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On Computable Real Functions

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In the present paper computability classes of functions are defined, mutual relations between these classes are investigated and some usual statistics of these classes are described.

There are many possibilities how to define computable real functions. In the literature we meet principally three approaches. The first of them is based on approximations of continuous real functions having computable regulators of continuity (cf. [4]), the second is based on approximations of continuous functions by recursive sequences of polynomials (cf. [7]) and the third deals with Borel measurable functions being inspired statistically (cf. [1], [2], [3]). In the present paper we compare all three approaches (part I and II) and present some examples of various types of computable real functions. These examples are usual statistics (part III); this fact is due to our aim to apply results of computability and complexity theory to statistics (cf. [1] – [5], [9]).

I. DEFINITIONS

For the first two parts of this paper we make the following limitation. We shall investigate only continuous functions of one real variable. The continuity is necessary for comparing various definitions, because some of them are based principally on the continuity condition (Def. C, D, E). The limitation to functions of one variable is not substantial; all what will be presented can be generalized for functions of more variables (see [5]), but in such a case the formal description is much more complicated.

We use freely the notion of computable functions on rational numbers, the way

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how to define them on the base of recursive functions on natural numbers seems to be clear (cf. e.g. [10]). Moreover, we use many times the equivalency between recursive functions and Markov normal algorithms.

For the sake of simplicity we shall study functions on a given nonempty closed interval I of real numbers with rational end points.

1.1. Denotations. We shall use $\mathbb{N}$ for natural numbers, $\mathbb{N}^+ = \mathbb{N} - \{0\}$, $\mathbb{Q}$ for rational and $\mathbb{R}$ for real ones. Denote by RI a set of real continuous functions on I.

1.2. Definitions

A. Define the subclass $C_1 \subseteq RI$ as follows:
$f \in C_1$ if 1) $f : \mathbb{Q} \cap I \to \mathbb{Q} \cap I$
2) f is computable on $\mathbb{Q} \cap I$.

B. Define the subclass $C_2 \subseteq RI$ as follows:
$f \in C_2$ if there is a computable function $b : (\mathbb{Q} \cap I) \times \mathbb{N}^+ \to \mathbb{Q} \cap I$ such that $(\forall n \in \mathbb{N})(\forall x \in \mathbb{Q} \cap I)(|b(x, n) - f(x)| < 1/n)$. The function b is called a computable approximant of f.

These definitions are due to Hájek and Havránek (cf. [2], [3]). They are perhaps the most straightforward definition of computable real functions. The continuity condition is here superfluous.

The following four definitions require the assumption of continuity of the function f. The class C_5 (or C_3) can be derived from the class C_1 (or C_2 respectively) by adding the assumption of existence of a recursive regulator of uniform continuity of the function f (or of an aproximant of the function f). The class C_4 – defined later – is equivalent to the class C_3 (see [7]). This definition is related to applications since it corresponds to the usual realization of continuous functions in computers by means of polynomial approximations. The class C_6 is equivalent to the class C_3 (see 2.7.).

C. Define the subclass $C_3 \subseteq RI$ as follows: $f \in C_3$ if there are two computable functions $b : (\mathbb{Q} \cap I) \times \mathbb{N}^+ \to \mathbb{Q} \cap I$ and $g : (\mathbb{Q} \cap I)^2 \times \mathbb{N}^+ \to \mathbb{N}^+$ such that
1) $(\forall n \in \mathbb{N})(\forall x \in \mathbb{Q} \cap I)(|b(x, n) - f(x)| < 1/n)$,
2) $(\forall x, y, z, v \in \mathbb{Q} \cap I)$(if $x < y$, $z < v$ and $|y - z| < 1/g(x, v, n)$ then $|b(y, n) - b(z, n)| < 1/n$). ([1])

Remark. The function g is called a computable regulator of uniform continuity on $\langle x, v \rangle$; clearly its existence is equivalent to the existence of computable regulator on the whole I.

D. Define the subclass $C_4 \subseteq RI$ as follows: $f \in C_4$ if there exists a recursively enumerable sequence of polynomials $\{P_n(x)\}$ and a computable function g such that $(\forall M \in \mathbb{N}^+)(\forall x \in I)$ if $n \geqq g(M)$ then $|f(x) - P_n(x)| < 1/M$ ([7]).

Note. A sequence of polynomials $\{P_n(x)\}$ is recursivelly enumerable if there exist recursive functions a, b, c, d such that

$$\forall n \in \mathbb{N}^+ \; P_n(x) = \sum_{j=0}^{d(n)} (-1)^{(c(n))_j} \frac{(a(n))_j}{(b(n))_j} x^j$$

where $(b(n))_j > 0$ for $j \leqq d(n)$.

E. Define the subclass $C_5 \subseteq RI$ as follows: $f \in C_5$ if
1) $f : \mathbb{Q} \cap I \to \mathbb{Q} \cap I$ is computable,
2) there exists a computable function $g : \mathbb{N}^+ \to \mathbb{N}^+$ such that $(\forall n \in \mathbb{N}^+)$ (if $|x - y| < 1/g(n)$ then $|f(x) - f(y)| < 1/n$.

F. Define the subclass $C_6 \subseteq RI$ as follows: $f \in C_6$ if
1) there exists a computable sequence $\{S_n, n \in \mathbb{N}\}$ such that each S_n is a computable compact part of $I \times I$,
2) $\bigcap S_n = f$. (P. Hájek's specialization of some ideas of D. S. Scott [8] and P. Weirauch).

Remark: The notion of computable part can be specified as follows: S_n is a computable part if it is a finite union of rectangulars with rational vertices.

II. HIERARCHY OF COMPUTABLE REAL FUNCTIONS

2.1. A naturally arriving question is to find mutual relations among the just defined classes $C_1, \ldots, C_6$. From the Definitions A to E it is clear that the following inclusions hold:

$$C_5 \subseteq C_1 \subseteq C_2 \,, \quad C_5 \subseteq C_3 \subseteq C_2 \,.$$

These inclusions are generally proper as will be shown later.

2.2. Lemma. There is an increasing function $g : \mathbb{N} \to \mathbb{N}$ such that for every computable function $\xi : \mathbb{N} \to \mathbb{N}$ there exists $n_0 \in \mathbb{N}$ such that $g(n_0) > \xi(n_0)$.

Proof. Let $\Psi(i, j)$ be a universal function for a set of partially recursive functions of one variable. Define $g(n) = \sum_{i=0}^{n} \phi(i) + n$ where $\phi(i)$ is an extension of the partially recursive function $\varphi(i) = \Psi(i, i) + 1$ on a set of all natural numbers.

This lemma will be used in a proof of existence of a function not belonging to any of the classes $C_1, \ldots, C_6$.

2.3. Theorem. Let I be a closed interval with rational end points. There is a function $f : I \to I$ such that
(1) f is continuous and increasing function on I,
(2) $f : \mathbb{Q} \cap I \to \mathbb{Q} \cap I$, (3) there is no computable regulator of uniform continuity of the function f, (4) the function f has no computable approximant.

Proof. Let us restrict ourselves to the interval $\langle 0, 1 \rangle$. Let g be the increasing function from the preceding lemma. Put $g'(n) = g(n) - g(0)$. The function f can now be defined as follows: $f(0) = 0$, $f(1/2^{g'(n)}) = 1/2^n$ for $n \in \mathbb{N}$,

$$f(x) = (1/2^{n+1}) + (x - (1/2^{g'(n+1)})(((1/2^n) - (1/2^{n+1}))/((1/2^{g'(n)}) - \\ - (1/2^{g'(n+1)})) \quad \text{for} \quad x \in (1/2^{g'(n+1)}, 1/2^{g'(n)}) .$$

The function f is continuous and increasing and it maps $I \cap \mathbb{Q}$ into $I \cap \mathbb{Q}$. (3) By contradiction. Let exist a computable regulator of uniform continuity $\pi : \mathbb{N} \to \mathbb{N}$. It is clear that $f(x) \leqq 2^{-n}$ if $x \leqq 2^{-g'(n)}$. Hence if $x \leqq 2^{-\pi(n)}$ then $x \leqq 2^{-g'(n)}$. Hence $\pi(n) \geqq g'(n)$. By Lemma 2.2 π is not recursive. (4) By contradiction. Let there exist a computable approximant of f. The function f is increasing and thus $f \in C_3$ (see [5]). Note that if $f \in C_3$, then there is a computable regulator of uniform continuity of the function f (see [6]), this is a contradiction.

2.4. Theorem. There is a function $f : \langle 0, 1\rangle \to \langle 0, 1\rangle$ such that (1) $f : \langle 0, 1\rangle \cap \cap \mathbb{Q} \to \mathbb{Q}$, (2) f is a continuous function, (3) f *rest* $\mathbb{Q} \cap \langle 0, 1\rangle$ is a computable function, (4) there exists no computable regulator of uniform continuity of the function f.

Proof. Let $\mathfrak{A}$ be a universal Markov algorithm and Ψ a corresponding universal function for a set of partially recursive functions of one variable. Put $\mathfrak{A}[n] = \mathfrak{A}[n, n]$, $\varphi(n) = \Psi(n, n)$. The domain of a diagonalization of a universal algorithm $M = \{n: !\mathfrak{A}[n]\} = \{n: !\varphi(n)\}$ is known to be a nonrecursive recursively enumerable set. Consider a "stepwise" algorithm $\mathfrak{B}[n, k]$ to the algorithm $\mathfrak{A}$ such that a) $\mathfrak{B}[n, k]$ is complete, b) $\mathfrak{B}[n, k] = 1$ if $\mathfrak{A}$ is applicable to n and stopped exactly on the k-step, $\mathfrak{B}[n, k] = 0$ else. Denote a partially recursive function corresponding to $\mathfrak{B}[n, k]$ by $b(n, k)$. Let $\langle 0, 1\rangle$ be covered by a system of intervals $\mathcal{U}_{n,k} = \langle(1/2^n) + (1/2^{n+k+1}), (1/2^n) + (1/2^{n+k})\rangle$, $n = 1, 2, 3, \ldots$, $k = 0, 1, 2, \ldots$ and let the function f be constructed as follows: f *rest* $\mathcal{U}_{n,k}$ = a pick of the height $1/2^n$ on $\mathcal{U}_{n,k}$ if $b(n, k) = 1$, f *rest* $\mathcal{U}_{n,k} = 0$ else. The function f evidently satisfies requirements (1), (2), (3). Prove (4) by contradiction: Let a computable regulator r exist. This fact is equivalent to a predicate "$\exists j \in \mathbb{N}\ b(c, j) = 1$ if $\sum_{i=0}^{r(c)} b(c, j) = 1$". When checking the initial segment of M the predicate comes true if $c \in M$. This is a contradiction as M is not recursive.

Remark. By a pick of the height $1/2^n$ on $\langle a, b\rangle$ we mean the function $g(x) = (x - a)/(b - a)\, 2^{n-1}$ if $x \in \langle a, (a + b)/2\rangle$, $g(x) = (b - x)/2^{n-1}\,(b - a)$ if $x \in \langle (a + b)/2, b\rangle$.

Consequence: There is a continuous function f such that $f \in C_1$ and $f \notin C_5$.

2.5. Fact. There exists a function f such that $f \in C_3$ and $f \notin C_1$.

Proof. Put $f(x) = \sqrt{2}$. Obviously $f \notin C_1$ because of Definition A. We shall show that $f \in C_3$. Put $y_{n+1} = (1/2)(y_n + (2/y_n))$; $y_0 = 2$. So $y_n - 2 < 4(3/5)^{2n}/(1 - (2/5)^{2n})$ (see [11]).

Remark. Another example of a function belonging to C_3 and not belonging to C_1 can be the function $f(x) = e^x$, $x \in \langle 0, 1\rangle$.

2.6. Theorem. There exists a function g such that $g \in C_2$ and $g \notin C_3$.

Proof. Use the function f constructed in 2.3 and define a function g as follows: $g(x) = f(x)$ for $x \notin \langle 3/4, 1\rangle$, $g(3/4) = 0$, $g(1) = 0$, $g(7/8) = e$. Between these points the function g is defined by interpolation (i.e. on the interval $\langle 3/4, 1\rangle$). Approximants of the function g will be denoted by $b'(x, n)$ and constructed with the use of approximants of the function f as follows: $b'(x, n) = b(x, n)$ for $x \notin \langle 3/4, 1\rangle$, $b'(3/4, n) = 0$, $b'(1, n) = 0$, $b'(7/8, n) = \sum_{k=0}^{n+3} x^k/k!$. Between these points the approximant b' is defined by interpolation. The function b' is a computable approximant of the function g having no computable regulator of uniform continuity (see 2.3). If there is another approximant of g, then there is a computable regulator of uniform continuity of the function f by [6], § 3 and it is contradiction (2.3).

2.7. Theorem. The class C_3 is equal to C_6.

Proof. Let $f \in C_3$. Let us create $1/g(n)$ – net on the interval I, where g is a computable regulator of uniform continuity of an approximant b of the function f. Define the lower limit as $b(i/g(n), n) - 2/n$ and the upper limit as $b(i/g(n), n) + 2/n$. S_n is defined as a union of rectangulars with these vertices. It holds that if $x \in \langle i/g(n), (i+1)/g(n)\rangle$ then $|b(i/g(n), n) - f(x)| \leqq |b(i/g(n), n) - b(x, n)| + |b(x, n) - f(x)| < 2/n$. It is clear that S_n covers the function f and $\bigcap_n S'_n = f$. Let $f \in C_6$. Define $S'_{n+1} = \bigcap_{r=1}^{n} S_r$. It holds that $S'_{n+1} \subseteq S'_n$ and $\bigcap S'_n = f$. Denote a subsequence of such S'_n that $\varrho(S'_n) < 1/n$ by S''_n, where $\varrho(S_n)$ is minimum of the rectangular distance between the upper and lower limit. It is $S''_n = \{(x_i, u_i, n), (x_i, v_i, n)\}$ where (x_i, u_i) and (x_i, v_i) are vertices of rectangulars. Denote $a_i = \max(u_i, u_{i-1})$, $c_i = \min(v_i, v_{i-1})$ for each $i \in \mathbb{N}$ where u_i are the points of a lower limit and v_i are the points of an upper limit. Define $b(x_i, n) = (a_i + c_i)/2$. Between these points the function b is defined by interpolation. It is clear that $f \in C_3$.

2.8. Fact. For the classes C_1, C_3, C_5 it holds that $C_1 \cap C_3 = C_5$.

Proof. Directly from definitions (see moreover V2, §3 of [6]).

III. SOME COMPUTABLE STATISTICS

In the preceding part we have studied the computatability for the function of one variable. Let us have a look at these notions applied to statistics understood in fact as measurable functions on a sample space. The sample space is considered to be a part of bounded "cube" in $\mathbb{R}^n$. An extension of the preceeding definitions to the functions of more than one variable is clear. By this way some classifications of statistical procedures has been created. Only the first step is clear. Later it is desirable to create a hierarchy with respect to the computational complexity (see [5], [9]). Such a hierarchy together with the usual statistical criteria could decide the applicability of particular statistics.

Now we are able only to describe some usual statistics of the just defined computability classes $C_1 - C_6$.

3.1. Consider the statistic $T(x) = (1/k) \sum_{i=0}^{k} x_i$, then $T \in C_5$: $T \in C_1$ is clear. A computable regulator of uniform continuity of the function T can be defined as $g(n) = n$ for metrics $\varrho(x, y) = \max |x_i - y_i|$, $i = 1, \ldots, k$.

3.2. For the statistic $H(x) = \sum_{i=0}^{k} x_i^2$ we have $H \in C_5$: a regulator of uniform continuity can be defined as $g(n) = n^{[\sum_{i=0}^{k} K_i]}$, where $K_i = \max 2|x_i|$, $x_i \in I_i$, $i = 1, \ldots, k$ ($[\cdot]$ denotes integer part).

3.3. Put $\bar{X} = (1/m) \sum_{i=0}^{m} x_i$, $\bar{Y} = (1/n) \sum_{i=0}^{n} y_i$. Then the statistic

$$t^2 = \operatorname{sg}(\bar{Y} - \bar{X})\, mn(m + n - 2)\, (\bar{Y} - \bar{X})^2 / (m + n) \left(\sum_{i=0}^{n} (y_i - \bar{Y})^2 + \right.$$
$$\left. + \sum_{i=0}^{m} (x_i - \bar{X})^2 \right)$$

belongs to C_5.

From 3.1 and 3.2 we know that $\bar{X}$, $\bar{Y}$, $\sum x_i^2$, $\sum y_i^2$ belong to C_5. The class C_5 is closed under composition (see [6]). Hence we need only to find a computable regulator $g(x, y, u, v) = c \,.\, \operatorname{sg}(y - x)\,(y - x)^2/(u + v + n(x^2 + y^2))$. It is easy to show with the use of diferential calculus that

$$|g(a + h) - g(a)| < K_1 h_1 + K_2 h_2 + K_3 h_3 + K_4 h_4 \,, \quad K_1, K_2, K_3, K_4 \in \mathbb{Q} \,.$$

Note. The statistic

$$t = (mn(m + n - 2))^{1/2}\, |\bar{Y} - \bar{X}| / ((m + n) (\sum_{i=0}^{n} (y_i - \bar{Y})^2 + \sum_{i=0}^{m} (x_i - \bar{X})^2))^{1/2}$$

belongs to C_3.

3.4. The function

$$\Phi(x) = (1/\surd(2\pi)) \int_{-\infty}^{x} \exp(-t^2/2)\, dt$$

belongs to C_3.

Put $x > 0$. Put

$$P(x) = (1/\surd 2\pi) \int_{-x}^{x} \exp(-t^2/2)\, dt\,.$$

It holds that

$$P(x) = P(0) + P'(0)\,.\,x/1! + P''(0)\,.\,x^2/2! + \ldots + R_m(x)$$

where $P^{(i)}(0)$ can be expressed with the use of the function $Q(x) = (1/\surd(2\pi))\,.\,\exp(-x^2/2)$ (see [11]). We have to find for each $n \in \mathbb{N}$ such an $m \in \mathbb{N}$ that $R_m(x) < 1/n$. So we obtain the approximants of the function P. It holds that $\Phi(x) = (1 + P(x))/2$, $x > 0$, $\Phi(0) = 1/2$, $\Phi(-x) = 1 - \Phi(x)$. Hence $\Phi \in C_2$ and because Φ is increasing, then $\Phi \in C_3$ (see V15 § 3) of [5]).

The next two lemmas can be used to prove a relation between a class of computable functions and some statistics.

3.5. Lemma. Let $f \in C_3$ be a strictly monotonous function. Then there is a strictly monotonous approximant of the function f.

Proof. Let f be an increasing function. Denote by $B(x, n)$ a strictly monotonous computable approximant of the function f. Take $n \in \mathbb{N}$ for fixed. Let create $1/g(m)$-net on the interval I. Denote points of the net $x_1, \ldots, x_p$. In the point $x_1 \in I$ define $B(x_1, n) = b(x_1, n_1)$, where $n_1 > 8n$. Now the point x_2 is taken and we have to find $n_2 \in \mathbb{N}$, $8n_1 < n_2 < 16n_1$ such that $b(x_2, n_2) > b(x_1, n_1)$. If there is no such $n_2 \in \mathbb{N}$ in the set $(8n, 16n) \cap \mathbb{N}$ then take new $n_1 := 2n_1$ and repeat the search as above. This loop stops after a finite number of steps. In the point $x_2 \in I$ define $B(x_2, n) = b(x_2, n_2)$. For $x \in \langle x_1, x_2 \rangle$ is $B(x, n)$ now interpolated linearly. For $m > n_2$ the function $B(x, n)$ satisfies $|B(x, n) - f(x)| < 1/n$.

3.6. Lemma. Let $f(x)$ be a strictly monotonous function defined on the interval $\langle 0, 1 \rangle$ with a strictly monotonous approximant $b(x, n)$. Then the inverse function $f^{-1}(y)$ has a monotonous approximant.

Proof. Denote by $f^{-1}(y, m)$ the approximant of function $f^{-1}(y)$, if $|f^{-1}(y) - f^{-1}(y, m)| < 1/m$ for each $m \in \mathbb{N}$. The approximant $f^{-1}(y, m)$ will be constructed as follows: Take $m \in \mathbb{N}$ for fixed. Create $1/2m$ – net on the interval $\langle 0, 1 \rangle$. Take some $n_0 \in \mathbb{N}$ and compute $b(0, n_0)$ and $b(1/2m, n_0)$. Now n_0 will be incremented till such $n_1 \in \mathbb{N}$ is found that $b(0, n_1) + 1/n_1 < b(1/2m, n_1) - 1/n_1$. If $y \in \langle 0, b(0, n_1) + 1/n_1)$ put $f^{-1}(y, m) = 0$. If $y \notin \langle 0, b(0, n_1) + 1/n_1)$ compute $b(2/2m, n_1)$ and again $n_2 \in \mathbb{N}$ is searched such that $n_2 > n_1$ and $b(1/2m, n_2) + 1/n_2 < b(2/2m, n_2) - 1/n_2$. If $y < b(1/2m, n_2) + 1/n_2$ put $f^{-1}(y, m) = 1/2m$.

If $y \geqq b(1/2m, n_2) + 1/n_2$ the same pattern is to be repeated. The loop stops after a finite number of steps.

The classes C_1, C_2 mentioned in the introduction do not require any continuity assumption. Thus the new classes C_1^*, C_2^* can be defined without continuity assumption for the function f.

3.7. The Wilcoxon statistic $S = \sum_{i=1}^{m} R_i$ belongs to C_1^*. ($S : \mathbb{N}^m \to \mathbb{N}$ is a computable and substantially discontinuous function.)

3.8. The Van der Warden statistic $S = \sum_{i=1}^{m} \Phi^{-1}(R_i/n + 1)$ belongs to C_2.

$S : \mathbb{N}^m \to \mathbb{R}$. The function Φ belongs to C_3 by 3.4. Φ is strictly monotonous and thus there is an approximant of Φ by 3.5. The function Φ^{-1} has a monotonous approximant by 3.6. That is $\Phi^{-1} \in C_3$. The inner function $R_i(x) = \text{card}\{j;\, x_j \leqq x_i\}$ is included in C_1^* because $R_i(x)$ is a substantially discontinuous function. The statistic S created by composition belongs to C_2^*.

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