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# Stabilization of Bilinear Systems by a Linear Feedback Control

WITOLD PEDRYCZ

The paper deals with the stabilization problem for an internally bilinear system by a linear feedback control $u = \mathbf{kx}$. Using Lyapunov's second method, sufficient conditions for the equilibrium point of the closed-loop system to be locally and globally stable are derived.

## 1. INTRODUCTION

In recent years, there has been considerable interest in bilinear systems [1, 2, 3] as appropriate mathematical model to represent the dynamical behaviour for a wide class of the engineering, biological, and economic systems. Recently many studies of the bilinear systems have been done from various points of view, e.g. controllability or observability, and a lot of problems connected with the optimal control have been solved.

This paper presents the problem of stabilization by means of a linear control $u = \mathbf{kx}$. In [4] sufficient conditions for stabilizing bilinear systems by means of the quadratic feedback control have been shown. It is worth to notice that technically it is not so easy to realize such a quadratic feedback control. By using Lyapunov's second method sufficient conditions for the equilibrium point of the closed-loop system are presented.

## 2. PROBLEM STATEMENT

In the following, we confine our attention to the case of single input bilinear systems represented as:

(1) $$\dot{\mathbf{x}}(t) = \mathbf{A}\,\mathbf{x}(t) + u(t)\,\mathbf{B}\,\mathbf{x}(t) + \mathbf{c}\,u(t)$$

where $\mathbf{x}(t)$ is $n$-dimensional state vector,
$u(t)$ is a scalar continuous function,
$\mathbf{c}$ is an $n$-dimensional column vector, and
$\mathbf{A}, \mathbf{B}$ are $n \times n$ dimensional constant matrices respectively.

We discuss a linear feedback control

$$u(t) = \mathbf{k}\,\mathbf{x}(t) \tag{2}$$

where $\mathbf{k}$ is $n$-dimensional row vector.

The matrix $\mathbf{A}$ is assumed not to be a stability matrix:

$$\underset{i=1,2,\ldots,n}{\exists} \quad \operatorname{Re} \lambda_i(\mathbf{A}) \geqq 0 \,, \tag{3}$$

i.e. at least one of its eigenvalues has non-negative real part.

Substituting eqn (2) into eqn (1) we get a closed-loop bilinear system:

$$\dot{\mathbf{x}}(t) = (\mathbf{A} + \mathbf{c}\mathbf{k})\,\mathbf{x}(t) + (\mathbf{k}\,\mathbf{x}(t))\,\mathbf{B}\,\mathbf{x}(t)\,. \tag{4}$$

## 3. PRELIMINARIES

In this section we introduce some definitions and a lemma which are necessary in the following section.

Let $\mathbf{A}$, $\mathbf{C}$ be $n \times n$, $n \times m$ dimensional matrices respectively, and $\mathbf{D}(k)$ is $n \times mk$ matrix given by

$$\mathbf{D}(k) = [\mathbf{C}, \mathbf{A}\mathbf{C}, \ldots, \mathbf{A}^{k-1}\mathbf{C}] \quad k = 1, 2, \ldots \,. \tag{5}$$

**Definition 1.** The pair $(\mathbf{A}, \mathbf{C})$ is called controllable iff $\mathbf{D}(k)$ is of full rank $n$.

Let $l$ be the controllability index of the pair $(\mathbf{A}, \mathbf{C})$ and assume

$$\operatorname{rank} \mathbf{D}(l) = r < n \,.$$

Then there exists a nonsingular matrix $\mathbf{T}$ such that

$$\mathbf{T}^{-1}\mathbf{A}\mathbf{T} = \begin{array}{cl} \left[\begin{array}{c|c} \mathbf{A}_{11}^* & \mathbf{A}_{12}^* \\ \hline \mathbf{0} & \mathbf{A}_{22}^* \end{array}\right] & \begin{array}{l} r \\ n-r \end{array} \\ \begin{array}{cc} \quad r & \quad n-r \end{array} & \end{array} \tag{6}$$

$$\mathbf{T}^{-1}\mathbf{C} = \begin{array}{cl} \left[\begin{array}{c} \mathbf{C}_1^* \\ \hline \mathbf{0} \end{array}\right] & \begin{array}{l} r \\ n-r \end{array} \end{array} \,. \tag{7}$$

**Definition 2.** The pair $(\boldsymbol{A}, \boldsymbol{C})$ is called a stabilizable pair iff

$$\underset{i=1,2,\ldots,n-r}{\forall} \operatorname{Re} \lambda_i(\boldsymbol{A}_{22}^*) < 0 \tag{8}$$

and the pair $(\boldsymbol{A}_{11}^*, \boldsymbol{C}_1^*)$ is a controllable pair.

**Lemma 1.** Let $\boldsymbol{P}, \boldsymbol{Q}$ be $n \times n$ symmetric positive definite matrices. Then there exists a positive constant $\alpha$ such that

$$\boldsymbol{Q} - 2\alpha \boldsymbol{P} \geqq \boldsymbol{0} \tag{9}$$

($\geqq$ denotes positive semi-definite).

Proof. For symmetric positive definite matrices $\boldsymbol{P}, \boldsymbol{Q}$ there exists a nonsingular transformation $\boldsymbol{x} = \boldsymbol{T}\boldsymbol{y}$ such that:

$$\boldsymbol{x}^{\mathrm{T}}(\boldsymbol{Q} - 2\alpha \boldsymbol{P})\, \boldsymbol{x} = \boldsymbol{y}^{\mathrm{T}}(\boldsymbol{W} - 2\alpha \boldsymbol{I})\, \boldsymbol{y} \tag{10}$$

where $\boldsymbol{x}, \boldsymbol{y} \in \boldsymbol{E}^n$ ($\boldsymbol{E}^n$ denotes $n$-dimensional Euclidean space) and

$$\boldsymbol{W} = \begin{bmatrix} \lambda_1(\boldsymbol{Q}\boldsymbol{P}^{-1}) & & 0 \\ & \cdots\cdots\cdots & \\ & \cdots\cdots\cdots & \\ 0 & & \lambda_n(\boldsymbol{Q}\boldsymbol{P}^{-1}) \end{bmatrix} = [W_i]_{i=1,2,\ldots,n} \tag{11}$$

Putting $\alpha = \frac{1}{2} \min\limits_{i=1,2,\ldots,n} W_i$ we obtain:

$$\boldsymbol{x}^{\mathrm{T}}(\boldsymbol{Q} - 2\alpha \boldsymbol{P})\, \boldsymbol{x} \geqq \boldsymbol{0}\,. \tag{12}$$

Now we state the solution of the stabilization problem.

## 4. PROBLEM SOLUTION

We present the solution of the problem in the following theorem.

**Theorem 1.**

1. If the pair $(\boldsymbol{A}, \boldsymbol{c})$ in eqn (1) is a stabilizable pair, then there exists a linear feedback control $u = \boldsymbol{k}\boldsymbol{x}$ such that the equilibrium point of the closed-loop system (4) is locally asymptotically stable.

2. If the pair $(\boldsymbol{A}, \boldsymbol{c})$ in eqn (1) is a stabilizable pair and there exist a symmetric positive definite matrix $\boldsymbol{P}$ and a row vector $\boldsymbol{k}$ which satisfy:

$$\boldsymbol{P}(\boldsymbol{A} + \boldsymbol{c}\boldsymbol{k}) + (\boldsymbol{A} + \boldsymbol{c}\boldsymbol{k})^{\mathrm{T}}\, \boldsymbol{P} < \boldsymbol{0} \tag{13}$$

$$PB + B^{T}P = 0 \tag{14}$$



then there exists a linear feedback control $u = \mathbf{kx}$ such the equilibrium point of the closed-loop system (4) is exponentially stable in large.

Proof.

1. Consider a scalar function $V(\mathbf{x}) = \mathbf{x}^{T}\mathbf{Px}$, where $\mathbf{P}$ is a symmetric matrix. The time derivative of $V$ along the solution of eqn (4) is given by:

$$\dot{V}(\mathbf{x}) = \mathbf{x} \operatorname{grad} V(\mathbf{x}) = \mathbf{x}^{T}[\mathbf{P}(\mathbf{A} + \mathbf{ck}) + (\mathbf{A} + \mathbf{ck})^{T} \mathbf{P}]\, \mathbf{x} + + (\mathbf{kx})\, \mathbf{x}^{T}(\mathbf{PB} + \mathbf{B}^{T}\mathbf{P})\, \mathbf{x}\,. \tag{15}$$

When the pair $(\mathbf{A}, \mathbf{c})$ is a stabilizable pair, $(\mathbf{A} + \mathbf{ck})$ becomes a stability matrix by chosing a proper $\mathbf{k}$. Then, there exists a symmetric positive definite marix $\mathbf{P}$ which is the unique solution of the linear equation

$$\mathbf{P}(\mathbf{A} + \mathbf{ck}) + (\mathbf{A} + \mathbf{ck})^{T} \mathbf{P} = -\mathbf{Q} \tag{16}$$

where $\mathbf{Q}$ is an arbitrary symmetric positive definite matrix. Since the matrix $\mathbf{R}$ defined as $\mathbf{R} = \mathbf{PB} + \mathbf{B}^{T}\mathbf{P}$ is symmetric, from Lemma 1 it follows that:

$$\mathbf{x}^{T}\mathbf{Qx} = \mathbf{y}^{T}\mathbf{y} \tag{17}$$

$$\mathbf{x}^{T}\mathbf{Rx} = \mathbf{y}^{T}\mathbf{Wy} \tag{18}$$

where $\mathbf{x} = \mathbf{Ty}$ and $\mathbf{W}$ is diagonal matrix with elements

$$W_{i} = \lambda_{i}(\mathbf{RQ}^{-1})\,. \tag{19}$$

Then

$$\dot{V}(\mathbf{x}) \equiv \dot{V}(\mathbf{y}) = \mathbf{y}^{T} \begin{bmatrix} -1 + W_{1}\mathbf{kx} & & 0 \\ & \ddots & \\ 0 & & -1 + W_{n}\mathbf{kx} \end{bmatrix} \mathbf{y}\,. \tag{20}$$

If

$$\forall_{i=1,2,\ldots,n} \quad -1 + W_{i}\mathbf{kx} < 0 \tag{21}$$

then $\dot{V}(\mathbf{x}) < 0$. Since inequality (21) is satisfied in the neighbourhood of the origin, it follows that the closed-loop system (4) is locally asymptotically stable.

2. Since the condition (14) shows that $W_i$ defined in the proof of (13) are zero, it follows that $V(\mathbf{x}) = \mathbf{x}^{T}\mathbf{Px}$ is a Lyapunov function. Therefore it is clear that (4) is asymptotically stable in the large.

We show now that the system described by eqn (4) is exponentially stable. After simple computation it could be seen that

$$\dot{V}(\mathbf{x}) = -\mathbf{x}^{\mathrm{T}}\mathbf{Q}\mathbf{x} < 0 \tag{23}$$

and, from Lemma 1, we obtain:

$$\dot{V}(\mathbf{x}) \leqq -2\alpha\mathbf{x}^{\mathrm{T}}\mathbf{P}\mathbf{x} = -2\alpha\, V(\mathbf{x})\,. \tag{24}$$

From (24) we have

$$V(\mathbf{x}) \leqq V(\mathbf{x}_0)\exp\left[-2\alpha(t - t_0)\right], \quad \mathbf{x}_0 = \mathbf{x}(t_0) \tag{25}$$

and

$$\|\mathbf{x}(t)\| \leqq \left[\frac{V(\mathbf{x}_0)}{\min\limits_{i=1,2,\dots,n} \lambda_i(\mathbf{P})}\right]^{1/2} \exp\left[-\alpha(t - t_0)\right] \tag{26}$$

with

$$\alpha = \tfrac{1}{2} \min_{i=1,2,\dots,n} \lambda_i(\mathbf{P})\,.$$

The problem connected with Theorem 1 is to determine the asymptotic stability domain for the origin.

Let us denote:

$$\Gamma = \{\mathbf{x} \in \mathbf{E}^n \mid -1 + W_i\mathbf{k}\mathbf{x} < 0\} \tag{27}$$

$$\Omega_l = \{\mathbf{x} \in \mathbf{E}^n \mid V(\mathbf{x}) < l,\ l > 0\}\,. \tag{28}$$

Both domains $\Gamma$ and $\Omega_l$ contain the origin. Therefore, if $\Omega_l \subset \Gamma$, $\Omega_l$ is an asymptotic stability domain. Since for $W_i = 0$ inequality (21) is obvious, we consider only the case $W_i \neq 0$.

Let there be the following equations:

$$-1 + W_i\mathbf{k}\mathbf{x} = 0$$

$$\mathbf{x}^{\mathrm{T}}\mathbf{P}\mathbf{x} = l\,.$$

Let $l_{\min}$ be the minimum value of $l$, such that there exists the solution of eqn (27) and (28). By a nonsingular transformation $\mathbf{x} = \mathbf{P}^{-1/2}\mathbf{z}$ we get

$$-1 + W_i\mathbf{k}\mathbf{P}^{-1/2}\mathbf{z} = 0 \tag{29}$$

$$\mathbf{z}^{\mathrm{T}}\mathbf{z} = l\,. \tag{30}$$

The distances $d_j$ between the origin and the hyperplanes represented by eqn (29) are given by:

$$d_j = \frac{1}{|W_j| \, \|\boldsymbol{k}\boldsymbol{P}^{-1/2}\|}, \quad j = 1, 2, \ldots, r, \quad r \leqq n \tag{31}$$

where $\|\cdot\|$ denotes the norm in Euclidean space $\boldsymbol{E}^n$.

$\Omega_l$ is an asymptotic stability domain if

$$l < \frac{1}{(\max\limits_{j=1,2,\ldots r} |W_j|)^2 \, \boldsymbol{P}^{-1/2} \boldsymbol{k}^{\mathrm{T}}} \tag{32}$$

## 5. CONCLUSIONS

Sufficient conditions for the equilibrium point of the closed-loop single input internally bilinear system to be locally and globally stable have been derived. The results obtained here could be extended to multi-input bilinear systems. The results may be useful from the practical point of view because of preferability and usefulness of a linear feedback control.

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*Prof. Witold Pedrycz, Department of Control and Information Engineering, Silesian Polytechnical University, ul. Pstrowskiego 16, 44-100 Gliwice. Poland.*