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Generalization of the Method of *D*-Decomposition

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The well-known Nejmark's method of D-decomposition is generalized for the cases of more than two parameters and of more general regions in the root plane than that of stability. An example of D-decomposition in the three-parametric space is given for illustration.

1. INTRODUCTION

The method of D-decomposition was used by Nejmark in order to find stability regions in mono- and biparametric spaces (cf. [1], [3]). The idea occurs of possible generalization in two ways:

1. The application of other criteria in the decomposition than that of stability, such as non-zero stability measure, constant damping and even other more intricate requirements.
2. The decomposition in spaces of more than two parameters; a very important case is that of the three-parametric space because of the wide use of three-parametric PID controllers.

The first of the mentioned ways does not require any principal modification of the original Nejmark's method in most cases. On the other hand, the boundaries of regions are usually too complicated in the three-parametric space, and even more in spaces of more than three parameters, to allow a successful use of the original Nejmark's procedure. The local analysis of the variability of roots, using their derivatives at the boundaries of the regions, often gives much better results.

The present paper deals with this problem in more detail. The considerations are illustrated by an example of the decomposition in the three-parametric space.

Let X be a space of m-dimensional vectors with real components $\{x_j\}$ $(j = 1, \dots, m)$.

The algebraic equation

$$P(p) = \sum_{k=0}^{n} a_k \cdot p^k = 0 \tag{1}$$

is assumed in normal form with $a_n = 1$. The other coefficients are generally linear functions of the parameters x_j:

$$a_k = b_{k0} + \sum_{j=1}^{m} b_{kj} \cdot x_j \,, \quad (j = 1; \dots, m) \,, \tag{2}$$

a_k, b_{k0}, b_{kj} being fixed real numbers. The introduction of (2) into (1) yields:

$$M_0(p) + \sum_{j=1}^{m} M_j(p) \cdot x_j = 0 \tag{3}$$

with

$$M_j(p) = \sum_{k=0}^{n} b_{kj} \cdot p^k \quad (j = 0; 1; \dots; m) \,. \tag{4}$$

The roots p may be real or pairs of complex conjugate values of the general form

$$p = \sigma + \mathrm{i} \cdot \omega \,. \tag{5}$$

Thus $M_j(p)$ can be generally written in the form:

$$M_j(p) = K_j(\sigma, \omega) + \mathrm{i} \cdot \omega \cdot L_j(\sigma, \omega) \,, \tag{6}$$

$K_j(\sigma, \omega)$, $L_j(\sigma, \omega)$ being linear combinations of products of the form $\sigma^u \cdot \omega^{2v}$ ($u \geqq 0$, $v \geqq 0$ being integers). Consequently both are even functions in respect of ω.

The decomposition of (3) into its real and imaginary part gives the relations:

$$\sum_{j=1}^{m} K_j(\sigma, \omega) \cdot x_j + K_0(\sigma, \omega) = 0 \,, \tag{7}$$

$$\omega \cdot \left(\sum_{j=1}^{m} L_j(\sigma, \omega) \cdot x_j + L_0(\sigma, \omega) \right) = 0 \,. \tag{8}$$

The roots are evidently functions of the parameters $x_1; \dots; x_m$:

$$p(x_1; \dots; x_m) = \sigma(x_1; \dots; x_m) + \mathrm{i} \cdot \omega(x_1; \dots; x_m) \,. \tag{5-a}$$

The differentiation of (3) with respect to x_j yields:

$$\frac{\partial p}{\partial x_j} = \frac{\partial \sigma}{\partial x_j} + \mathrm{i} \, . \frac{\partial \omega}{\partial x_j} = \frac{-M_j(p)}{P^{(1)}(p; x_1; \dots; x_m)}, \tag{9}$$

where

$$P^{(1)}(p; x_1; \dots; x_m) = M_0^{(1)}(p) + \sum_{j=1}^{m} M_j^{(1)}(p) \, . \, x_j \, , \tag{9-a}$$

$$M_j^{(1)}(p) = \frac{\mathrm{d}}{\mathrm{d}p} M_j(p) \quad (j = 0; 1; \dots; m) \, . \tag{9-b}$$

If p is a simple root and $x_1; \dots; x_m$ are its corresponding parameters, there is $M^{(1)}(p; x_1; \dots; x_m) \neq 0$. For multiple roots $M^{(1)}(p; x_1; \dots; x_m) = 0$ and (9) does not lead to an effective result.

A modified procedure is applicable in the case of multiple roots (cf.[2]). Varying the value of one of the parameters x_j by a finite value Δx_j we get from (1) the relation:

$$P(p + \Delta p) = \sum_{k=0}^{n} a_k(x_1; \dots; x_j + \Delta x_j; \dots; x_m) \, . \, (p + \Delta p)^k \, . \tag{10}$$

Taking into account the linearity of (2) and expanding (10) with the use of the binomial theorem, we get after some arrangements [2]:

$$\sum_{h=0}^{n} \Delta p^h \, . \, P^{(h)}(p) = -\Delta x_j \, . \sum_{h=0}^{n} \Delta p^h \, . \sum_{k=0}^{n} \binom{k}{h} \, . \, b_{kj} \, . \, p^{k-h} \, , \tag{11}$$

$P^{(h)}(p)$ being the derivative of $P(p)$ of the degree h. If p is an r-fold root the derivatives $P^{(h)}(p) = 0$ for all $h < r$.

The relation (11) can be used for the estimation of Δp for its sufficiently small value. Let h_1 be the lowest value of h giving non-zero value of derivative. Further let h_2 be the lowest value of h (if any) giving non-zero value of

$$Q(h_2) = \sum_{k=0}^{n} \binom{k}{h_2} \, . \, b_{kj} \, . \, p^{k-h_2} \, . \tag{11-a}$$

The dominant variation Δp can be than estimated for small values of Δp:

$$\Delta p \approx \left(\frac{-Q(h_2) \, . \, \Delta x_j}{P^{(h_1)}(p)} \right)^{1/(h_1 - h_2)} . \tag{12}$$

If there does not exist any value of h_2 giving non-zero $Q(h_2)$ or if $h_1 < h_2$ then the root is invariant with respect to the variations of x_j.

3. THE PROBLEMS OF SYNTHESIS

3.1. General

The relations (7), (8) bind the values of the parameters $x_1; \ldots; x_m$ with the values of the corresponding roots $p_1; \ldots; p_n$. The functions $K_j(\sigma, \omega)$, $L_j(\sigma, \omega)$ are too complicated to allow an efficient solution in respect of the roots. On the other hand they are linear in respect of the parameters $x_1; \ldots; x_m$, thus making possible a relatively simple analysis of the parametric space if the values of the roots are known or defined. The problems of this kind are known as the problems of synthesis.

3.2. Complete Synthesis

The problem of complete synthesis is defined as follows:

Values of the parameters $x_1; \ldots; x_m$ are to be found from given values of all roots p_k $(k = 1; \ldots; n)$ of the equation (1). Expressing $P(p)$ by means of its root factors we get the relation:

$$\prod_{k=1}^{n} (p - p_k) = \sum_{k=0}^{n} a_k \cdot p_k \,. \tag{13}$$

The comparison of corresponding coefficients of the left-hand and the right-hand sides gives the values of a_k for the system (2), thus giving:

$$\sum_{j=1}^{m} b_{kj} \cdot x_j = a_k - b_{k0} \tag{14}$$

for all k, up to $k = n - 1$.

The properties of such a system are well known and can be interpreted for the purpose of the complete synthesis as follows:

1. The system (14) has a solution if and only if the matrix of the system (the matrix of the left-hand side coefficients) and the augmented matrix (obtained by joining the column of negative right-hand values to the matrix of the system) are of the same rank. If the ranks of both matrices are different there does not exist any set of parametric values giving the given set of roots of the thus defined equation (1).
2. If the system has a solution and if the ranks of both matrices are m then the system has a unique solution. There exists exactly one set of parameters (one point in the parametric space X) giving the desired set of roots. The number of the linear equations (11) may be equal or superior to m. In the latter case some of the equations are lineary dependent.
3. If the system is solvable and the rank of both matrices r is inferior to m there exists at least one set of $(m - r)$ free parameters, the values of which may be chosen arbitrarily; the values of the remaining basic parameters can be then established from the system (14).

3.3. The Synthesis of Roots

The problem consists in establishing necessary and sufficient conditions for those parametric values, ensuring the existence of a given real root or of a given pair of complex roots. The problem may be extended to the requirement of simultaneous existence of more than one root, resp. of more than one pair of complex roots. Two cases are to be distinguished:

1. the synthesis of simple roots,
2. the synthesis of multiple roots.

There are two basic methods of solving this problem.

Method I

The root values are introduced into the relations (7), (8). The equations, linear in respect of the parameters, which are obtained in this way, define a subspace of X, ensuring the existence of the respective roots. If this subspace is represented by an empty set it is evident that the complete set of the chosen roots cannot be obtained with the given structure of the parametric relations (2). It must be kept in mind that the relation (8) is always fulfilled for any real root as ω has zero value. Thus the relation (8) does not represent any limiting condition in the synthesis of real roots, only the relation (7) being significant in this respect; this holds for the synthesis of simple real roots. – This method is not directly applicable in the synthesis of multiple roots.

Method II

The introduction of (2) into (1) gives an algebraic equation containing in explicite form the parameters $x_1; \ldots; x_m$ within its coefficients. Let us divide this polynomial by the product of all root factors, corresponding to the desired roots taking into account their eventual multiplicity. This division gives a resulting polynomial, as well as a residual polynomial, the coefficients of which are generally functions of the parameters. The existence of the given roots is equivalent with the requirement of zero value of any of the residual polynomials. The mathematical formulation of this condition gives a set of limitations, defining the chosen subspace of X, thus ensuring the existence of all required roots, including their eventual multiplicity.

3.4. Example of a Root Synthesis

3.4.1. *Formulation of the Polynomial*

May the polynomial be defined as follows:

$$P(p) = p^6 + 6p^5 + (12 - 2x_1 + x_2 + 3x_3) \,.\, p^4 +$$

$$+ (35 + 2x_1 + 2x_2 + x_3) \,.\, p^3 + (46 - 2x_1 + 5x_1 + 2x_3) \,.\, p^2 +$$
$$+ (32 + 6x_1 + x_2 + 4x_3) \,.\, p + (20 - 3x_2 + 2x_3) \,.$$

3.4.2. *Synthesis of a Pair of Complex Roots* $p_{1,2} = -1 \pm \mathrm{i}$

Root factor: $p^2 + 2p + 2$.
Resulting polynomial: $p^4 + 4p^3 + (2 - 2x_1 + x_2 + 3x_3) \,.\, p^2 + (23 + 6x_1 - 5x_3) \,.\, p + (-4 - 10x_1 + 3x_2 + 6x_3)$.
Conditions of the zero value of the residual polynomial:

$$14x_1 - 7x_2 + 2x_3 = 6 \,,$$
$$20x_1 - 9x_2 - 10x_3 = -28 \,.$$

These linear equations define a straight line in the three-dimensional parametric space, representing the subspace of $\boldsymbol{X}$, which ensures the existence of at least one given pair of roots.

3.4.3. *Synthesis of two pairs of roots*: $p_{1,2} = -1 \pm \mathrm{i}$; $p_{3,4} = -1 \pm 2\mathrm{i}$

Root factor: $p^4 + 4p^3 + 11p^2 + 14p + 10$.
Resulting polynomial: $p^2 + 2p + (-7 - 2x_1 + x_2 + 3x_3)$.
Zero residual polynomial:

$$10x_1 - 2x_2 - 11x_3 + 27 = 0 \,,$$
$$20x_1 - 6x_2 - 31x_3 + 85 = 0 \,,$$
$$34x_1 - 15x_2 - 38x_3 + 110 = 0 \,,$$
$$20x_1 - 13x_2 - 28x_3 + 90 = 0 \,.$$

This system of equations has a unique solution:

$$x_1 = 1 \,; \quad x_2 = 2 \,; \quad x_3 = 3 \,.$$

3.4.4 *Synthesis of a Pair of Double Roots* $p_{1,2,3,4} = -1 \pm \mathrm{i}$

Root factor: $p^4 + 6p^3 + 14p^2 + 16p + 8$.
Resulting polynomial: $p^2 + 2p + (-4 - 2x_1 + x_2 + 3x_3)$.
Zero residual polynomial:

$$10x_1 - 2x_2 - 11x_3 + 27 = 0 \,,$$
$$14x_1 - 3x_2 - 22x_3 + 58 = 0 \,,$$
$$22x_1 - 7x_2 - 20x_3 + 64 = 0 \,,$$
$$8x_1 - 7x_2 - 20x_3 + 36 = 0 \,.$$

This system of equations does not have any solution; the synthesis of the desired double roots is not feasible.

3.5. Synthesis of Root Regions

3.5.1. *Formulation of the Problem*

The problem consists in determining that part of the parametric space, which corresponds to the defined region of the complex root plane. The determination of stability regions in the parametric space is evidently a special case of this problem.

3.5.2. *General Way of Solution*

This is evidently a problem of mapping between the root plane and the parametric space. The solution can be found in two steps:

1. Mapping of the boundary of the region.
2. Determination of that side of the boundary image, which corresponds to the interior of the region.

Before starting the solution it is necessary to make sure that the entire region defined in the root plane allows mapping. For example mapping can be exactly defined only for such regions, which are symmetrical to the real axis of the root plane. This follows from the fact that the functions $K_j(\sigma, \omega)$, $L_j(\sigma, \omega)$ according to (6) are even functions of ω.

3.5.3. *Mapping of the Boundary*

Let us consider the mapping of that part of the region, which is represented by the region lying in the upper half of the root plane. According to what has been said before, this maps simultaneously the conjugate part lying in the lower half of the root plane. The part under consideration is generally limited by a part of the real axis and by a continuous curve of complex root values with positive imaginary parts.

Any single real root $p = \sigma$ of the root plane corresponds to one relation of the form (7), which defines a plane of the parametric space, ensuring the existence of this root. The differentiation of (7) in respect of p yields another relation, linear in respect of the parameters $x_1; \ldots; x_m$, i.e. another plane of the parametric space. As known from differential geometry, the intersection of both these planes defines a straight line, corresponding to the existence of a double root of the same value p. The variation of the value p along the boundary lying on the real axis of the root plane gives a system of straight lines, forming a skew surface in the m-dimensional parametric space Thus any point of this skew surface corresponds to the mapping of a double real root.

Together with any simple complex root the corresponding conjugate root is defined, too. The relations (7), (8) define a straight line in the parametric space, ensuring the existence of at least one pair of the given complex conjugate roots. The variation of p along the complex boundary gives again a system of straight lines in the parametric space, representing another skew surface as the result of mapping of the complex part of the boundary of the root plane. This skew surface begins and ends on the skew surface of the double real roots, the final straight lines giving the intersection of the image of the "complex" curve with the image of the real axis of the root plane.

The singular case of the linear dependence of (7), (8) may occur, defining a plane in the parametric space and thus partitioning supplementarily the parametric space.

Double complex roots are again defined by simultaneous validity of (7), (8) and their derivatives. Higher derivatives give higher multiplicity of the roots.

The skew surface and the singular planes may form own and mutual intersections, thus partitioning the parametric space often in a very complicated manner. Geometrical interpretation surpasses in most cases the capabilities of intuitive imagination. The analytic discussion of the local behaviour is usualy the more efficient way of treatment.

3.5.4. *Local Investigation of the Boundaries*

The boundary may be expressed in the complex root plane in parametric form, using an auxiliary real parameter λ:

$$\sigma = f(\lambda)\,, \quad \omega = g(\lambda)\,. \tag{15}$$

The sense of increasing values of λ defines a direction of circulation on the boundary with one point of discontinuity. These functions are assumed continuous in respect of σ, ω and at least by parts smooth. Thus the functions (15) have at least one derivative everywhere, except, perhaps, a finite number of points. In any point of smoothness there exists a tangent of the boundary with the angle of inclination φ_t:

$$\cos \varphi_t = \frac{\dfrac{\mathrm{d}f(\lambda)}{\mathrm{d}\lambda}}{\sqrt{\left[\left(\dfrac{\mathrm{d}f(\lambda)}{\mathrm{d}\lambda}\right)^2 + \left(\dfrac{\mathrm{d}g(\lambda)}{\mathrm{d}\lambda}\right)^2\right]}}\,; \quad \sin \varphi_t = \frac{\dfrac{\mathrm{d}g(\lambda)}{\mathrm{d}\lambda}}{\sqrt{\left[\left(\dfrac{\mathrm{d}f(\lambda)}{\mathrm{d}\lambda}\right)^2 + \left(\dfrac{\mathrm{d}g(\lambda)}{\mathrm{d}\lambda}\right)^2\right]}}\,. \tag{16}$$

The normal at this point has the angle of inclination

$$\varphi_n = \varphi_t \pm \frac{\pi}{2}\,. \tag{17}$$

Let us choose the sign in (17) in such a way that the normal be directed to the interior of the region.

May the set of parametric values $\{x_1; \ldots; x_m\}$ define a single root $p(\lambda)$ on the skew surface, which maps the "complex" root boundary. Then according to (9) a derivative exists in respect of any parameter x_j $(j = 1; \ldots; m)$, which can be transformed to the polar form:

$$\frac{\partial p}{\partial x_j} = \frac{\partial \sigma}{\partial x_j} + i \,.\, \frac{\partial \omega}{\partial x_j} = D_j \,.\, \exp(\psi_j \,.\, i)\,, \tag{18}$$

where $D_j \geqq 0$ is the modulus; ψ_j the argument of the derivative.

The value of $\cos(\psi_j - \varphi_n)$ represents a criterion of the movement of the respective root, produced by the variation of the respective parameter x_j. A positive increment of the parameter shifts the root in the direction of the interior if $\cos(\psi_j - \varphi_n) > 0$; in the direction of the outside if $\cos(\psi_j - \varphi_n) < 0$; and in the direction of the tangent if $\cos(\psi_j - \varphi_n) = 0$. The value of D_j is a measure of the "sensitivity" of this movement.

Similarily the value of $\cos(\psi_j - \varphi_t)$ makes possible the estimation of the movement of the root in respect of the direction of the tangent of the boundary.

3.5.6. *Mapping of the Real Part of the Boundary*

The mapping of the real part of the boundary merits special attention.

Any point of the real part of the boundary allows two ways of interpretation:

1. as a single real root;
2. as a double real root.

In both cases the values of $M_j(p)$ $(j = 0; 1; \ldots; m)$ according to (3) and (4) are real.

A definite real derivative, given by (9), exists in the former case, describing the movement of the root along the real axis [2]. This can be interpreted as a position "apart from the boundary of the region of complex roots".

The latter case happens if the root and the parameters fulfill the two conditions:

$$M_0(p) + \sum_{j=1}^{m} M_j(p) \,.\, x_j = 0\,, \tag{19}$$

$$M_0^{(1)}(p) + \sum_{j=1}^{m} M_j^{(1)}(p) \,.\, x_j = 0\,, \tag{20}$$

where

$$M_j^{(1)}(p) = \frac{d}{dp} M_j(p) \quad (j = 0; 1; \ldots; m)\,. \tag{21}$$

The discussion of the proximity of the skew surface (19), (20) can be done by means of the method given above (formulae (11), (12)).

3.5.7. *Conclusions on Root Region Synthesis*

For the general boundary in the root plane, consisting of a real and a complex part, two skew surfaces are defined as mapping to the parametric space, one corresponding to the real part, the other to the complex part of the boundary.

The former is a part of the unlimited skew surface, mapping the entire real axis of the root plane, as represented by double real roots; it separates the parametric space into two halfspaces: that of the pairs of real roots (equal or different) and that of the pairs of complex conjugate roots.

The latter skew surface starts and ends on the former one by straight lines, which are common to both skew surfaces, thus separating the parametric halfspace of the complex roots into the image of the interior and that of the exterior of the synthetized root region.

This division of the parametric space may be quite complicated, the separating skew surfaces intersecting oneself, as well as each other. Moreover the separation may comprise singular planes, corresponding to linear dependence of the defining equations.

The examination of the nature of the boundaries and of the consequences of parametric variations was given in 3.5.5 and 3.5.6.

4. EXAMPLE OF DECOMPOSITION IN THE THREE-PARAMETRIC SPACE

4.1. The System under Examination

The decomposition, given further, concerns the system defined by the algebraic equation:

(A)
$$p^4 + (5 + 0{\cdot}2x_1) . p^3 + (11 + x_1 + 0{\cdot}2x_2) . p^2 + + (15 + x_2 + 0{\cdot}2x_3) . p + x_3 = 0 .$$

The introduction of $p = \sigma + i . \omega$ gives, after the decomposition into the real and imaginary part, the following basic relations:

(B-1)
$$((-0{\cdot}6\sigma - 1) . \omega^2 + (0{\cdot}2\sigma^3 + \sigma^2)) . x_1 + (-0{\cdot}2\omega^2 + (0{\cdot}2\sigma^2 + \sigma)) . x_2 + (0{\cdot}2\sigma + 1) . x_3 = = -\omega^4 + (6\sigma^2 + 15\sigma + 11) . \omega^2 - (\sigma^4 + 5\sigma^3 + 11\sigma^2 + 15\sigma) ,$$

(B-2)
$$\omega((-0{\cdot}2\omega^2 + (0{\cdot}6\sigma^2 + \sigma)) . x_1 + (0{\cdot}4\sigma + 1) . x_2 + 0{\cdot}2x_3) = = \omega((4\sigma + 5) . \omega^2 - (4\sigma^3 + 15\sigma^2 + 22\sigma + 15)) .$$

The differentiation of (A) with respect to the parameters gives:

(C-1) $$\frac{\partial p}{\partial x_1} \, . \, Q(p; x_1; x_2; x_3) = -(0{\cdot}2p^3 + p^2) \, ,$$

(C-2) $$\frac{\partial p}{\partial x_2} \, . \, Q(p; x_1; x_2; x_3) = -(0{\cdot}2p^2 + p) \, ,$$

(C-3) $$\frac{\partial p}{\partial x_3} \, . \, Q(p; x_1, x_2; x_3) = -(0{\cdot}2p + 1) \, ,$$

where

(C-4) $$Q(p; x_1; x_2; x_3) = (4p^3 + 15p^2 + 22p + 15) + \\ + (0{\cdot}6p^2 + 2p) \, . \, x_1 + (0{\cdot}4p + 1) \, . \, x_2 + 0{\cdot}2x_3 \, .$$

4.2. Real Roots

The introduction of $\omega = 0$ into (B-1) gives the condition of the existence of at least one real root $p = \sigma$ in the form:

(D-1) $$(0{\cdot}2\sigma^3 + \sigma^2) \, . \, x_1 + (0{\cdot}2\sigma^2 + \sigma) \, . \, x_2 + (0{,}2\sigma + 1) \, . \, x_3 = \\ = -(\sigma^4 + 5\sigma^3 + 11\sigma^2 + 15\sigma) \, .$$

The relation (B-2) is always fulfilled by virtue of $\omega = 0$. The zero value of the remaining factor (B-2) gives:

(D-2) $$(0{\cdot}6\sigma^2 + 2\sigma) \, . \, x_1 + (0{\cdot}4\sigma + 1) \, . \, x_2 + 0{\cdot}2x_3 = \\ = -(4\sigma^3 + 15\sigma^2 + 22\sigma + 15) \, ,$$

which is the derivative of (D-1) in respect of σ. The simultaneous fulfilment of both (D-1) and (D-2) gives the relations for the skew surface of the double roots with σ as parameter; this skew surface separates the parametric space into the halfspace of real roots and that of complex roots.

Another differentiation of (D-2) with respect to σ gives:

(D-3) $$P^{(2)}(\sigma) = (12\sigma^2 + 30\sigma + 22) + (1{\cdot}2\sigma + 2) \, . \, x_1 + 0{\cdot}4x_2 \, .$$

For $\sigma = 0$ the expression (D-1) defines the plane

(E-1) $$x_3 = 0$$

and (D-2) defines another plane:

(E-2) $$x_2 + 0{\cdot}2x_3 = -15 \, ,$$

 their intersection giving the parametric set of at least double zero roots:

(E-3) $$x_2 = -15\,, \quad x_3 = 0\,.$$

The introduction of (E-3) into (D-3) gives:

(E-4) $$P^{(2)}(0) = 16 + 2x_1\,.$$

Thus the parametric values of $x_1 = -8$; $x_2 = -15$; $x_3 = 0$ give a threefold zero root, the remaining non-zero root being $p_4 = -3{\cdot}4$.

Other values of x_1, combined with $x_2 = -15$; $x_3 = 0$, give two zero roots, the remaining two roots being different from zero. For example $x_1 = -3{\cdot}2843$ .. gives a twofold zero root and another twofold real root $p_{3,4} = -2{\cdot}1716$ ...; similarily for $x_1 = 53{\cdot}2843$ ... a twofold zero root and another twofold real root $p_{3,4} = -7{\cdot}8284$ ... is obtained. These points of the parametric space are points of intersection of the skew surface of double real roots with itself.

Let us now perform the examination of the point $(-3{,}2843;\ -15;\ 0)$ on the variability of the roots with the variation of parameters. Applying (11), (12) we get:

$$P^{(2)}(0) = 9{,}4315\,,$$
$$\Delta p_{x1} = 0\,,$$
$$\Delta p_{x2} = -0{\cdot}1060\,.\,\Delta x_2\,,$$

only one of the pair of roots being subject to variation

$$\Delta p_{x3} = \pm 0{\cdot}1456\,.\,\sqrt{(\Delta x_3)}\,.\,\mathrm{i} \quad \text{for} \quad \Delta x_3 > 0\,,$$
$$\Delta p_{x3} = \pm 0{\cdot}1456\,.\,\sqrt{|\Delta x_3|} \quad \text{for} \quad \Delta x_3 < 0\,.$$

This indicates that the positive increment Δx_3 is directed to the region of pairs of complex roots, the negative increment to the direction of pairs of real unequal roots.

The analysis of the same point in respect to the real root $p = \sigma = -2{\cdot}1716$ gives:

$$P^{(2)}(-2{\cdot}1716) = 9{\cdot}4315$$

with the variability:

$$\text{For } x_1 : \Delta p(x_1) \approx \pm 0{\cdot}5318\,\sqrt{(-\Delta x_1)},$$
$$\text{for } x_2 : \Delta p(x_2) \approx \pm 0{\cdot}3609\,\sqrt{(+\Delta x_2)},$$
$$\text{for } x_3 : \Delta p(x_3) \approx \pm 0{\cdot}2449\,\sqrt{(-\Delta x_3)}.$$

In all cases both roots undergo variations with the variation of any of the parameters x_1; x_2; x_3. For Δx_1 the positive increment is directed to the region of complex roots; the same is valid for the positive increment of Δx_3; but the shift is directed to the region of complex roots for negative increments Δx_2.

$p_{1,2} = \sigma$

σ	Ruling straight line			
0		$x_2 = -15;$		$x_3 = 0$
$-0{\cdot}25$	$x_1 - 2$	$x_2 = 23{\cdot}232;$	$x_1 - 16$	$x_3 = -6{\cdot}364$
$-0{\cdot}5$	$x_1 -$	$x_2 = 9{\cdot}367;$	$x_1 - 4$	$x_3 = -4{\cdot}878$
$-0{\cdot}75$	$x_1 - 0{\cdot}667$	$x_2 = 5{\cdot}383;$	$x_1 - 1{\cdot}778$	$x_3 = -3{\cdot}573$
-1	$x_1 - 0{\cdot}5$	$x_2 = 3{\cdot}75;$	$x_1 -$	$x_3 = -2{\cdot}5$
$-1{\cdot}25$	$x_1 - 0{\cdot}4$	$x_2 = 2{\cdot}931;$	$x_1 - 0{\cdot}64$	$x_3 = -1{\cdot}722$
$-1{\cdot}5$	$x_1 - 0{\cdot}333$	$x_2 = 2{\cdot}372;$	$x_1 - 0{\cdot}444$	$x_3 = -1{\cdot}327$
$-1{\cdot}75$	$x_1 - 0{\cdot}286$	$x_2 = 1{\cdot}789;$	$x_1 - 0{\cdot}327$	$x_3 = -1{\cdot}435$
-2	$x_1 - 0{\cdot}25$	$x_2 = 0{\cdot}972;$	$x_1 - 0{\cdot}25$	$x_3 = -2{\cdot}222$
$-2{\cdot}25$	$x_1 - 0{\cdot}222$	$x_2 = -0{\cdot}288;$	$x_1 - 0{\cdot}198$	$x_3 = -3{\cdot}946$
$-2{\cdot}5$	$x_1 - 0{\cdot}2$	$x_2 = -2{\cdot}25;$	$x_1 - 0{\cdot}16$	$x_3 = -7$
$-2{\cdot}75$	$x_1 - 0{\cdot}182$	$x_2 = -5{\cdot}290;$	$x_1 - 0{\cdot}132$	$x_3 = -12{\cdot}006$
-3	$x_1 - 0{\cdot}167$	$x_2 = -10$	$x_1 - 0{\cdot}111$	$x_3 = -20$

Tab. I contains the analytic expression of some ruling straight lines of the skew surface of double real roots.

4.3. Complex Boundaries

In most cases the complex boundaries consist of simple roots, except perhaps some isolated multiple roots. The examination does not present difficulties and the method of differentiation (formula (9)) is applicable in establishing the variations of the roots. Some examples are given in the following.

For the limit of stability ($\sigma = 0$) the following equations are obtained:

$$(\text{F-1}) \qquad -\omega^2 \,.\, x_1 - 0{\cdot}2\omega^2 \,.\, x_2 + x_3 = -\omega^4 + 11\omega^2 \,,$$

$$(\text{F-2}) \qquad -0{\cdot}2\omega^2 \,.\, x_1 + x_2 + 0{\cdot}2x_3 = 5\omega^2 - 15 \,.$$

Tab. II gives the equations of several ruling straight lines of the corresponding skew surface. It is worth noting, that the separation applies only for the halfspace of complex roots, but is not valid for the halfspace of unequal real roots.

The further example gives a skew surface of constant damping ($\sigma = \lambda$; $\omega = 1{\cdot}5\lambda$):

$$(\text{F-3}) \qquad (-1{\cdot}15\lambda^3 - 1{\cdot}25\lambda^2) \,.\, x_1 + (-0{\cdot}25\lambda^2 + \lambda) \,.\, x_2 + (0{\cdot}2\lambda + 1) \,.\, x_3 = \\ = 7{\cdot}4375\lambda^4 + 28{\cdot}75\lambda^3 + 13{\cdot}75\lambda^2 - 15\lambda \,,$$

$$(\text{F-4}) \qquad (0{\cdot}15\lambda^2 + 2\lambda) \,.\, x_1 + (0{\cdot}4\lambda + 1) \,.\, x_2 + 0{\cdot}2 \,.\, x_3 = \\ = 5\lambda^3 - 3{\cdot}75\lambda^2 - 22\lambda - 15 \,.$$

Tab. II.

$\sigma = 0$

ω	Ruling straight line
0	$x_2 = -15;$ $x_3 = 0$
0·25	$x_2 = -14{\cdot}788;$ $x_1 - 16$ $x_3 = -7{\cdot}980$
0·5	$x_2 = -14{\cdot}146;$ $x_1 - 4$ $x_3 = -7{\cdot}921$
0·75	$x_2 = -13{\cdot}068$ $x_1 - 1{\cdot}778$ $x_3 = -7{\cdot}824$
1	$x_2 = -11{\cdot}538;$ $x_1 -$ $x_3 = -7{\cdot}692$
1·25	$x_2 = -\ 9{\cdot}540;$ $x_1 - 0{\cdot}64$ $x_3 = -7{\cdot}529$
1·5	$x_2 = -\ 7{\cdot}053;$ $x_1 - 0{\cdot}444$ $x_3 = -7{\cdot}339$
1·75	$x_2 = -\ 4{\cdot}053;$ $x_1 - 0{\cdot}327$ $x_3 = -7{\cdot}127$
2	$x_2 = -\ 0{\cdot}517;$ $x_1 - 0{\cdot}25$ $x_3 = -6{\cdot}897$
2·25	$x_2 = +\ 3{\cdot}577;$ $x_1 - 0{\cdot}198$ $x_3 = -6{\cdot}653$
2·5	$x_2 = +\ 8{\cdot}25;$ $x_1 - 0{\cdot}16$ $x_3 = -6{\cdot}4$
2·75	$x_2 = +13{\cdot}523;$ $x_1 - 0{\cdot}132$ $x_3 = -6{\cdot}142$
3	$x_2 = +19{\cdot}412;$ $x_1 - 0{\cdot}111$ $x_3 = -5{\cdot}882$

Tab. III.

$\sigma = \lambda;\quad \omega = \pm 1{\cdot}5\,.\,\lambda$

λ	Ruling straight line
0	$x_2 = -15;$ $x_3 = 0$
$-0{\cdot}25$	$x_1 - 2$ $x_2 = 22{\cdot}375;$ $x_1 - 4{\cdot}923$ $x_3 = -6{\cdot}309$
$-0{\cdot}5$	$x_1 -$ $x_2 = 7{\cdot}889;$ $x_1 - 1{\cdot}231$ $x_3 = -4{\cdot}609$
$-0{\cdot}625$	$x_1 - 0{\cdot}8$ $x_2 = 5{\cdot}209;$ $x_1 - 0{\cdot}788$ $x_3 = -3{\cdot}740$
$-0{\cdot}75$	$x_1 - 0{\cdot}667$ $x_2 = 3{\cdot}581;$ $x_1 - 0{\cdot}547$ $x_3 = -2{\cdot}848$
$-0{\cdot}875$	$x_1 - 0{\cdot}571$ $x_2 = 2{\cdot}574;$ $x_1 - 0{\cdot}402$ $x_3 = -1{\cdot}923$
-1	$x_1 - 0{\cdot}5$ $x_2 = 1{\cdot}978;$ $x_1 - 0{\cdot}308$ $x_3 = -0{\cdot}959$
$-1{\cdot}25$	$x_1 - 0{\cdot}4$ $x_2 = 1{\cdot}588;$ $x_1 - 0{\cdot}197$ $x_3 = +1{\cdot}122$
$-1{\cdot}5$	$x_1 - 0{\cdot}333$ $x_2 = 1{\cdot}892;$ $x_1 - 0{\cdot}137$ $x_3 = +3{\cdot}448$
$-1{\cdot}75$	$x_1 - 0{\cdot}286$ $x_2 = 2{\cdot}623;$ $x_1 - 0{\cdot}100$ $x_3 = +6{\cdot}042$
-2	$x_1 - 0{\cdot}25$ $x_2 = 3{\cdot}611;$ $x_1 - 0{\cdot}077$ $x_3 = +8{\cdot}889$

Tab. III gives some equations of the ruling straight lines of this skew surface. The case of constant measure of stability ($\sigma = -1$) leads to the equations:

$$(-0{\cdot}4\omega^2 + 0{\cdot}8)\,.\,x_1 + (-0{\cdot}2\omega^2 - 0{\cdot}8)\,.\,x_2 + 0{\cdot}8\,.\,x_3 = -\omega^4 + 2\omega^2 + 8\,, \tag{F-5}$$

$$(-0{\cdot}2\omega^2 - 1{\cdot}4)\,.\,x_1 + 0{\cdot}6\,.\,x_2 + 0{\cdot}2\,.\,x_3 = \omega^2 - 4\,. \tag{F-6}$$

Some equations of the ruling straight lines are summarized in Tab. IV.

Tab. IV.

$\sigma = -1$

ω	Ruling straight line
0	$x_1 - 0{\cdot}5\,x_2 = 3{\cdot}75;\quad x_1 - \quad x_3 = -2{\cdot}5$
0·25	$x_1 - 0{\cdot}5\,x_2 = 3{\cdot}715;\quad x_1 - 0{\cdot}941\,x_3 = -2{\cdot}451$
0·5	$x_1 - 0{\cdot}5\,x_2 = 3{\cdot}606;\quad x_1 - 0{\cdot}8\,x_3 = -2{\cdot}308$
0·75	$x_1 - 0{\cdot}5\,x_2 = 3{\cdot}405;\quad x_1 - 0{\cdot}64\,x_3 = -2{\cdot}075$
1	$x_1 - 0{\cdot}5\,x_2 = 3{\cdot}088;\quad x_1 - 0{\cdot}5\,x_3 = -1{\cdot}765$
1·25	$x_1 - 0{\cdot}5\,x_2 = 2{\cdot}624;\quad x_1 - 0{\cdot}390\,x_3 = -1{\cdot}388$
1·5	$x_1 - 0{\cdot}5\,x_2 = 1{\cdot}978;\quad x_1 - 0{\cdot}308\,x_3 = -0{\cdot}959$
1·75	$x_1 - 0{\cdot}5\,x_2 = 1{\cdot}114;\quad x_1 - 0{\cdot}246\,x_3 = -0{\cdot}492$
2	$x_1 - 0{\cdot}5\,x_2 = 0;\quad x_1 - 0{\cdot}2\,x_3 = 0$
2·25	$x_1 - 0{\cdot}5\,x_2 = -1{\cdot}395;\quad x_1 - 0{\cdot}165\,x_3 = +0{\cdot}504$
2·5	$x_1 - 0{\cdot}5\,x_2 = -3{\cdot}097;\quad x_1 - 0{\cdot}138\,x_3 = +1{\cdot}011$
2·75	$x_1 - 0{\cdot}5\,x_2 = -5{\cdot}126;\quad x_1 - 0{\cdot}117\,x_3 = +1{\cdot}512$
3	$x_1 - 0{\cdot}5\,x_2 = -7{\cdot}5;\quad x_1 - 0{\cdot}1\,x_3 = +3{\cdot}08$

4.4. Examination of a Region

Let us define the boundary of the region in parametric form:

(G) $$\sigma = -2 + 2\,.\cos \lambda\pi\,, \quad \omega = 2\,.\sin \lambda\pi$$

and the interior of this circle let represent the defined region. The corresponding part of the parametric space is limited by the skew surface of the real interval of the root plane $\langle -4; 0\rangle$ and by the skew surface defined by the half of the circle in the root plane (skew surface C). The parametric equations of the skew surface C are obtained by introducing (G) into (B-1), (B-2). The direction of the interior of the circle is given by the angle of the normal:

(H) $$\varphi_n = -\lambda\,.\,\pi\,.$$

This example is applied to illustrate the principles given in 3.5.5. The analysis for three values of the parameter λ and for the arbitrarily chosen value $x_3 = 10$ is given in Tab. V.

Tab. V.

$\sigma = -2 + 2 \, . \cos \lambda\pi; \quad \omega = 2 \, . \sin \lambda\pi$

λ φ_n $p_{1,2}$		0·25 −0·25π −0·586 ±1·414 . i	0·5 −0·5π −2 ±2 . i	0·75 −0·75π −3·414 ±1·414 . i	
Ruling straight line		$x_1 - 0{\cdot}853 \, . \, x_2 =$ $= 3{\cdot}076$ $x_1 - 0{\cdot}427 \, . \, x_3 =$ $= -3{\cdot}451$	$x_1 - 0{\cdot}25 \, . \, x_2 =$ $= 4{\cdot}519$ $x_1 - 0{\cdot}125 \, . \, x_2 =$ $= 4{\cdot}615$	$x_1 - 0{\cdot}146 \, . \, x_2 =$ $= -0{\cdot}241$ $x_1 - 0{\cdot}073 \, , \, x_3 =$ $= -10{\cdot}157$	
$x_1; x_2; x_3$ p_3 p_4		0·817; −2·647; 10 −1·996 +0·533 . i −1·996 −0·533 . i	5·865; 5·385; 10 −1·087 +0·264 . i −1·087 −0·264 . i	−9·425; −62·714; 10 +3·505 +0·209	
$\frac{\partial p_1}{\partial x_j}$	$j = 1$ 2 3	−0·076 −0·176 . i −0·087 +0·090 . i +0·076 +0·030 . i	−0·289 +0·083 . i +0·093 +0·052 . i −0·010 −0·036 . i	−0·037 +0·065 . i +0·016 −0·012 . i −0·005 +0·001 . i	
$\frac{\partial p_1}{\partial x_j}$	$j = 1$ 2 3	0·192 . exp (−1·978 . i) 0·125 . exp (+2·342 . i) 0·082 . exp (+0·378 . i)	0·301 . exp (2·862 . i) 0·106 . exp (0·506 . i) 0·038 . exp (−1·851 . i)	0·075 . exp (2·087 . i) 0·020 . exp (−0·661 . i) 0·005 . exp (2·873 . i)	
$\cos (\psi_j - \varphi_n)$ $j = 1$ $j = 2$ $j = 3$		 cos (−1·192) = +0·369 cos (3·127) = −1·000 cos (1·164) = +0·396	 cos (−1·851) = −0·276 cos (2·076) = −0·484 cos (−0·280) = +0·961	 cos (−1·840) = −0·266 cos (1·695) = −0·124 cos (−1·054) = +0·494	
				p_3	p_4
$\frac{\partial p_3}{\partial x_j}$	$j = 1$ $j = 2$ $j = 3$	−0·024 +0·611 . i +0·087 −0·283 . i −0·076 +0·121 . i	+0·189 +0·339 . i −0·093 −0·335 . i +0·010 +0·311 . i	−0·127 −0·036 −0·010	+0·001 +0·004 +0·021

5. CONCLUDING REMARKS

The preceding considerations have shown the possibility of generalization of the method of D-decomposition to parametric spaces of more than two dimensions and of more general regions of the root plane. The problem may be formulated as the problem of mapping of pairs of roots (pairs of complex conjugate roots; pairs of double real roots) on the parametric space. Every pair of such roots is mapped into a straight line of the parametric space with the root values as parameters of the coefficients of the equations, which are linear in respect of the structural parameters $x_1; \ldots; x_m$. The set of these systems of two linear equations, formed along a closed boundary, gives a skew surface in the parametric space. The considered region of the root plane may be defined as the interior or the exterior of the closed boundary and effective methods are described to obtain the image in the parametric space. Single real roots are mapped into planes of the parametric space.

Performing such a mapping for several single roots or pairs of roots we get that part of the parametric space, which corresponds to the simultaneous occurence of all considered roots, as the intersection of those parts of the parametric space that are limited by the images of singular roots or pairs of roots. This intersection may be an empty set of the parametric space, thus indicating the incompatibility of the given roots in respect of the given algebraic equation with coefficients represented by linear functions of the structural parameters.

The result of the mapping has a quite clear geometrical sense. But the situation is complicated in the three-dimensional parametric space, and the more in parametric spaces of more than three dimensions, the space being decomposed by the skew surfaces and by the singular planes usually in a very intricate manner. Therefore an analytic solution is more convenient in most cases, making possible the discussion of any point of any boundary. The respective methods have been described.

The preceding discussion is aimed at the methodology and at the analytical and geometrical interpretation of the matter. Some arbitrary assumptions, such as the choice of a constant value of a parameter, used in the illustrative example, do not give a clear idea of the practical applicability of the method. As a matter of fact the multidimensional decomposition is a powerful tool for the solution of many problems of important practical value, such as the design and setting of multiparametric controllers with inherent interaction and other problems of this kind. The problems of practical applicability of the results and methods are worth more detailed consideration and further elaboration.

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