On Finitary Linear Systems

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An abstract operator approach is introduced, permitting a unified study of discrete- and continuous-time linear systems. As an application, an algorithm is given for deciding if a linear system can be built from any fixed linear component. Finally, a criterion is given for reachability of the abstract systems introduced, giving thus a unified proof of known reachability results for discrete-time, continuous-time, and delay-differential systems.

INTRODUCTION

We give in this paper an abstract operator approach which permits a unified study of some problems, like realization, for various kinds of linear systems. This approach improves the one presented in Sontag [7].

Systems will be defined as interconnections of a basic system ϱ — e.g., an integrator, unit delay, or any fixed linear system — with coefficients in a fixed operator ring — representing themselves lumped or distributed elements. A general realization theorem: "finitely realizable = 'rational' " is proved in this context.

Assuming that both a desired linear behavior (input/output map) f and a fixed system Σ are given, a natural question is: can f be realized using Σ as a basic component? We give in Section B.2 an algorithm which serves to answer this question.

Refining the definition of systems allows the explicit consideration of "time" and "pointwise states". We prove in such a context a general criterion of pointwise reachability.

The problems in this paper are directly related to the theory of *linear systems over commutative rings*; for instance, the characterization of finitely realizable behaviors

as rational implies the possibility of effectively constructing realizations, e.g. via Hankel-matrix techniques. See Sontag [8] for a survey of the topic of systems over rings, and Kamen [5] for previous work on the application of systems over rings to the study of distributed systems.

A. ADMISSIBLE TRIPLES

1. Definitions and Elementary Properties

(1.1) Definition. A triple (A, B, ϱ), where B is a commutative ring, A is a B-module and ϱ is a B-endomorphism of A, is called *admissible* if the following conditions hold:

(a) ϱ is transcendental (over B), i.e., for all $b_0, \ldots, b_n$ in B, $\sum b_i \varrho^i \neq 0$ in $\text{End}_B A$.

(b) Given any $b_1, \ldots, b_n$ in B and any u in A, there is one and only one y in A satisfying

$$y + b_1 \varrho y + \ldots + b_n \varrho^n y = \varrho u \ .$$

Assume (A, B, ϱ) is admissible. Denote by ϱA the image of ϱ; it is clearly a B-module. The restriction of ϱ (which we shall denote also by ϱ) is a B-endomorphism of ϱA. It follows from uniqueness in (b) above that $(\varrho A, B, \varrho)$ is also admissible. The smallest subring of $\text{End}_B (A)$ containing ϱ is (isomorphic to) the polynomial ring in ϱ, and we denote it by $B[\varrho]$. By restriction, we also write $B[\varrho]$ for the corresponding subring of $\text{End}_B (\varrho A)$.

(1.2) Lemma. Assume that B is a commutative ring, A is a B-module and ϱ is a transcendental B-endomorphism of A. Then the following statements are equivalent:

(a) (A, B, ϱ) is admissible.

(b) For all n in N, all F in $B^{n \times n}$, and all v in A^n, the equation

$$X = \varrho(FX + v)$$

has a unique solution X in A^n, where $\varrho : A^n \to A^n$ is the coordinatewise extension of ϱ.

(c) There is a subring $B[(\varrho)]$ of $\text{End}_B (\varrho A)$, containing $B[\varrho]$, which can be naturally identified to the ring of rational power series in ϱ.

Proof. (a) $\Rightarrow$ (c). Observe that condition (b) in (1.1) asserts that all polynomials in ϱ having the identity as independent term are bijective on ϱA. Indeed, given $v = \varrho u$ in A, there is a unique y in A with $y + b_1 \varrho y + \ldots + b_n \varrho^n y = v$; but $y = \varrho(u - b_1 y - \ldots - b_n \varrho^{n-1} y) \in \varrho A$, so unique solutions exist in ϱA. The meaning as a map of rational series p/q with p, q in $B[\varrho]$ and $q(0) = $ unit is then clear.

(c) $\Rightarrow$ (b). The given equation is equivalent to finding X in $(I - \varrho F) X = \varrho v \in$ $\in (\varrho A)^n$. As det $(I - \varrho F)$ is invertible in $B[[\varrho]]$, it is a unit in $\mathrm{End}_B (\varrho A)$. So $I - \varrho F$ is an invertible matrix and there is a unique solution, as wanted.

(b) $\Rightarrow$ (a). As in ordinary differential equations, given the equation $y + \ldots$ $\ldots + b_n \varrho^n y = \varrho u$, solve for y_1 in the simultaneous equations

$$y_1 = \varrho(-b_1 y_1 - \ldots - b_n y_n + u),$$

$$y_2 = \varrho y_1,$$

$$\vdots$$

$$y_n = \varrho y_{n-1}. \qquad \qquad \square$$

The elements of $B[[\varrho]]$ will be called *rational* maps.

(1.3) Definition. The subring of $B[[\varrho]]$ corresponding to those power series of order > 0 will be called the ring of *causal rational* endomorphisms. It will be denoted by $B_c[[\varrho]]$.

(1.4) Observation. Causal rational maps s are then those which admit a factorization $s = pq^{-1}$, with $p(0) = 0$, $q(0) =$ unit. Also, any such s admits a factorization $s_1 \circ \varrho$, with s_1 a rational map. But then, we can define an $\hat{s}$ in $\mathrm{End}_B (A)$ by the same formula (where now $\varrho : A \to \varrho A$), and $\hat{s}$ clearly extends s. By abuse of notation, we shall identify s and $\hat{s}$, and hence think also of $B_c[[\varrho]]$ as a subring of $\mathrm{End}_B (A)$, the precise meaning being clear from the context. Accordingly, a B-homomorphism $f : A^m \to A^p$ will be called *causal rational* when each coordinate map belongs to (the set of extensions of) $B_c[[\varrho]]$.

2. Substitution of Rational Maps

We shall assume throughout this section that B is an integral domain. By (A, B, ϱ) we shall denote always an arbitrary admissible triple.

Given $s = b_1 \varrho + b_2 \varrho^2 + \ldots$ in $B_c[[\varrho]]$ and $p = a_0 + a_1 \varrho + \ldots + a_n \varrho^n$ in $B[\varrho] \subseteq \mathrm{End}_B (\varrho A)$, denote by $p_s := a_0 + a_1 s + \ldots + a_n s^n$ the *substitution* of s in p. Observe that $p(0) = a_0 = p_s(0)$, so (in $B_c[[\varrho]]$) p is invertible if and only if p_s is a unit. Given an arbitrary $f = pq^{-1}$ in $B[[\varrho]]$, define $f_s := p_s q_s^{-1}$. It is easy to see that if $f = a_0 + a_1 \varrho + a_2 \varrho^2 + \ldots$ then $f_s = a_0 + a_1 b_1 \varrho + (a_1 b_2 + a_2 b_1^2) \varrho^2 + \ldots$. The assignment $\alpha : f \mapsto f_s$ is a ring endomorphism of $B[[\varrho]]$. In fact, it is injective, because B has no zero divisors. Moreover, if f is in $B_c[[\varrho]]$, f_s is also causal, so α induces an endomorphism of $B_c[[\varrho]]$. If $f : A^m \to A^n$ is causal rational, define f_s by substituting in each coordinate.

(2.1) Lemma. Assume s is in $B_c[\![\varrho]\!]$. Then (A, B, s) is an admissible triple.

Proof. Algebraic independence of s is clear. Let now p in $B[\varrho]$ be such that the element $p(0)$ is a unit. Then (b) in (1.1) amounts to proving that for every u in A there is a unique y with $p_s y = su$. But this follows from the fact that su is in ϱA and p_s is an isomorphism in $\text{End}_B(\varrho A)$. $\qquad\square$

(2.2) Definition. Given f in $B_c[\![\varrho]\!]$, define the *rank* of f, $r(f)$, as the smallest of the numbers $\max\{\deg p, \deg q\}$ for all possible factorizations $f = pq^{-1}$ with p, q in $B[\varrho]$, $p(0) = 0$.

If B is a unique factorization domain, then the above minimum can be found directly from an irreducible factorization of f.

(2.3) Observation. Assume now that $f = pq^{-1} \in B_c[\![\varrho]\!]$, with $p = a_n\varrho^n + \ldots$ $\ldots + a_1\varrho$ and $q = b_m\varrho^m + \ldots + 1$, $a_n b_m \neq 0$. Given $u = rs^{-1}$ with $r(0) = 0$, $s(0) = 1$, define $P := a_n r^n s^{k-n} + \ldots + a_1 rs^{k-1}$ and $Q := b_m r^m s^{k-m} + \ldots + s^k$, where k is the greatest of m and n. It is clear that $f_s = PQ^{-1}$, and a simple calculation shows that $\max\{\deg P, \deg Q\} = k \cdot \max\{\deg r, \deg s\}$.

We have proved then that $r(f_u) \leqq r(f)\, r(u)$. Actually, more is true in an important special case:

(2.4) Proposition. Suppose B is a field. Then, for any f, u in $B_c[\![\varrho]\!]$, $r(f_u) = r(f)\, r(u)$.

Proof. The result will follow if we prove that for p, q, r, s as in (2.3), if $(p, q) = (r, s) = 1$ then also $(P, Q) = 1$. Assume on the contrary that there is some polynomial t dividing both P and Q. Let K be an algebraic extension of B containing a root a of t. Then $P(a) = Q(a) = 0$.

Claim: $s(a) \neq 0$. Indeed, assume $s(a) = 0$ and suppose that $m > n$. Then $Q = b_m r^m + s(b_{m-1} r^{m-1} + \ldots)$, so $0 = Q(a) = b_m r(a)^m$ and $r(a) = 0$. But $(r, s) = 1$ implies that they can not have common roots on any extension. A contradiction is also arrived at if $n > m$.

Write $b := r(a)/s(a) \in K$. Then $0 = P(a) = s(a)^k p(b)$, so $p(b) = 0$. Also, $0 = Q(a) = s(a)^k q(b)$, so $q(b) = 0$. As before, this contradicts the fact that $(p, q) = 1$. $\qquad\square$

The main result is:

(2.5) Theorem. Assume that B is a field. Given (say, as quotients of polynomials) $f : A^m \to A^p$ and $s : A \to A$ both causal rational, there is an algorithm which will either find a $g : A^m \to A^p$ causal rational satisfying $f = g_s$, or it will decide that no such g exists.

Proof. It is enough to prove the theorem for $m = p = 1$. Observe also that if g
as above exists, then it is *unique*: indeed, this is merely another way of saying that the
map $h \to h_s$ is injective. Denote $m := r(f)$, $n := r(s)$; both numbers are calculated
by standard methods from the representations of f and s. Denote $k := m/n$. From
(2.4) it follows that, if a g exists, we must have $r(g) = k$.

Assume $f = \sum c_i \varrho^i$, $s = \sum d_j \varrho^j$. We are trying to find $g = \sum x_i \varrho^i$ satisfying

(∗)
$$c_1 = x_1 d_1 \,,$$
$$c_2 = x_1 d_2 + x_2 d_1^2 \,,$$
$$\text{etc.}$$

Denote by C the column vector with the entries $c_1, \ldots, c_{2m}$ and by D the (triangular)
matrix for the corresponding d_i, so that the $2m$ first rows of (∗) correspond to the
equation $C = DX$, with X a $2m$-vector.

Claim: g exists if and only if both (a) there is a solution X of $C = DX$, and (b)
this solution satisfies rank $H_{m+1,m} =$ rank $H_{m,m} = k$, where

$$
H_{i,j} = \begin{Vmatrix}
x_1 & x_2 & x_3 \ldots x_j \\
x_2 & x_3 \ldots \ldots \\
x_3 & \vdots & \vdots \\
\vdots & \vdots & \vdots \\
x_i & \ldots \ldots \ldots x_{i+j-1}
\end{Vmatrix} \,.
$$

Proof of necessity. If there is such a g, then condition (a) is obviously satisfied;
(b) follows from the fact that the rank of $H_{k,k}$ is $k \le m$, and is equal to the rank of
the (infinite) Hankel matrix of g.

Sufficiency. Assume there is an X satisfying (a) and (b). By Kalman, Falb and
Arbib [3, p. 331], there is (and one can construct) a (unique) g of rank k with first $2m$
coefficients equal to $x_1, \ldots, x_{2m}$. We are only left to prove that $f = g_s$. But $r(g_s) =
= kn = m = r(f)$, and from $C = DX$ it follows that the first $2m$ terms of g_s and f
coincide. Equality is then a consequence of well-known facts from realization theory
over fields. ☐

It is not claimed, of course, that the previous algorithm is efficient, but only that
it gives the required answer.

(2.6) Corollary. Theorem (2.5) holds for B an arbitrary completely integrally
closed domain.

Proof. If K is the field of fractions of B, view f, s as elements of $K_c[(\varrho)]$ and solve
for a g via (2.5). Then there will be a solution in $B_c[(\varrho)]$ if and only if g itself is in
$B_c[(\varrho)]$, by uniqueness.

From Eilenberg [2, Th. 12.2], it is easy to show that this will happen if and only
if the coefficients of the minimal polynomial of g (over k) and the entries of X are
all in B. ☐

3. Examples

(3.1) Finite dimensional constant linear systems. In this case let A consist of all locally integrable functions $f : [0, \infty) \to R$ with $f(0) = 0$. For any such f, let $\varrho(f)$ in A be given by $\varrho(f)(t) = \int_0^t f(s) \, ds$. Then, with the pointwise R-space structure for A, it follows from basic theorems in differential equations that (A, R, ϱ) is admissible. This example shows the improvement over the setup in Sontag [7], where C_+^∞ had to be used for A.

(3.2) Variations of (3.1). It might be wanted to allow for impulses as inputs, but still essentially use functions for the states. For example, in the context of Mikusiński's operational calculus, define $C := $ the convolution algebra of continuous real (or complex)-valued functions with domain $[0, \infty)$. Define $\varrho := s^{-1}$ (the "integration" operator), and $A := s^2 C$. Observe that the elements of ϱA can be identified with (equivalence classes of) piecewise continuous locally integrable functions. Admissibility is well-known (see for example Brand [1, p. 545]).

(3.3) Compositions of a fixed Volterra operator. Here define A as in (3.1). Assume $k(s, t)$ is a (fixed) continuous real valued function for all $0 \leqq t \leqq s$. Define ϱ by $(\varrho x)(s) := \int_0^s k(s, t) x(t) \, dt$, for all x in A. We claim that (A, R, ϱ) is admissible. As B is here a field, we only need to check (b) in (1.1). Write $H := -(b_1 \varrho + \ldots \ldots + b_n \varrho^n)$, then it is known that H is also a Volterra operator, say with kernel $h(s, t)$. Given u in A, let $v := \varrho u$; then v is continuous. The conclusion in (1.1.6) is equivalent to proving that, *for all positive a*, there is a unique solution, in $C[0, a]$, of the equation $y(s) - \int_0^s h(s, t) y(t) \, dt = v(s)$. But this last fact follows from Taylor [9, p. 168].

(3.4) Certain infinite dimensional systems. Let X be a real Banach space, and B a commutative algebra of linear bounded operators $X \to X$. Define A as the additive group of locally Bochner-integrable functions $[0, \infty) \to X$ (see Ladas, Lakshmikanthan [4, p. 10]). Let ϱ be the operator $f \mapsto \int_0^t f(s) \, ds$. Admissibility follows from local existence of solutions of differential equations plus the fact that ϱA consists entirely of (strongly) continuous functions.

(3.5) All the examples (discrete-time, cellular, delay-differential, etc.) in Sontag [7], with ϱ here corresponding to ϱ^{-1} there. For instance, retarded delay-differential equations are introduced as follows. Let A consist of all locally integrable functions $f : R \to R$ with support bounded to the left, i.e. with $f(\tau) = 0$ for all τ sufficiently small. Let $\varrho(f)(t) := \int_{-\infty}^t f(\tau) \, d\tau$, $\sigma_a := a$-second shift operator $\sigma_a u(t) := u(t - a)$, and let $B := R[\sigma_{a_1}, \ldots, \sigma_{a_r}]$, for some fixed set of positive rationally independent real numbers $a_1, \ldots, a_r$. Admissibility is a consequence of the theory of delay equations; note that the definition of A is equivalent to setting initial conditions equal to zero in a suitably chosen interval.

1. Definitions and Realization Theorem

We shall assume throughout that (A, B, ϱ) is a fixed (but arbitrary) admissible triple.

(1.1) Definition. An (A, B, ϱ)-*system* Σ is given by a triple of B-matrices (F, G, H) of dimensions $n \times n$, $n \times m$ and $p \times n$ respectively. The *input* map $L : A^m \to A^n$ is the group homomorphism which assigns to each u in A^m the unique solution x in A^n of the equation $x = \varrho(Fx + Gu)$. The *result* is $f_\Sigma := H \circ L : A^m \to A^p$.

We can interpret Σ as given by an interconnection of "black boxes" with input-output functions ϱ (for example, integrators), interconnected via adders and operators in B (e.g. amplifiers or delays). The inputs act via similar connections, and outputs of Σ are obtained by forming suitable combinations of the outputs of these "black boxes". Other very different interpretations are possible in the case of "cellular systems", etc.

(1.2) Definition. The *rank* of Σ, $r(\Sigma)$ is, by definition, the integer n. A system Σ is said to be of *minimal rank* when it has smallest rank among all Σ with the same result.

(1.3) Theorem. Assume $f : A^m \to A^n$. Then f is the result of some (A, B, ϱ)-system if and only if f is causal rational.

Proof. The problem is that of proving there is a one-to-one correspondence between elements of $B_c[(\varrho)]^{n \times m}$ and maps of the form

$$\sum_{n \geq 0} H(I - \varrho F)^{-1} \varrho G = \sum_{n \geq 0} HF^n G \varrho^{n+1} .$$

But the results in realization theory for discrete-time B-systems give the theorem. $\square$

2. Substitution of Systems

Given an (A, B, ϱ)-system Σ, we want to define the substitution of a scalar (i.e. $m = p = 1$) system Σ_1 into Σ as that system obtained by replacing all "black boxes" marked "ϱ" in Σ by copies of Σ_1. In other words, Σ should consist of suitable connections of the basic system Σ_1. *For simplicity, we shall assume $m = p = 1$ for all systems in this section.*

(2.1) Definition. If $\Sigma_1 = (F, G, H)$ and $\Sigma = (U, V, W)$ are two (A, B, ϱ)-systems, the (A, B, ϱ)-system $\Sigma(\Sigma_1) = (C, D, E)$, of rank $= r(\Sigma) \cdot r(\Sigma_1)$, is defined as follows.

For notational convenience, let the indices run over all pairs (i, j) with $1 \leqq i \leqq r(\Sigma_1)$, $1 \leqq j \leqq r(\Sigma)$). Then, let

$$
\begin{aligned}
C_{(i,j)(r,s)} &:= G_j U_{ir} H_s && \text{if} \quad i \neq r, \\
&:= G_j U_{ir} H_s + F_{js} && \text{if} \quad i = r. \\
D_{(i,j)} &:= G_j V_i . \\
E_{(i,j)} &:= W_i H_j .
\end{aligned}
$$

(2.2) Lemma. Let f be the result of Σ and s the result of Σ_1. Then the result of $\Sigma(\Sigma_1)$ is $f_s = W(I - sU)^{-1} sV$.

Proof. Assume u is in A; we want to find the output of $\Sigma(\Sigma_1)$ given that the input is u. By definition of $\Sigma(\Sigma_1)$, for each (i, j) we have for $x = Lu$:

$$
\begin{aligned}
x_{(i,j)} &= \varrho \big(\sum_{(r,s)} C_{(i,j)(r,s)} x_{(r,s)} + D_{(i,j)} u \big) = \\
&= \varrho \big(\sum_s F_{js} x_{(i,s)} + G_j (\sum_r U_{ir} z_r + V_i u) \big) ,
\end{aligned}
$$

where we have denoted $z_r := \sum_s H_s x_{(r,s)} = H x_r$ and x_r is the column vector in A^n, $n = r(\Sigma_1)$, with entries $x_{(r,1)}, \ldots, x_{(r,n)}$. With the further notation $v_i := \sum_r U_{ir} z_r + V_i u$, the equations above can be written:

$$
x_i = \varrho (F x_i + G v_i) .
$$

Solving for x_i, we have $x_i = (I - \varrho F)^{-1} \varrho G v_i$, hence for every i, $z_i = H x_i = s v_i$. Denote by z the column vector in A^q, $q = r(\Sigma)$, with entries $z_1, \ldots, z_q$. Then it follows from the definition of the v_i that z satisfies the equation $z = s(Uz + Vu)$, so $z = (I - sU)^{-1} sVu$. The output of $\Sigma(\Sigma_1)$ is then

$$
\begin{aligned}
y = Ex &= \sum_{(i,j)} E_{(i,j)} x_{(i,j)} = \sum_i W_i (\sum_j H_j x_{(i,j)}) = \\
&= \sum_i W_i z_i = Wz = W(I - sU)^{-1} sVu = f_s u ,
\end{aligned}
$$

as wanted. □

(2.3) Theorem. Assume that B is a field and that Σ, Σ_1 are minimal rank (A, B, ϱ)-systems. Then, $\Sigma(\Sigma_1)$ is also minimal.

Proof. It is well known that minimality for any system Σ over a field is equivalent to the relation $r(\Sigma) = r(f_\Sigma)$. But, by definition of $\Sigma(\Sigma_1)$, we have

$$
r(\Sigma(\Sigma_1)) = r(\Sigma) r(\Sigma_1) = r(f) r(s) = r(f_s)
$$

using (A.2.4). □

(2.4) Theorem. Assume B is a completely integrally closed domain. Given the (A, B, ϱ)-systems Σ_1, Σ_2, there is an algorithm which either constructs a Σ such that $\Sigma(\Sigma_1)$ has the same input/output behavior as Σ_2 or decides that no such Σ exists.

Proof. A direct consequence of Corollary $(A.2.6)$: if g is found, any realization
algorithm gives a Σ with $f_\Sigma = g$. $\qquad\square$

(2.5) Observation. In some cases, we might want a stronger result. For example, take the delay-differential case, where B is a finitely generated algebra over a field K (here, the reals). From (2.4) we know how to decide if a given delay-differential system Σ_2 can be "simulated" by the use of a system Σ_1 and connections in B, i.e. admitting other delays. But we might want to ask whether the same result can be achieved using only scalar interconnections (i.e. amplifiers). This is readily solved: from the fact that a field is (trivially) completely integrally closed, it must only be checked that the g found is in $K_c[(\varrho)]$, and for this it is enough to check its minimal polynomial. $\qquad\square$

C. TIME-SYSTEMS

The system descriptions in the previous sections do not allow explicit consideration of time, so many properties, like reachability, cannot be even defined in that generality. We show now one way of introducing time into our framework.

(1.1) Definition. A monoid T *has left common divisors* iff for each t, t' in T there are a τ in T and a, b in T such that $t = \tau a, t' = \tau b$.

(1.2) Examples. (i) $T := $ any group.

(ii) $T := $ any $\wedge$-semilattice, with $xy := x \wedge y$; in particular any totally-ordered set T, like the reals or the integers.

In what follows, T is an arbitrary but fixed monoid as above, and k is a fixed field. Let k^T denote the set of all functions $T \to k$; this set has a natural $k[T]$-module structure, namely, each t in T acts as a shift $\sigma_t : \sigma_t(w)(a) := w(at)$.

(1.3) Definition. (A, B, ϱ) is a *time triple* iff it is admissible and A is a $k[T]$-submodule of k^T and $B[\varrho]$ is a subring of $\mathrm{End}_{k[T]}(A)$.

The above is of course just a way of stating that ϱ and all operators in B are shift-invariant. We fix now (A, B, ϱ) as above.

(1.4) Definition. Given an (A, B, ϱ) system $\Sigma = (F, G, H)$, let

$$REACH_\Sigma := \{x \text{ in } k^n, \ x = (L_\Sigma(w))(t), \ w \text{ in } A^m, \ t \text{ in } T\} .$$

Σ *is pointwise reachable* iff $REACH_\Sigma = k^n$.

(1.5) Lemma. $REACH_\Sigma$ *is a subspace of* k^n.

Proof. Let $x = L(w)(t), y = L(w')(t')$. Choose τ, a, b as in (1.1). Then $rx + y = L(r\sigma_a(w) + \sigma_b(w'))(\tau)$, for any r in k.

(1.6) Theorem. Σ is pointwise reachable if and only if there is no v in k^n such that

$$(*) \qquad\qquad v'[G, FG, \ldots, F^{n-1}G] = 0 \,.$$

Proof. Assume that $REACH_\Sigma \neq k^n$. Then there exists a v in k^n such that $REACH_\Sigma \subseteq \ker v$, when v is regarded as a map $k^n \to k$. Let $\boldsymbol{v}$ be the constant function v, i.e. $\boldsymbol{v} : A^n \to A$, $(\boldsymbol{v}x)(t) := v\,x(t)$. Then

$$(**) \qquad\qquad 0 = v'L = v'(I - \varrho F)^{-1} \varrho G \,.$$

Expanding $(I - \varrho F)^{-1}$ in powers of ϱ^{-1} gives the result $(*)$.

The converse is equally clear, since $(*)$ implies $(**)$ via the Cayley-Hamilton Theorem. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad$ $\square$

(1.7) Applications. The above theorem gives a unified proof of the well-known reachability criteria for both continuous- and discrete-time finite-dimensional linear systems. In the delay-differential case, introduced in example A.3.5, F and G are polynomial matrices. Since a polynomial is zero iff each of its coefficients is zero, $(*)$ can also be expressed — in a rather involved way! — as a rank condition on constant real matrices. In the latter form, a special case ($r = 1$, only one delay in F, and no delays in G) was known before; see Kirillova and Čurakova [6]. The general result was announced with an outline of its proof in Sontag [7].

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