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Polynomial Design of Deadbeat Control Laws

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A new method to calculate deadbeat control laws for linear multivariable systems is presented. The method is based on solving a simple linear equation in polynomial matrices. It offers a straightforward and computationally efficient alternative to the state-space methods based on reachability subspaces.

INTRODUCTION

Deadbeat control is a typical example of linear control strategies in discrete-time systems. The standard objective is to drive any initial state of the system to zero in a shortest time possible. There are some variations of this standard problem, namely, to zero just the output of the system or to impose an amplitude-type constraint on the control sequence. The deadbeat control strategy is also widely used when designing servo systems. As a matter of fact, the servo problem can be formulated in terms of the standard problem for an augmented system.

A natural framework for solving the standard deadbeat problem are state-space techniques based on reachable subspaces. They were pioneered by Kalman and Bertram [4]; see also Kalman [3] for coverage of the single-input-output reversible case. The design procedure for multivariable systems is much more complicated and is described partly in Ackermann [1] and fully in Mullis [7].

On the other hand, the servo or follow-up deadbeat problem for specific reference inputs was first solved in the frequency domain by Tou [8], Chang [2], Volgin [9] and others. A complete multivariable solution was given in Kučera [5, 6]. The design procedure is based essentially on solving linear polynomial equations.

The purpose of this paper is to show that polynomial equations can also be used to advantage when solving the standard deadbeat control problem. They provide simple and effective means of finding all deadbeat control laws in parametric form.

The method is not restricted to non-dynamical control laws and remains intact for systems defined over arbitrary fields.

STATE SPACE PROCEDURE

Consider an n dimensional, discrete-time, completely controllable system

$$x_{t+1} = Fx_t + Gu_t \tag{1}$$

where G is $n \times m$ and has rank m. The objective of deadbeat control is to find a linear state-variable control law

$$u_t = -Lx_t \tag{2}$$

such that $(F - GL)^k x = 0$ for every state x which can be driven to the origin in time k.

Following Mullis [7], we shall first define the notion of an ordered selection. Let $\{A_k\}$ be a sequence of $n \times m$ matrices and let $q_k = \operatorname{rank}[A_1 \dots A_k] - \operatorname{rank}[A_1 \dots \dots A_{k-1}]$. An ordered selection for $\{A_k\}$ is then a sequence of matrices $\{S_k\}$ with S_k $m \times q_k$ for which

$$\operatorname{range}[A_1S_1 \dots A_kS_k] = \operatorname{range}[A_1 \dots A_k]$$

for each k. The choice of S_k need be no more complicated than eliminating any column of $[A_1 \dots A_k]$ which is linearly dependent on the columns which precede it. For completeness, we recall that the reachability index r of (F, G) is the smallest integer k for which

$$\operatorname{rank}[F^{k-1}G \dots FG \;\; G] = \operatorname{rank}[F^kG \dots FG \;\; G]\,.$$

The design procedure is recursive, in general. Let $F_0 = F$ and let $S_{k1}, \dots, S_{kk}$ be an ordered selection for $[F_{k-1}^{k-1}G \dots F_{k-1}G \;\; G]$. Define matrices F_k, L_k and M_k by

$$\begin{aligned} M_k[F_{k-1}^{k-1}GS_{k1} \dots GS_{kk}] &= [S_{k1} \; 0 \dots 0] \\ L_k &= M_kF_{k-1}^k \\ F_k &= F_{k-1} - GL_k\,. \end{aligned} \tag{3}$$

Then

$$L = L_1 + L_2 + \dots + L_r \tag{4}$$

where r is the reachability index of (F, G), is a deadbeat control law.

An important special case occurs when F is invertible. The recursive procedure can then be shortcut by setting $F_{r-1} = F$. However, we may miss some solutions.

(5) **Example.** Consider the system (1) with

$$F = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad G = \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix}$$

and find all state-variable deadbeat control laws (2).

Applying the recursive procedure (3), we obtain

$$L_1 = \begin{bmatrix} \beta & 1+\beta & 0 \\ \alpha & \alpha & 1 \end{bmatrix}, \quad F_1 = \begin{bmatrix} -\beta & -\beta & 0 \\ 1 & 1 & 0 \\ -\alpha & -\alpha & 0 \end{bmatrix}$$

$$L_2 = \begin{bmatrix} 1-\beta & 1-\beta & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad F_2 = \begin{bmatrix} -1 & -1 & 0 \\ 1 & 1 & 0 \\ -\alpha & -\alpha & 0 \end{bmatrix}$$

for any reals α, β. Since $r = 2$, we have

(6)
$$L = \begin{bmatrix} 1 & 2 & 0 \\ \alpha & \alpha & 1 \end{bmatrix}$$

and every state can be driven to zero in no more than two steps.

POLYNOMIAL EQUATION APPROACH

We shall now demonstrate how matrix polynomial equations can be effectively used to obtain all deadbeat control laws. With the notation

$$x = x_0 + x_1 d + x_2 d^2 + \dots$$
$$u = u_0 + u_1 d + u_2 d^2 + \dots$$

for the state and control sequence, respectively, an equivalent description of system (1) reads

(7)
$$Ax = Bu + x_0$$

where

$$A = I_n - dF, \quad B = dG$$

are $n \times n$ and $n \times m$ polynomial matrices in the delay operator d. We are looking for all deadbeat control laws of the form

(8)
$$u = -Qz, \quad x = Pz$$

where z is an internal variable and P and Q are $n \times n$ and $m \times n$ polynomial matrices, respectively.

Substituting (8) into (7), we obtain

$$(AP + BQ)\, z = x_0$$

which results in

$$x = P(AP + BQ)^{-1}\, x_0$$

and

$$u = -Q(AP + BQ)^{-1}\, x_0 .$$

Now, both x and u are to be finite sequences, i.e., polynomials in d, to satisfy the deadbeat requirement. Since this is to hold true for every x_0, the P and Q must satisfy the equation

$$AP + BQ = I_n . \tag{9}$$

Complete controllability of our system implies (and is implied by) left coprimeness of A and B. Hence equation (9) has always a solution. However, not all solutions qualify for a deadbeat control law. Since

$$x = Px_0 , \quad u = -Qx_0 \tag{10}$$

and x is to be of least possible degree, we must take the solution which minimizes the degree of every column of P (and hence of Q). Incidentally, such a solution may not be unique.

It is interesting to realize that this polynomial approach is straightforward, not recursive, and that it yields all state-variable deadbeat control laws, including the dynamical ones, in parametric form.

(11) **Example.** Consider again the system (1) with

$$F = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad G = \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix}$$

and find all state-variable deadbeat control laws (8).

From the matrices

$$A = \begin{bmatrix} 1 & -d & 0 \\ -d & 1-d & 0 \\ 0 & 0 & 1-d \end{bmatrix}, \quad B = \begin{bmatrix} d & 0 \\ 0 & 0 \\ 0 & d \end{bmatrix}$$

and solve equation (9). The general solution reads

$$P = \begin{bmatrix} 1 - d^2 & -d & 0 \\ d + d^2 & 1 + d & 0 \\ 0 & 0 & 1 \end{bmatrix} - \begin{bmatrix} d - d^2 & 0 \\ d^2 & 0 \\ 0 & d \end{bmatrix} \begin{bmatrix} t_{11} & t_{12} & t_{13} \\ t_{21} & t_{22} & t_{23} \end{bmatrix}$$

$$Q = \begin{bmatrix} 2d + d^2 & 2 + d & 0 \\ 0 & 0 & 1 \end{bmatrix} + \begin{bmatrix} 1 - d - d^2 & 0 \\ 0 & 1 - d \end{bmatrix} \begin{bmatrix} t_{11} & t_{12} & t_{13} \\ t_{21} & t_{22} & t_{23} \end{bmatrix}$$

the $t_{11}, t_{12}, \ldots, t_{23}$ being arbitrary polynomials. A simple analysis reveals that the minimum-degree solution is obtained on setting

$$t_{11} = 1\,, \quad t_{12} = 0\,, \quad t_{13} = 0$$
$$t_{21} = \alpha\,, \quad t_{22} = \beta\,, \quad t_{23} = 0$$

for any reals α, β and becomes

$$P = \begin{bmatrix} 1 - d & -d & 0 \\ d & 1 + d & 0 \\ -\alpha d & -\beta d & 1 \end{bmatrix}$$

$$Q = \begin{bmatrix} 1 + d & 2 + d & 0 \\ \alpha - \alpha d & \beta - \beta d & 1 \end{bmatrix}.$$

Thus all deadbeat control laws are characterized by the transfer matrix

$$R = QP^{-1} = \begin{bmatrix} 1 & 2 & 0 \\ \alpha + (\alpha - \beta)\, d & \beta + (\alpha - \beta)\, d & 1 \end{bmatrix}.$$

The resulting deadbeat response is given by (10)

$$x = \begin{bmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \end{bmatrix} + \begin{bmatrix} -\xi_1 - \xi_2 \\ \xi_1 + \xi_2 \\ -\alpha\xi_1 - \beta\xi_2 \end{bmatrix} d$$

for any reals ξ_1, ξ_2 and ξ_3 constituting the initial state.

The polynomial solution is more general than the one using state-space techniques in that it includes dynamical control laws. The non-dynamical laws are obtained on putting $\alpha = \beta$ in (12); the result is then identical with the state-space solution given in (6).

CONCLUDING REMARKS

An alternative method of designing state-variable deadbeat control laws for system (1) has been proposed. The procedure consists in solving linear equation (9) in polynomial matrices.

Clearly, the deadbeat control law assigns the characteristic polynomial λ^n to the closed-loop system. However, this polynomial must be split into invariant polynomials in such a way as to make λ^r the minimal invariant polynomial. In the polynomial design, any solution of equation (9) guarantees the first property. The second requirement is then accomplished simply by taking the minimum-degree solution.

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