - \lambda_i^0}{\lambda_i^0}\right)^{-1/2},$$

$$0 < \lambda_i, \lambda'_i < 2\lambda_i^0; \; i = 1, 2, \ldots .$$

Proof. The notion of infinite tensor product of Hilbert spaces is given in Guichardet [4]. Because the elements $\{K_{R_0}(\cdot, R); R \in \mathscr{R}_0\}$ generates $H(K_{R_0})$, the isomorphism between $H(K_{R_0})$ and $\bigotimes_{i=1}^{\infty} H(K_i)$ is a consequence of the fact that

$$\langle K_{R_0}(\cdot, R), K_{R_0}(\cdot, R')\rangle_{H(KR_0)} = K_{R_0}(R, R') = \prod_{i=1}^{\infty} K_i(\lambda_i, \lambda'_i) =$$

$$= \prod_{i=1}^{\infty} \langle K_i(\cdot, \lambda_i), \; K_i(\cdot, \lambda'_i)\rangle_{H(K_i)} = \langle \bigotimes_{i=1}^{\infty} K_i(\cdot, \lambda_i), \; \bigotimes_{i=1}^{\infty} K_i(\cdot, \lambda'_i)\rangle_{\bigotimes_{i=1}^{\infty} H(K_i)} .$$

Elementary decomposable vectors ([4]) – generating elements of $\bigotimes_{i=1}^{\infty} H(K_i)$ – are of the type $h = \bigotimes_{i=1}^{\infty} h_i$, where $h_i = f_i \in H(K_i)$ for every $i \in J$, J being a finite subset of $I = \{1, 2, \ldots\}$ and $h_i = 1$ for $i \in I - J$. The function

$$g(R) = g(\{\lambda_i\}_{i=1}) = \prod_{i \in J} f_i(\lambda_i) ; \quad R \in \mathscr{R}_0$$

is an isomorphic image of such vector h. Now let $x_i = \bigotimes_{j=1}^{\infty} h_j$, where $h_j = 1$ for $j \neq i$ and $h_j = f_i \in H(K_i)$ for $j = i$; $i = 1, 2, \ldots$. Then the function $g(R) = g(\{\lambda_i\}_{i=1}^{\infty}) = \sum_{i=1}^{\infty} f_i(\lambda_i)$; $R \in \mathscr{R}_0$ belongs to $H(K_{R_0})$ if and only if the series $\sum_{i=1}^{\infty} x_i$ converges in $\bigotimes_{i=1}^{\infty} H(K_i)$. The necessary and sufficient condition for this is that the series $\sum_{i,j=1}^{\infty} \langle x_i, x_j \rangle_{\otimes_i H(K_i)}$ converges, where

$$\langle x_i, x_j \rangle_{\otimes_i H(K_i)} = \|f_i\|^2_{H(K_i)} \quad \text{if} \quad i = j$$

and

$$\langle x_i, x_j \rangle_{\otimes_i H(K_i)} = \langle 1, f_i \rangle_{H(K_i)} \cdot \langle 1, f_j \rangle_{H(K_j)}$$

for $i \neq j$.

Example 2. Let $f_i(\lambda_i) = \lambda_i\, e_i(s)\, e_i(t)$; s, t fixed points in $[0, T]$. Then

$$\sum_{i,j=1}^{\infty} \langle x_i, x_j \rangle = \big(\sum_{i=1}^{\infty} \lambda_i^0\, e_i(s)\, e_i(t)\big)^2 + 2 \sum_{i=1}^{\infty} (\lambda_i^0)^2\, e_i^2(s)\, e_i^2(t) < \infty$$

and from the preceding results we can conclude that $U = \sum_{i=1}^{\infty} X_i^2\, e_i(s)\, e_i(t)$, where $X_i = \int_0^T X(s)\, e_i(s)\, \mathrm{d}s$ is the locally best unbiased estimate of the functional $f_{s,t}(R) = R(s, t)$; $R \in \mathscr{R}_0$.

Example 3. Let $X(t) = X_1 \sin t + X_2 \cos t$; $0 \leqq t \leqq 2\pi$, where X_i, are independent $\mathrm{N}(0, \lambda_i)$; $i = 1, 2$, distributed random variables. Let $U = X_1^2 \sin s \sin t + X_2^2 \cos s \cos t$ and let $V = X(s)\, X(t)$. Then $E_R[U] = E_R[V] = R(s, t)$ for all $R \in \mathscr{R}_0$, but it can be easily computed that $E_R[V^2] - E_R[U^2] = \lambda_1 \lambda_2 [\sin(s + t)]^2 \geqq 0$ for all $R \in \mathscr{R}_0$.

5. ESTIMATION OF COMPONENTS OF COVARIANCE FUNCTION

Let us assume that an unknown covariance function of a Gaussian stochastic process X is of the form $R_\alpha(s, t) = \sum_{i=1}^{n} \alpha_i\, R_i(s, t)$, where $\alpha_i > 0$ $i = 1, 2, \ldots, n$ are unknown parameters and $R_i(s, t)$ are known linearly independent covariance functions of the form $R_i(s, t) = \sum_{k=1}^{\infty} \lambda_{k,i}\, e_k(s)\, e_k(t)$ with $\lambda_{k,i} > 0$ $k = 1, 2, \ldots$ and $\sum_{k=1}^{\infty} \lambda_{k,i} < \infty$ for $i = 1, 2, \ldots, n$. Under these conditions $R_\alpha(s, t) = \sum_{k=1}^{\infty} \lambda_k\, e_k(s)\, e_k(t)$, where we have denoted $\lambda_k = \sum_{i=1}^{n} \lambda_{ki} \alpha_i$; $k = 1, 2, \ldots$. According to results of preceding chapters $P_{R\alpha}$ is equivalent to $P_{R\alpha^0}$ for fixed vector $\alpha_0' = (\alpha_1^0, \ldots, \alpha_n^0)$ if and only if $|(\lambda_k - \lambda_k^0)/\lambda_k^0| < 1$ for all k and

$$\sum_{k=1}^{\infty} \left(\frac{\lambda_k - \lambda_k^0}{\lambda_k^0}\right)^2 < \infty\,, \quad \text{where} \quad \lambda_k^0 = \sum_{i=1}^{n} \lambda_{k,i}\alpha_i^0\,; \quad k = 1, 2, \ldots.$$

If these conditions are fulfiled, then

$$K_{R\alpha^0}(R_\alpha, R_{\alpha'}) = E_{PR\alpha^0}\left[\frac{\mathrm{d}P_{R\alpha}}{\mathrm{d}P_{R\alpha^0}} \cdot \frac{\mathrm{d}P_{R\alpha'}}{\mathrm{d}P_{R\alpha^0}}\right] = \prod_{k=1}^{\infty}\left(1 - \frac{\lambda_k - \lambda_k^0}{\lambda_k^0} \cdot \frac{\lambda_k' - \lambda_k^0}{\lambda_k^0}\right)^{-1/2}.$$

The following examples illustrate the situations that may occur.

Example 4. Let $\lambda_{k,1} = 1/k^2$ and $\lambda_{k,2} = 1/(k^2 + k)$; $k = 1, 2, \ldots$ and let $\alpha^0 = (1, 1)$. Because the series

$$\sum_{k=1}^{\infty}\left(\frac{(\alpha_1 - 1)/k^2 + (\alpha_2 - 1)/(k^2 + k)}{1/k^2 + 1/(k^2 + k)}\right)^2$$

diverges for all $\alpha' = (\alpha_1, \alpha_2) \neq \alpha^0$ the measures $P_{R\alpha}$ and $P_{R\alpha^0}$ are orthogonal for $\alpha \neq \alpha^0$.

Example 5. For $\lambda_{k,1} = 1/k^2$; $\lambda_{k,2} = 1/k^4$, $k = 1, 2, \ldots$ and $\alpha^0 = (1, 1)$, $\alpha = (1, b)$ where $0 < b < 2$, the series

$$\sum_{k=1}^{\infty}\left(\frac{(b - 1)/k^4}{1/k^2 + 1/k^4}\right)^2 = \sum_{k=1}^{\infty}\left(\frac{b - 1}{k^2 + 1}\right)^2$$

converges and $P_{R\alpha}$ is equivalent to $P_{R\alpha^0}$. If $\alpha = (c, 1)$ with $0 < c < 2$, then the series

$$\sum_{k=1}^{\infty}\left(\frac{(c - 1)/k^2}{1/k^2 + 1/k^4}\right)^2 = \sum_{k=1}^{\infty}\left(\frac{k^2(c - 1)}{k^2 + 1}\right)^2$$

diverges for $c \neq 1$ and for such α's that $P_{R\alpha}$ and $P_{R\alpha^0}$ are orthogonal.

For a discrete stochastic process $Y = \{X_k^2\}_{k=1}^{\infty}$, where $X_k = \int_0^T X(s)\, e_k(s)\, \mathrm{d}s$; $k = 1, 2, \ldots$ are independent random variables we have the following model:

$$E_{R\alpha}[X_k^2] = \sum_{i=1}^{n} \lambda_{ki}\alpha_i = \sum_{i=1}^{n} \alpha_i \boldsymbol{a}_i\,,$$

where $\boldsymbol{a}_i = (\lambda_{1i}, \lambda_{2i}, \ldots)$ are such that $\boldsymbol{a}_i \in l^1$, that is $\sum_{k=1}^{\infty} \lambda_{ki} < \infty$; $i = 1, 2, \ldots, n$. But at least one of the vectors $\boldsymbol{a}_i$; $i = 1, 2, \ldots, n$ does not belong to the space $H(R_{\alpha^0}^Y)$, where $R_{\alpha^0}^Y(i, j)$; $i, j = 1, 2, \ldots$ is a covariance function of the process Y by given α^0. Actually, let all $\boldsymbol{a}_i$; $i = 1, 2, \ldots, n$ belong to $H(R_{\alpha^0}^Y)$, then

$$\|\boldsymbol{a}_i\|^2_{H(RY\boldsymbol{\alpha}^0)} = \sum_{k=1}^{\infty} \frac{\lambda_{ki}^2}{D^2_{\boldsymbol{\alpha}^0}[X_k^2]} = \tfrac{1}{2}\sum_{i=1}^{\infty}\left[\frac{\lambda_{ki}}{\sum_{j=1}^{n}\alpha_j^0\lambda_{kj}}\right]^2 < \infty$$

for $i = 1, 2, \ldots, n$, from which we have:

$$\lim_{k\to\infty} \frac{\lambda_{ki}}{\sum_{j=1}^{n}\alpha_j^0\lambda_{kj}} = 0 \quad \text{for} \quad i = 1, \ldots, n\,,$$

and this implies

$$\lim_{k\to\infty} \frac{\sum_{i=1}^{n}\alpha_i^0\lambda_{ki}}{\sum_{j=1}^{n}\alpha_j^0\lambda_{kj}} = 0\,, \quad \text{a contradiction}\,.$$

Because of this fact we cannot use the methods of linear regression analysis given by Parzen [9] to estimate the vector $\boldsymbol{\alpha}$.

An unbiased estimate of vector $\boldsymbol{\alpha}' = (\alpha_1, \ldots, \alpha_n)$ can be found by the method of least squares. The series $\sum_{k=1}^{\infty}[X_k^2 - \sum_{i=1}^{n}\lambda_{ki}\alpha_i]^2$ converges with probability one for every $\alpha_i > 0$; $i = 1, 2, \ldots, n$ because $\sum_{k=1}^{\infty} E_{\boldsymbol{\alpha}}|X_{k_i}^2 - \sum_{i=1}^{n}\lambda_{ki}\alpha_i|^2 = 2\sum_{k=1}^{\infty}(\sum_{i=1}^{n}\lambda_{ki}\alpha_i)^2 < \infty$. Next

$$\sum_{k=1}^{\infty}\left|\frac{\partial}{\partial\alpha_j}(X_k^2 - \sum_{i=1}^{n}\lambda_{ki}\alpha_i)^2\right| \leqq 2\sum_{k=1}^{\infty}(X_k^2\lambda_{kj} + \sum_{i=1}^{n}\lambda_{ki}\lambda_{kj}\alpha_i) < \infty$$

with probability one for every $\alpha_i > 0$; $i = 1, 2, \ldots, n$, because $\sum_{k=1}^{\infty} E_{\boldsymbol{\alpha}}|X_k^2\lambda_{kj}| = \sum_{k=1}^{\infty}\lambda_k\lambda_{kj} < \infty$, so that we can write

$$\frac{\partial}{\partial\alpha_j}\sum_{k=1}^{\infty}(X_k^2 - \sum_{i=1}^{n}\lambda_{ki}\alpha_i)^2 = \sum_{k=1}^{\infty}(X_k^2 - \sum_{i=1}^{n}\lambda_{ki}\alpha_i)\,\lambda_{kj} = 0$$

for $j = 1, 2, \ldots, n$ and unbiased estimate $\hat{\boldsymbol{\alpha}}$, found by the method of least squares, is a solution of a system of normal equations

$$A\boldsymbol{\alpha} = \begin{pmatrix} (\boldsymbol{a}_1, Y)_{l^2} \\ \vdots \\ (\boldsymbol{a}_n, Y)_{l^2} \end{pmatrix},$$

where the matrix $A = \|(\boldsymbol{a}_i, \boldsymbol{a}_j)_{l^2}\|$ $i, j = 1, 2, \ldots, n$ with $(\boldsymbol{a}_i, \boldsymbol{a}_j)_{l^2} = \sum_{k=1}^{\infty}\lambda_{ki}\lambda_{kj}$ and

$(\boldsymbol{a}_j, Y)_{l^2} = \sum_{k=1}^{\infty} \lambda_{kj} X_k^2$. For the matrix A we have

$$\sum_{i,j=1}^{n} (\boldsymbol{a}_i, \boldsymbol{a}_j)_{l^2} c_i c_j = \sum_{k=1}^{\infty} \left(\sum_{i=1}^{n} \lambda_{ki} c_i \right)^2 \geqq 0$$

and from the assumption of linear independence of covariance functions R_i; $i = 1, 2, \ldots, n$ we get $\sum_{i=1}^{n} \lambda_{ki} c_i = 0$ for all $k = 1, 2, \ldots$ if and only if $c_i = 0$; from which can be deduced that A is nonsingular and

$$\hat{\boldsymbol{\alpha}} = A^{-1} \begin{pmatrix} (\boldsymbol{a}_1, Y)_{l^2} \\ \vdots \\ (\boldsymbol{a}_n, Y)_{l^2} \end{pmatrix}.$$

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